# A NON-ARCHIMEDIAN MEASURE IN THE SPACE OF REAL SEQUENCES

L. W. COHEN

**1. Introduction.** Let $S$ be the set of real sequences $X=(x_n)$. For $X, Y \in S$ we define $X+Y=(x_n+y_n)$, 0 as the sequence $x_n=0$ and introduce order by writing $X>0$ when for some $m$, $x_n=0$ for $n<m$ and $x_m>0$. Thus $S$ may be considered as an ordered abelian group with a non-archimedian order. Let $S$ be topologized by considering the open intervals

$$(X, Y)=\{Z|X<Z<Y\}$$

as a basis for the open sets. Then $S$ is a topological group. We note that $S$ is not locally compact. We wish to define a measure on $S$ which is invariant with respect to translations of measurable sets by elements in $S$ and which assigns a nonzero measure to the sets in a basis for the topology in $S$. It is evident from a consideration of the spheres in Hilbert space that such a measure can not in general be real valued for spaces which are not locally compact. In the example studied here the range of the measure function is a subset of $S$.

The ring of measurable sets which serves as the domain of the measure function is generated by a class of sets called intervals. We shall show that these intervals are a basis for the topology of $S$ defined by the open intervals. They have some properties of the real half-open intervals $a' \leqq x < a''$ which are useful in deriving the properties of a measure function.

For a positive integer $p$ and real numbers

$$a_1, \cdots, a_{p-1}, a'_p, a''_p$$

let $I_p=I(a_1, \cdots, a_{p-1}; a'_p, a''_p)$ be the set of $X=(x_n) \in S$ such that

$$x_n=a_n\,, \qquad \text{for } n<p\,,$$

$$a'_p \leqq x_p < a''_p$$

$$-\infty < x_n < +\infty\,, \qquad n>p\,.$$

If $p=1$ there are no conditions on the $x_n$ for $n<p$. If $a''_p \leqq a'_p$ then $I_p$ is empty. That the sets $I_p$ and the open intervals $(X, Y)$ are equivalent as bases for neighborhood topologies is shown as follows:

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Consider

$$X=(x_n)\in I(a_1, \cdots, a_{p-1}; a'_p, a''_p) .$$

Then

$$x_n=a_n, \text{ for } n<p, \text{ and } a'_p\leqq x_p<a''_p .$$

Now consider $X'_n=(x'_n)$, $X''=(x''_n)$ where

$$x'_n=x_n=x''_n \qquad \text{for } n\leqq p ,$$

$$x'_{p+1}<x_{p+1}<x''_{p+1} .$$

Clearly

$$X', X''\in I(a_1, \cdots, a_{p-1}; a'_p, a''_p) ,$$

$$X'<X<X'' .$$

Now if $Y=(y_n)\in(X', X'')$ then

$$x'_n=y_n=x''_n=a_n \qquad \text{for } n<p ,$$

$$a'_p\leqq x'_p=x_p=y_p=x''_p<a''_p$$

and so $Y\in I(a_1, \cdots, a_{p-1}; a'_p, a''_p)$. Hence

$$X\in(X', X'')\subset I(a_1, \cdots, a_{p-1}; a'_p, a''_p) .$$

Conversely, consider $X=(x_n)\in(X', X'')$ where $X'=(x'_n)<X''=(x''_n)$. From the definition of order in $S$ it follows that there is an integer $p$ such that

$$x'_n=x_n=x''_n \text{ for } n<p , \ x'_p<x''_p$$

and one of the following is true:

(1) $x'_p<x_p<x''_p$ ,
(2) $x'_p<x_p=x''_p$ ,
(3) $x'_p=x_p<x''_p$ .

If (1) is true let

$$a_n=x_n \text{ for } n<p, \ a'_p=x_p, \ a''_p=x''_p .$$

It follows that

$$X\in I(a_1, \cdots, a_{p-1}; a'_p, a''_p)\subset(X', X'') .$$

Suppose (2) is true. Since $X<X''$, there is a smallest integer $q>p$ such that $x_q<x''_q$. Now let

$$a_n=x_n \text{ for } n<q, \ a'_q=x_q \text{ and } a''_q=x''_q .$$

It follows that

$$X \in I(a_1, \cdots, a_{q-1}; a'_q, a''_q) \subset (X', X'') .$$

Suppose (3) is true. Since $X' < X$, there is a smallest integer $q > p$ such that $x'_q < x_q$. Let

$$a_n = x_n \text{ for } n < q, \ a'_q = x_q, \ a''_q = x_q + 1 .$$

Again it follows that

$$X \in I(a_1, \cdots, a_{q-1}; a'_q, a''_q) \subset (X', X'') .$$

The equivalence of the two bases is established.

