ON THE NUMBER OF DISSIMILAR LINE-SUBGRAPHS
OF A GIVEN GRAPH

FRANK HARARY

1. Introduction. Two enumeration formulas are obtained in this paper. The first provides a solution for the counting polynomial which gives the number of dissimilar line-subgraphs of a given graph, and is a generalization of the formula due to Pólya for the number of graphs which appears in [2]. The second serves to find the number of graphs in which a prescribed subgraph is distinguished or rooted. The special case in which a single point of the graph is distinguished is called a *rooted graph* [3, p. 76]. The number of rooted graphs also appears in [2].

Both of these results utilize in an essential way the classical enumeration theorem of Pólya [4] which enables one to express the configuration-counting series in terms of the configuration group and the figure-counting series. In order that these results be self-contained, we briefly review Pólya's Theorem, specialized to one variable. Let *figure* be an undefined term. To each figure there is assigned a nonnegative integer called its *content*. Let φ_k denote the number of different figures of content k. The figure-counting series $\varphi(x)$ is defined by

$$\varphi(x) = \sum_{k=0}^{\infty} \varphi_k x^k.$$

Let Γ be a permutation group of degree s and order h. A *configuration* of length s is a sequence of s figures. The content of a configuration is the sum of the contents of its figures. Two configurations are Γ-*equivalent*, if there is a permutation of Γ sending one into the other. Let F_k denote the number of Γ-inequivalent figures of content k. The configuration-counting series $F(x)$ is defined by

$$F(x) = \sum_{k=0}^{\infty} F_k x^k.$$

The permutation group Γ will be called the *configuration group*.

Pólya's Theorem expresses $F(x)$ in terms of $\varphi(x)$ and Γ, using the cycle index of Γ. Let $h_{j_1 j_2 \cdots j_s}$ be the number of elements of Γ of type $(j_1, j_2, \cdots, j_s)$, that is, having j_k cycles of length k, $(k = 1, \cdots, s)$ so that

(1) $1 \cdot j_1 + 2 \cdot j_2 + \cdots + s \cdot j_s = s$.

Let $f_1, f_2, \cdots, f_s$ be indeterminates. Then $Z(\Gamma)$, the *cycle index* of Γ, is defined as in [4, p. 159] by

(2) $Z(\Gamma) = \dfrac{1}{h} \sum_{(j)} h_{j_1 j_2 \cdots j_s} f_1^{j_1} f_2^{j_2} \cdots f_s^{j_s}$,

where the sum is taken over all partitions (j) of s satisfying (1). For any series $f(x)$, let $Z(\Gamma, f(x))$ be the series obtained from $Z(\Gamma)$ on replacing each indeterminate f_k by $f(x^k)$.

POLYA'S THEOREM [4, p. 163]. *The configuration-counting series is obtained when the figure-counting series is substituted into the cycle index of the configuration group*

(3) $F(x) = Z(\Gamma, \varphi(x))$.

2. Enumeration of the dissimilar line-subgraphs of a given graph.

A *graph* $\mathscr{G}$ consists of a finite set II of points $A_1, A_2, \cdots, A_p$ together with a given subset Δ of the collection of all unordered pairs of distinct points. The elements of Δ are called the *lines* of $\mathscr{G}$. Two points of $\mathscr{G}$ are *adjacent* if there is a line of $\mathscr{G}$ containing them. Two graphs are *isomorphic* if there is a one-to-one correspondence between their point sets which preserves adjacency. An *automorphism* of a graph $\mathscr{G}$ is an isomorphism of $\mathscr{G}$ with itself. It is well known that the collection of all automorphisms of $\mathscr{G}$ constitutes a group [3, p. 5]. Let $\Gamma_0(\mathscr{G})$ be the group of the graph $\mathscr{G}$. Then $\Gamma_0(\mathscr{G})$ is a permutation group of degree $p=$ the number of points of $\mathscr{G}$. We shall call $\Gamma_0(\mathscr{G})$ the *point-group* of $\mathscr{G}$.

A *line-subgraph* or briefly $\angle$-*subgraph* of $\mathscr{G}$ is a graph whose point set is that of $\mathscr{G}$ and whose line set is a subset of the line set of $\mathscr{G}$. Two points A, B of $\mathscr{G}$ are *similar* if there is an automorphism of $\mathscr{G}$ mapping A onto B. Two lines AB, CD of $\mathscr{G}$ are similar if there is an automorphism of $\mathscr{G}$ mapping $\{A, B\}$ onto $\{C, D\}$. Two $\angle$-subgraphs of $\mathscr{G}$ are similar if there is an automorphism of the graph $\mathscr{G}$ which maps the lines of one onto the lines of the other. Similarity being an equivalence relation, *the number of dissimilar $\angle$-subgraphs of $\mathscr{G}$* refers to the number of similarity classes of $\angle$-subgraphs.