For each interval $I_p$ the element $(x_n) \in S$ where

$$x_p = \max [a''_p - a'_p, 0] \text{ and } x_n = 0 \text{ if } n \neq p$$

is called the length of $I_p$ and is denoted by $\mu(I_p)$. Clearly $\mu(I_p) \geqq 0$ in $S$ and the equality holds if and only if $I_p$ is empty. It will be shown that: The intervals $I_p$ generate a ring over which the function $\mu$ can be extended to an additive, nonnegative function with values in $S$. If $M$ is a set in the ring and $X+M$ is the set of $X+Y$ for $Y \in M$ then $\mu(M) = \mu(X+M)$. The function $\mu$ may be called an invariant measure on the ring.

**2. Properties of Intervals $I_p$.** Consider two intervals

$$I_p = I(a_1, \cdots, a_{p-1}; a'_p, a''_p) , \qquad I_q = I(b_1, \cdots, b_{q-1}; b'_q, b''_q) .$$

The following two lemmas are immediate consequences of the definition of interval.

LEMMA 1. *$0 \neq I_q \subset I_p$ if and only if $p \leqq q$, and* $a_n = b_n$, $\quad n < p$,

$$a'_p \leqq b_p < a''_p , \qquad p < q ,$$

$$a'_p \leqq b'_p < b''_p \leqq a''_p , \qquad p = q .$$

LEMMA 2. *If $p < q$ and $I_p \cap I_q \neq 0$ then $I_q \subset I_p$.*

*Proof.* Since $p < q$ and there is some $X = (x_n) \in I_p \cap I_q$, we have

$$a_n = x_n = b_n , \qquad n < p ,$$

$$a'_p \leqq x_p = b_p < a''_p .$$

It follows from Lemma 1 that $I_q \subset I_p$.

LEMMA 3. *If* $I_p \cap I_q \neq 0$ *then* $I_p \cap I_q = I_r$ *where* $r = \max[p, q]$.

LEMMA 4. *The union of a finite number of intervals is the union of a finite number of disjoint intervals.*

*Proof.* The statement is true for a single interval. Assume that the statement is true for the union of any $m$ intervals. Consider

$$(1) \qquad I_{p_i},\ i=1, \cdots, m+1 .$$

If the intervals (1) are disjoint the statement is true for them. Suppose that for $h \neq j$, $I_{p_h} \cap I_{p_j} \neq 0$. If $p_h < p_j$ then, by Lemma 2, $I_{p_j} \subset I_{p_h}$. Then the intervals (1) have the same union as some $m$ of them and the statement follows from the assumption. If $p_h = p_j = p$ then, since $I_{p_h} \cap I_{p_j} \neq 0$, we have

$$I_{p_h} = I(a_1, \cdots, a_{p-1}; a'_p, a''_p) , \qquad I_{p_j} = I(a_1, \cdots, a_{p-1}; b'_p, b''_p) ,$$

and the real half open intervals $[a'_p, a''_p)$, $[b'_p, b''_p)$ have a nonempty intersection. If

$$c'_p = \min(a'_p, b'_p), \ \ c''_p = \max(a''_p, b''_p)$$

then $[a'_p, a''_p) \cup [b'_p, b''_p) = [c'_p, c''_p)$ and

$$I_{p_h} \cup I_{p_k} = I(a_1, \cdots, a_{p-1}; c'_p, c''_p) = I_p .$$

The intervals (1) have the same union as the $m$ intervals $I_p, I_{p_i}$ where $i \neq h, j$, and the statement again follows from the assumption. Induction completes the proof.

LEMMA 5. *If* $I_{p_i}$, $i=1, \cdots, m$, *are disjoint nonempty subintervals of* $I_p$ *and* $I_p = \bigcup_{i=1}^{m} I_{p_i}$ *then* $p_i = p$ *for* $i=1, \cdots, m$, *and*

$$\mu(I_p) = \sum_{i=1}^{m} \mu(I_{p_i}) .$$

*Proof.* Let

$$I_p = I(a_1, \cdots, a_{p-1}; a'_p, a''_p)$$
$$I_{p_i} = I(a_{i1}, \cdots, a_{i, p_i - 1}; a'_{p_i}, a''_{p_i}) , \qquad i=1, \cdots, m.$$

Since $0 \neq I_{p_i} \subset I_p$, we have $p \leqq p_i$, and

$$a_{in} = a_n , \qquad n < p ,\ i=1, \cdots, m,$$
$$a'_p \leqq a_{ip} < a''_p , \qquad p_i > p ,$$
$$a'_p \leqq a'_{p_i} < a''_{p_i} \leqq a''_p , \qquad p_i = p .$$

Consider the half-open intervals $[a'_{p_i}, a''_{p_i})$ for $p_i=p$ and the numbers $a_{ip}$ for $p_i>p$. Let $c_1, \cdots, c_k$ be the distinct numbers among those $a_{ip}$. Since $\bigcup_{i=1}^m I_{p_i}=I_p$ and the $I_{p_i}$ are disjoint,

$$[a'_p, a''_p)=\Big(\bigcup_{p_i=p}[a'_{p_i}, a''_{p_i})\Big)\cup\Big(\bigcup_{j=1}^{k}[c_j]\Big)$$

and the summands are disjoint sets. But a half-open real interval is not such a union unless there are no sets $[c_j]$ consisting of single points. Hence $p_i=p$ for $i=1, \cdots, m$ and

$$a''_p-a'_p=\sum_{i=1}^{m}(a''_{p_i}-a'_{p_i}) . \tag{1}$$

If $\mu(I_p)=(x_n)$, $\mu(I_{p_i})=(x_{in})$ then, since $p_i=p$ and $I_{p_i}\neq 0$,

$$x_n=x_{in}=0 , \qquad n\neq p,\ i=1, \cdots, m,$$

$$x_p=a''_p-a'_p ,$$

$$x_{ip}=a''_{p_i}-a'_{p_i} , \qquad i=1, \cdots, m,$$

and it follows from (1) that

$$\sum_{i=1}^{m}\mu(I_{p_i})=\Big(\sum_{i=1}^{m}x_{in}\Big)=(x_n)=\mu(I_p) .$$

LEMMA 6. *If $I_{p_i}$, $i=1, \cdots, m$, and $J_{q_j}$, $j=1, \cdots, n$, are two sets of disjoint intervals with the same union then*

$$\sum_{i=1}^{m}\mu(I_{p_i})=\sum_{j=i}^{n}\mu(J_{p_j}) .$$

*Proof.* Since, by Lemma 2, the intersection of two intervals is an interval, possibly empty, the sets $I_{p_i}\cap J_{q_j}$ are disjoint intervals. Since the $I_{p_i}$ and the $J_{q_j}$ have the same union, we have

$$I_{p_i}=\bigcup_{j=1}^{n}(I_{p_i}\cap J_{q_j}) , \qquad i=1, \cdots, m,$$

$$J_{q_j}=\bigcup_{i=1}^{m}(I_{p_i}\cap J_{q_j}) , \qquad j=1, \cdots, n.$$

Applying Lemma 5 and recalling that $\mu(I_p)=0\in S$ if $I_p$ is empty, we obtain

$$\mu(I_{p_i})=\sum_{j=1}^{n}\mu(I_{p_i}\cap J_{q_j}) ,$$

$$\mu(J_{q_j})=\sum_{i=1}^{m}\mu(I_{p_i}\cap J_{q_j}) .$$

Since $S$ is an abelian group,

$$\sum_{i=1}^{m} \mu(I_{p_i}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu(I_{p_i} \cap J_{q_j}) = \sum_{j=1}^{n} \mu(J_{q_j}) .$$

In order to obtain properties of differences of unions of intervals

$$\bigcup_{i=1}^{m} I_{p_i} - \bigcup_{j=1}^{n} J_{q_j}$$

it will be sufficient to consider the special class $\mathscr{D}$ of sets

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i} ,$$

$$I_{p_i} \text{ disjoint, } I_{p_i} \subset I_p, \ i=1, \cdots, m.$$

Since $I_{p_i} \subset I_p$, either $p_i \geqq p$ or $I_{p_i}=0$.[1] A set $E \in \mathscr{D}$ is called *proper* if, among the $I_p$, $I_{p_i}$ used to represent it, $p_i > p$.