The problem solved by Theorem 1 is the following. Given an arbitrary graph $\mathscr{G}$ with p points and q lines, how many dissimilar $\angle$-subgraphs does $\mathscr{G}$ have containing k lines, $k=0, 1, \cdots, q$? Using the terminology of Pólya's Theorem, the configurations to be counted

are the ℓ-subgraphs of $\mathscr{G}$ which are inequivalent with respect to the automorphisms of $\mathscr{G}$. The figures out of which these configurations are built are the pairs of distinct points of $\mathscr{G}$. The content of a figure is 0 if the two points are non-adjacent and is 1 if they are adjacent. Hence the figure-counting series is $\varphi(x) = 1 + x$.

The content of an ℓ-subgraph is the number k of lines it contains. With the basic graph $\mathscr{G}$ fixed throughout this discussion, let g_k be the number of dissimilar ℓ-subgraphs having k lines. Then the configuration-counting series or counting polynomial $g(x)$ is given by

$$g(x) = \sum_{k=0}^{q} g_k x^k.$$

In order to apply Pólya's Theorem to express $g(x)$, it remains to find the configuration group.

We require the permutation group Γ of degree q which acts on the lines of $\mathscr{G}$, having the property that two ℓ-subgraphs are similar if and only if they are Γ-equivalent. Clearly this group consists of all permutations of the set Δ induced by the automorphisms of $\mathscr{G}$, that is, by the elements of $\Gamma_0(\mathscr{G})$. Let $\Gamma_1(\mathscr{G})$, the *line-group* of $\mathscr{G}$, denote this group; then $\Gamma = \Gamma_1(\mathscr{G})$, a permutation group of degree q. Collecting these results and applying Pólya's Theorem, we have the following theorem.

THEOREM 1. *The counting polynomial for the number of dissimilar ℓ-subgraphs of $\mathscr{G}$ is obtained by substituting $1 + x$ into the cycle index of the line-group of $\mathscr{G}$*

$$(4) \qquad g(x) = \sum_{k=0}^{q} g_k x^k = Z(\Gamma_1(\mathscr{G}), \ 1 + x).$$

This specializes at once to the counting polynomial $g_p(x)$ of [2] for the number of different graphs of p points when one takes for $\mathscr{G}$ the *complete graph* of p points, that is, the graph in which any two distinct points are adjacent. For two graphs are different if they are not isomorphic, which means that they are dissimilar ℓ-subgraphs of the complete p point graph.

We next apply Theorem 1 to all the graphs of 4 points. These are drawn in Figure 1, in which $\mathscr{G}_{kn}$ denotes the nth graph of 4 points and k lines. In order to describe the line-groups of these graphs concisely, the following notations are employed. Let E_n, S_n, D_n be the identity, symmetric, and dihedral groups of degree n respectively. Let $G_1 \times G_2$ denote the direct product of the permutation groups G_1 and G_2 which act on two disjoint sets of objects. Let K_4 (the Klein four-group)
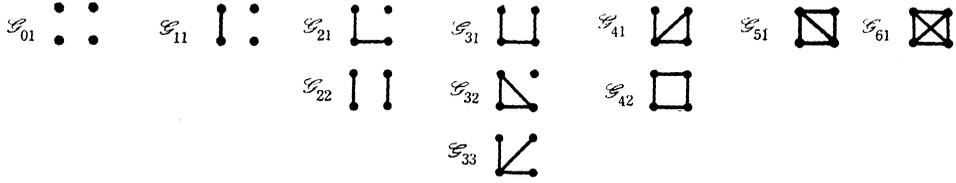
Figure 1

denote the permutation group whose elements are

$$(a)(b)(c)(d), \quad (ab)(cd), \quad (ad)(bc), \quad (ac)(bd).$$

Finally, let $P(S_n)$ be the pair group of S_n, that is, the group of degree $n(n-1)/2$ which is abstractly isomorphic with S_n, but acts on the pairs of the n objects permuted by S_n.

In Table 1 below, each graph $\mathscr{G}$ of 4 points is indicated in the first column, its line-group $\Gamma_1(\mathscr{G})$ in the second column, the cycle index $Z(\Gamma_1(\mathscr{G}))$ in the third column, and the counting polynomial $g(x)$ $=Z(\Gamma_1(\mathscr{G}), 1+x)$ in the last column.

Table 1

$\mathscr{G}$	$\Gamma_1(\mathscr{G})$	$Z(\Gamma_1(\mathscr{G}))$	$g(x)=Z(\Gamma_1(\mathscr{G}),\ 1+x)$
$\mathscr{G}_{01}$	$1=E_0$	1	1
$\mathscr{G}_{11}$	$E=E_1$	f_1	$1+x$
$\mathscr{G}_{21}$	S_2	$(f_1^2+f_2)/2$	$1+x+x^2$
$\mathscr{G}_{22}$	S_2	$(f_1^2+f_2)/2$	$1+x+x^2$
$\mathscr{G}_{31}$	$S_2\times E$	$(f_1^3+f_1f_2)/2$	$1+2x+2x^2+x^3$
$\mathscr{G}_{32}$	S_3	$(f_1^3+3f_1f_2+2f_3)/6$	$1+x+x^2+x^3$
$\mathscr{G}_{33}$	S_3	$(f_1^3+3f_1f_2+2f_3)/6$	$1+x+x^2+x^3$
$\mathscr{G}_{41}$	$S_2\times E_2$	$(f_1^4+f_1^2f_2)/2$	$1+3x+4x^2+3x^3+x^4$
$\mathscr{G}_{42}$	D_4	$(f_1^4+3f_2^2+2f_1^2f_2+2f_4)/8$	$1+x+2x^2+x^3+x^4$
$\mathscr{G}_{51}$	$K_4\times E$	$(f_1^5+3f_1f_2^2)/4$	$1+2x+4x^2+4x^3+2x^4+x^5$
$\mathscr{G}_{61}$	$P(S_4)$	$(f_1^6+9f_1^2f_2^2+8f_3^2+6f_2f_4)/24$	$1+x+2x^2+3x^3+2x^4+x^5+x^6$

We conclude this section with Figure 2 which illustrates the dissimilar ℓ-subgraphs for the case $\mathscr{G} = \mathscr{G}_{51}$.

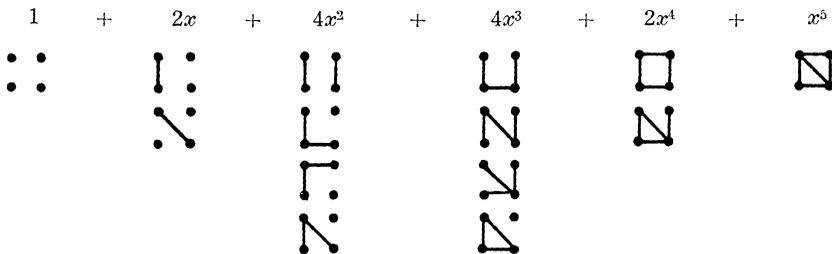$$1 \quad + \quad 2x \quad + \quad 4x^2 \quad + \quad 4x^3 \quad + \quad 2x^4 \quad + \quad x^5$$

Figure 2

3. Enumeration of $\mathscr{G}$-rooted graphs. A *rooted graph* is one in which one of the points is singled out from the others. The distinguished point is called the *root R*. Two rooted graphs $\mathscr{G}_1$, $\mathscr{G}_2$ with roots R_1, R_2 are isomorphic if there exists an isomorphism of $\mathscr{G}_1$ with $\mathscr{G}_2$ considered as ordinary unrooted graphs, which maps R_1 onto R_2.

In this section the notion of rooted gragh will be generalized. Two graphs having the same point set Π as $\mathscr{G}$ will be isomorphic as $\mathscr{G}$-rooted graphs, speaking loosely, if they are similar when viewed from $\mathscr{G}$. A $\mathscr{G}$-*rooted graph H* is one, defined on the same point set as $\mathscr{G}$, in which the graph $\mathscr{G}$ is distinguished. It is not necessary that $\mathscr{G}$ be an /-subgraph of H (as will be seen in Figure 3). The counting polynomial whose coefficients give the number of such $\mathscr{G}$-rooted graphs of p points and k lines for $k=0, \cdots , p(p-1)/2$ is the result of Theorem 2.