LEMMA 7. *If $E \in \mathscr{D}$ then $E$ is the union of a finite number of disjoint proper elements of $\mathscr{D}$.*

*Proof.* If $E \in \mathscr{D}$ then

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i}$$

where

$$I_p = I(a_1, \cdots, a_{p-1}; a'_p, a''_p) ,$$

$$I_{p_i} = I(a_{i1}, \cdots, a_{i,p_i-1}; a'_{p_i}, a''_{p_i}) , \qquad i=1, 2, \cdots, m,$$

and the $I_{p_i}$ are disjoint subsets of $I_p$. Hence $p_i \geqq p$ and $a_{in}=a_n$ for $n<p$. If $p_i=p$ then $\sigma_i=[a'_{p_i}, a''_{p_i}) \subset [a'_p, a''_p)=\sigma$ and the $\sigma_i$ are disjoint.

$$\sigma - \bigcup_{p_i=p} \sigma_i = \bigcup_{j=1}^{h} \tau_j$$

where the $\tau_j=[b'_j, b''_j)$ are disjoint. Let

$$I_p^j = I(a_1, \cdots, a_{p-1}; b'_j, b''_j) , \quad \alpha_j = \{i | a_{i_p} \in \tau_j \text{ and } p_i > p\} , \quad j=1, \cdots, h.$$

The $\alpha_j$ are disjoint; and $I_{p_i} \subset I_p^j$ if and only if $p_i > p$ and $i \in \alpha_j$. The sets

$$E_j = I_p^j - \bigcup_{i \in \alpha_j} I_{p_i} \qquad j=1, \cdots, h,$$

[1] It will be assumed that the $I_{p_i}$ in a representation of a set $E$ are not empty. This does not sacrifice any generality.

are disjoint proper elements of $\mathscr{D}$ whose union is $E$. This is so because

$$I_p - \bigcup_{p=p_i} I_{p_i} = \bigcup_{j=1}^{h} I_p^j$$

and every $I_{p_i}$ with $p_i > p$ is in some $I_p^j$.

LEMMA 8. *If*

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i}\ , \qquad F = J_p - \bigcup_{j=1}^{n} J_{q_j}$$

*are proper sets in* $\mathscr{D}$ *then* $E \cap F = 0$ *if and only if* $I_p \cap J_p = 0$.

*Proof*. Since $E \subset I_p$, $F \subset J_p$ it is clear that $E \cap F = 0$ if $I_p \cap J_p = 0$. Suppose $I_p \cap J_p \neq 0$. Let

$$I_p = I(a_1, \cdots, a_{p-1}; a'_p, a''_p)\ , \qquad J_p = I(b_1, \cdots, b_{p-1}; b'_p, b''_p)\ ,$$

$$I_{p_i} = I(a_{i1}, \cdots, a_{i,p_i-1}; a'_{p_i}, a''_{p_i})\ , \qquad J_{p_j} = I(b_{j1}, \cdots, b_{j,p_j-1}; b'_{p_j}, b''_{p_j})\ ,$$

$$i = 1, \cdots, m,\ j = 1, \cdots, n.$$

Since $E$ and $F$ are proper, $p_i$, $q_j > p$. Since $I_p \cap J_p \neq 0$, we have $a_n = b_n$, $n < p$, and $[a'_p, a''_p) \cap [b'_p, b''_p) = [c', c'') \neq 0$. The half-open interval $[c', c'')$ contains a number $x \neq a_{ip}, b_{jp}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. If $X = (x_n)$ where $x_p = x$ and $x_n = a_n$, for $n < p$, then $X \in E \cap F$. Hence if $E \cap F = 0$ then $I_p \cap J_p = 0$.

For

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i} \in \mathscr{D}$$

we define $\mu(E) \in S$ by

$$\mu(E) = \mu(I_p) - \sum_{i=1}^{m} \mu(I_{p_i})\ .$$

It is to be noted that a set $E$ may have two representations;

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i} = J_q - \bigcup_{j=1}^{n} J_{q_j}$$

and the uniqueness of $\mu(E)$ must be proved (cf. corollary to Lemma 11). In order to do this and to prove the additivity of $\mu$ as a function on $\mathscr{D}$ to $S$ we make some definitions which are useful.

If

$$I_p = I(a_1, \cdots, a_{p-1}; a'_p, a''_p)$$

we call $p$ the *rank* of $I_p$, $a_n$ the $n$th *point component* of $I_p$ and $[a'_p, a''_p)$

the *interval component* of $I_p$. Given a set of nonempty intervals $I_{p_i}, \cdots, I_{p_m}$ the number $N$ of distinct ranks $p_i$ is called the *spread* of the set of intervals. For example, if $E$ is a proper set in $\mathscr{D}$, then the spread of $E$ is 1 if and only if $E$ is an interval $I_p$.

LEMMA 9. *If*

(a) $I_{p_i}$, $i=1, \cdots, m$, *are nonempty, disjoint intervals,*

(b) $E_j = J_{q_j} - \bigcup_{k=1}^{k_j} J_{q_{jk}}$, $j=1, \cdots, h$, *are nonempty, disjoint, proper sets in* $\mathscr{D}$,

(c) $\bigcup_{i=1}^m I_{p_i} = \bigcup_{j=1}^h E_j$

*then*

$$\sum_{i=1}^m \mu(I_{p_i}) = \sum_{j=1}^h \mu(E_j) .$$

*Proof*. Let $N$ be the spread of the set of intervals $I_{p_i}, J_{q_j}, J_{q_{jk}}$. If $N=1$, $p_i = q_j = p$ and the sets $E_j$ are the intervals $J_{q_j}$ since the $E_j$ are proper. The conclusion follows from Lemma 6.

Assume that $N > 1$ and that the lemma is proved if the spread of the set of intervals in (a), (b) is $N-1$.

First we show that if $p = \min(p_1, \cdots, p_m)$, $q = \min(q_1, \cdots, q_h)$ then $p=q$. Suppose $p<q$. There is some $p_r = p$. The $p$th component of $I_{p_r}$ is a half-open interval $\sigma$ and the $p$th component of $J_{q_j}$ is a point $b_j$. There is a number $x \in \sigma - \{b_1, \cdots, b_h\}$. If $X = (x_n)$ where $x_p = x$ and $x_n$, $n<p$, is the $n$th component of $I_{p_r}$ then

$$X \in I_{p_r} - \bigcup_{j=1}^h J_{q_j} \subset \bigcup_{i=1}^m I_{p_i} - \bigcup_{j=1}^h E_j$$

contrary to (c). Hence $q \leq p$. Suppose $q<p$. There is some $q_r = q$ and

$$E_r = J_{q_r} - \bigcup_{k=1}^{k_r} J_{q_{rk}} \neq 0 , \qquad q_{rk} > q_r .$$

The $q$th component of $J_{q_r}$ is a nonempty half-open interval $\tau$, the $q$th components of $J_{q_{rk}}$, $k=1, \cdots, k_r$, and of $I_{p_i}$ are points, say $c_1, \cdots, c_s$. There is a number $x \in \tau - \{c_1, \cdots, c_s\}$. If $X=(x_n)$ where $x_q = x$ and $x_n$, $n<q$, is the $n$th component of $J_{q_r}$,

$$X \in \left(J_{q_r} - \bigcup_{k=1}^{k_r} J_{q_{rk}}\right) - \bigcup_{i=1}^m I_{p_i} \subset \bigcup_{j=1}^h E_j - \bigcup_{i=1}^m I_{p_i} ,$$

contrary to (c). Hence $p=q$.

Next, we show that

$$\bigcup_{p_i = p} I_{p_i} = \bigcup_{q_j = p} J_{q_j} . \tag{1}$$

Let

$$A' = \bigcup_{p_i = p} I_{p_i}\,, \qquad A'' = \bigcup_{q_j = p} J_{q_j}\,.$$

Suppose $A'' - A' \neq 0$. For some $q_r = p$, there is

$$X = (x_n) \in J_{q_r} - \bigcup_{p_i = p} I_{p_i}\,.$$

Let

$\sigma =$ the interval component of $J_{q_r}$,

$\sigma_i =$ the interval component of $I_{p_i}$ where $p_i = p$,

$\alpha = \{i \mid J_{q_r} \cap I_{p_i} \neq 0 \text{ and } p_i = p\}$ .