Let the arbitrary graph $\mathscr{G}$ with p points and q lines be fixed throughout this discussion. Two graphs $\mathscr{G}_1$, $\mathscr{G}_2$, both having the same point set Π as $\mathscr{G}$, are *isomorphic as $\mathscr{G}$-rooted graphs* if there is an isomorphic mapping of $\mathscr{G}_1$ with $\mathscr{G}_2$ which is also an automorphism of $\mathscr{G}$. Let h_k be the number of different (that is, non-isomorphic) $\mathscr{G}$-rooted graphs of k lines and let

$$h(x)= \sum_{k=0}^{p(p-1)/2} h_k x^k .$$

As for /-subgraphs, a $\mathscr{G}$-rooted graph is a configuration whose figures are the pairs of distinct points of $\mathscr{G}$, and the figure-counting series is again $1+x$. It follows from the definition of isomorphism of two $\mathscr{G}$-rooted graphs that *the configuration group Γ is the pair group of the point-group of $\mathscr{G}$.* We define *the pair group $P(H)$ of a permutation group H* of degree s as the permutation group of degree $s(s-1)/2$ which acts on the set of all pairs of distinct elements of the object set of H, each of the elements of $P(H)$ being induced by an element of H. Thus $P(H)$ and H are isomorphic as abstract groups.

We have seen that for $\mathscr{G}$-rooted graphs, the figure series is $1+x$ and the configuration group $\Gamma=P(\Gamma_0(\mathscr{G}))$. We now apply Pólya's Theorem.

THEOREM 2. *The counting polynomial for $\mathscr{G}$-rooted graphs is obtained when one substitutes $1+x$ into the cycle index of the pair group of the point-group of $\mathscr{G}$:*

(5) $$h(x)= \sum_{k=0}^{p(p-1)/2} h_k x^k =Z(P(\Gamma_0(\mathscr{G})), 1+x) .$$

It is interesting that this formula is a generalization of both the number of ordinary graphs and the number of rooted graphs [2]. For the choice of $\mathscr{G}$ as the complete graph of p points, we have $P(\Gamma_0(\mathscr{G}))$ $=\Gamma_1(\mathscr{G})$, so that, as remarked after Theorem 1, $h(x)=g_p(x)=$ the counting polynomial for all p point graphs. On the other hand, if $\mathscr{G}$ is taken as a graph with two components, one a single isolated point and the other a complete graph of $p-1$ points, then

$$h(x)=Z(P(\Gamma_0(\mathscr{G})),\ 1+x)=Z(P(S_{p-1}\times E),\ 1+x)$$

which is exactly the formula in [2] for $G_p(x)$, the counting polynomial for the rooted graphs of p points.

After writing $Z(\Gamma_0(\mathscr{G}))$, the expression for $Z(P(\Gamma_0(\mathscr{G})))$ is obtained readily using a formula from [2], which appears as equation (8) below.

For convenience, let Γ be any permutation group and

$$(6) \qquad\qquad Z_2(\Gamma)=Z(P(\Gamma)).$$

Then, referring to (2),

$$(7) \qquad\qquad Z_2(\Gamma)=\frac{1}{h}\sum_{(j)} h_{j_1 j_2 \cdots j_s} Z_2(f_1^{j_1} f_2^{j_2} \cdots f_s^{j_s})$$

where, from [2], one has

$$(8) \qquad Z_2(f_1^{j_1} f_2^{j_2} \cdots f_s^{j_s}) = \sum_{n=0}^{[(s-1)/2]} f_{2n+1}^{n j_{2n+1}+(2n+1)\binom{j_{2n+1}}{2}} \cdot \sum_{n=1}^{[s/2]} (f_n f_{2n}^{n-1})^{j_{2n}} f_{2n}^{2n\binom{j_{2n}}{2}}$$
$$\cdot \sum_{1\leq q<r\leq s} f_{m(q,\,r)}^{j_q j_r d(q,\,r)},$$

in which $m(q,r)$ and $d(q,r)$ denote least common multiple and greatest common divisor respectively. Combining (7) and (8) with (2), one has a formula for $Z_2(\Gamma_0(\mathscr{G}))$ once $Z(\Gamma_0(\mathscr{G}))$ is known.

We conclude by applying Theorem 2 to the 4 point graphs given in Figure 1. Let $\mathscr{G}'$ denote the *complement* of $\mathscr{G}$, that is, the graph whose point set coincides with that of $\mathscr{G}$, in which two points are adjacent if and only if they are not adjacent in $\mathscr{G}$. Since $\Gamma_0(\mathscr{G}')=$ $\Gamma_0(\mathscr{G})$, we see that the number of $\mathscr{G}$-rooted graphs equals the number of $\mathscr{G}'$-rooted graphs. Hence it is appropriate to divide the graphs of Figure 1 into pairs $\mathscr{G}, \mathscr{G}'$. Note that $\mathscr{G}_{31}$ and its complement are isomorphic, but otherwise each of these graph pairs contains distinct graphs. The results for $h(x)$ are given in Table 2.