Then

$$x_p \in \sigma - \bigcup_{i \in \alpha} \sigma_i$$

and so there is a nonempty, half-open interval $\tau$ such that

$$\tau \subset \sigma - \bigcup_{i \in \alpha} \sigma_i\,.$$

The $p$th components of the $I_{p_i}$, $p_i > p$, and of $J_{q_{rk}}$, $k = 1, \cdots, k_r$ are finite in number, say $c_1, \cdots, c_s$. Hence there is a number $y$ such that

$$y \in \tau - \{c_1, \cdots, c_s\}\,.$$

If $Y = (y_n)$ where $y_p = y$ and $y_n$, $n < p$, is the $n$th component of $J_{q_r}$,

$$Y \in \Bigl(J_{q_r} - \bigcup_{k=1}^{k_r} J_{q_{rk}}\Bigr) - \bigcup_{i=1}^{m} I_{p_i} \subset \bigcup_{j=1}^{h} E_j - \bigcup_{i=1}^{m} I_{p_i}\,,$$

contrary to (c). A similar argument shows that $A' - A'' \neq 0$ leads to a contradiction. Hence (1) is proved.

Since the $E_j$ are disjoint proper sets in $\mathscr{D}$ it follows from Lemma 8 that $I_{q_r} \cap I_{q_s} = 0$ if $p = q_r = q_s$ and $r \neq s$. Hence, from (1) and Lemma 6,

$$\sum_{p_i = p} \mu(I_{p_i}) = \sum_{q_j = p} \mu(J_{q_j})\,. \tag{2}$$

From (c) and (1)

$$\Bigl(\bigcup_{p_i > p} I_{p_i}\Bigr) \cup \Bigl(\bigcup_{q_j = p} J_{q_j}\Bigr) = \Bigl(\bigcup_{q_j = p} \Bigl(J_{q_j} - \bigcup_{k=1}^{k_j} J_{q_{jk}}\Bigr)\Bigr) \cup \Bigl(\bigcup_{q_j > p} E_j\Bigr)\,. \tag{3}$$

It follows from (a), (1) that the two unions on the left are disjoint and from (b) that the two unions on the right are disjoint. Hence

$$\Bigl(\bigcup_{p_i > p} I_{p_i}\Bigr) \cup \Bigl(\bigcup_{q_j = p} \bigcup_{k=1}^{k_j} J_{q_{jk}}\Bigr) = \bigcup_{q_j > p} E_j\,. \tag{4}$$

The ranks of the intervals $I_{p_i}$, $J_{q_j}$, $J_{q_{jk}}$ occurring in (4) exclude $p$ since $p_i > p$, $q_{jk} > q_j = p$ on the left and $q_{jk} > q_j > p$ on the right. Hence the spread of the set of intervals in (4) is $N-1$. Since the $E_j$ are disjoint it follows from Lemma 8 that the $J_{q_j}$, $q_j = p$, are disjoint. Since for each $j$, the $J_{q_{jk}}$ are disjoint in $k$ and $J_{q_{jk}} \subset J_{q_j}$, the $J_{q_{jk}}$ are disjoint in $j$, $k$ for $q_j = p$. It follows from (1), (a) that the intervals on the left of (4) are disjoint. Thus the set of nonempty intervals on the left of (4) satisfy (a) of the lemma, the set of $E_j$ on the right satisfy (b), and (4) is (c) for the intervals involved. Since the spread is $N-1$, we have, by the assumption of the lemma for $N-1$,

$$\sum_{p_i > p} \mu(I_{p_i}) + \sum_{q_j = p} \sum_{k=1}^{k_j} \mu(J_{q_{jk}}) = \sum_{q_j > p} \mu(E_j) \ . \tag{5}$$

Combining (2), (5), it follows that

$$\sum_{i=1}^{m} \mu(I_{p_i}) = \sum_{p_i > p} \mu(I_{p_i}) + \sum_{p_i = p} \mu(I_{p_i}) = \sum_{q_j = p} \left( \mu(J_{q_j}) - \sum_{k=1}^{k_j} \mu(J_{q_{jk}}) \right) + \sum_{q_j > p} \mu(E_j)$$

$$= \sum_{q_j = p} \mu(E_j) + \sum_{q_j > p} \mu(E_j) = \sum_{j=1}^{h} \mu(E_j) \ .$$