We illustrate the process of calculating the polynomials in Table 2 with $\mathscr{G}_{22}$ or $\mathscr{G}_{22}'$.

Table 2

$\mathscr{G}$	$h(x)$
$\mathscr{G}_{01}$ or $\mathscr{G}_{61}$	$g_4(x)=1+x+2x^2+3x^3+2x^4+x^5+x^6$
$\mathscr{G}_{11}$ or $\mathscr{G}_{51}$	$1+3x+6x^2+3x^3+6x^4+3x^5+x^6$
$\mathscr{G}_{21}$ or $\mathscr{G}_{41}$	$1+4x+9x^2+12x^3+9x^4+4x^5+x^6$
$\mathscr{G}_{22}$ or $\mathscr{G}_{42}$	$1+2x+4x^2+5x^3+4x^4+2x^5+x^6$
$\mathscr{G}_{31}$	$1+4x+9x^2+12x^3+9x^4+4x^5+x^6$
$\mathscr{G}_{32}$ or $\mathscr{G}_{33}$	$G_4(x)=1+2x+4x^2+6x^3+4x^4+2x^5+x^6$

$$\Gamma_0(\mathscr{G}_{42})=\Gamma_0(\mathscr{G}_{22})=D_4$$

$$Z(D_4)=\frac{1}{8}(f_1^4+3f_2^2+2f_1^2f_2+2f_4).$$

Applying (8), one gets

$$Z_2(D_4)=\frac{1}{8}(f_1^6+5f_1^2f_2^2+2f_2f_4),$$

so that the resulting polynomial is

$$h(x)=1+2x+4x^2+5x^3+4x^4+2x^5+x^6 .$$

The coefficients in this polynomial count the $\mathscr{G}$-rooted graphs of Figure 3. We first redraw the diagrams of graphs $\mathscr{G}_{22}$ and $\mathscr{G}_{42}$ so that they appear as complements on the same point set.
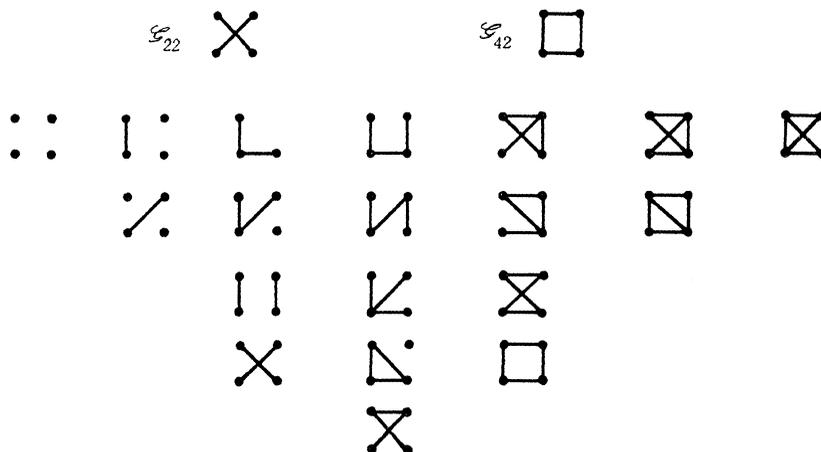

Figure 3

Finally we observe that both of these counting problems have counterparts in the theory of binary relations [1]. They serve to enumerate dissimilar symmetric subrelations over the same field as a given symmetric irreflexive relation and the "$\mathscr{R}$-rooted" symmetric irreflexive

relation $\mathscr{R}$. The extensions to relations which are not necessarily symmetric or irreflexive correspond to the analogous counting problems for directed graphs and for graphs in the wider sense in which loops or slings[1] [3, p. 3] may occur, and offer no difficulty.

REFERENCES

1. R.L. Davis, *The number of structures of finite relations*, Proc. Amer. Math. Soc., **4** (1953), 486–495.
2. F. Harary, *The number of linear, directed, rooted, and connected graphs*, Trans. Amer. Math. Soc., **78** (1955), 445–463.
3. D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936; reprinted New York, 1950.
4. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen, und chemische Verbindungen*, Acta Math., **68** (1937), 145–254.

UNIVERSITY OF MICHIGAN

---

[1] A sling is a line joining a point with itself.