LEMMA 10. *For $E \in \mathscr{D}$, $\mu(E) = 0 \in S$ if $E$ is empty and*

$$\mu(E) = \sum_{j=1}^{n} \mu(E_j)$$

*if $E = \bigcup_{j=1}^{n} E_j$ where the $E_j$ are nonempty, disjoint, proper sets in $\mathscr{D}$.*

*Proof.* If $E \in \mathscr{D}$, then

$$E = I_p - \bigcup_{i=1}^{m} I_{p_i}$$

where the $I_{p_i}$ are disjoint subsets of $I_p$. If $E$ is empty, then

$$\bigcup_{i=1}^{m} I_{p_i} = I_p$$

and it follows from Lemma 5 that

$$\mu(E) = \mu(I_p) - \sum_{i=1}^{m} \mu(I_{p_i}) = 0 \in S \ .$$

If $E = \bigcup_{j=1}^{n} E_j$ where the $E_j$ are nonempty, disjoint, proper sets in $\mathscr{D}$ then

$$I_p = \left( \bigcup_{j=1}^{n} E_j \right) \cup \left( \bigcup_{i=1}^{m} I_{p_i} \right)$$

and the intervals in the set $\{I_p, E_j, I_{p_i} \neq 0\}$ satisfy the conditions of

Lemma 9. Since $\mu(I_{p_i})=0$ if $I_{p_i}$ is empty, it follows that

$$\mu(I_p)=\sum_{j=1}^{n}\mu(E_j)+\sum_{i=1}^{m}\mu(I_{p_i}) ,$$

$$\mu(E)=\mu(I_p)-\sum_{i=1}^{m}\mu(I_{p_i})=\sum_{j=1}^{n}\mu(E_j) .$$

LEMMA 11. *If* $E\in\mathscr{D}$ *and* $E_1, \cdots, E_m$ *are disjoint elements of* $\mathscr{D}$ *such that*

$$E=\bigcup_{i=1}^{m}E_i$$

*then*

$$\mu(E)=\sum_{i=1}^{m}\mu(E_i) .$$

*Proof.* It follows from Lemma 10 that the statement is true if $E=0$ and that if $E\neq 0$ only $E_i\neq 0$ need be considered. By Lemma 7,

$$E_i=\bigcup_{j=1}^{j_i}E_{ij} , \qquad i=1, \cdots, m$$

where the $E_{ij}$, $j=1, \cdots, j_i$, are disjoint, nonempty, proper elements of $\mathscr{D}$. Since the $E_i$ are disjoint, the $E_{ij}$ are disjoint in $i, j$. Now

$$E=\bigcup_{i=1}^{m}\bigcup_{j=1}^{j_i}E_{ij} .$$

By Lemma 10,

$$\mu(E)=\sum_{i=1}^{m}\sum_{j=1}^{j_i}\mu(E_{ij})=\sum_{i=1}^{m}\mu(E_i) .$$

COROLLARY. *For* $E\in\mathscr{D}$, $\mu(E)$ *is unique.*

This follows from Lemma 11 with $m=1$.

LEMMA 12. *For* $E\in\mathscr{D}$, $\mu(E)\geqq 0$ *in the order in S.*

*Proof.* If $E=0$, $\mu(E)=0$. If $E$ is a nonempty, proper set in $\mathscr{D}$ then

$$E=I_p-\bigcup_{i=1}^{m}I_{p_i}$$

and $p_i>p$. Now $\mu(I_p)=(x_n)$, $\mu(I_{p_i})=(x_{in})$, $i=1, \cdots, m$, and

$$x_p > 0\,, \qquad x_n = 0\,, \qquad\qquad n \neq p$$

$$x_{in} = 0\,, \qquad\qquad n \leqq p < p_i,\ i = 1, \cdots, m.$$

Since

$$\mu(E) = \mu(I_p) - \sum_{i=1}^{m} \mu(I_{p_i}) = (x_n - \sum_{i=1}^{m} x_{im})\,,$$

$$x_n - \sum_{i=1}^{m} x_{in} = 0\,, \qquad\qquad n < p\,,$$

$$x_p - \sum_{i=1}^{m} x_{ip} = x_p > 0\,,$$

it follows that $\mu(E) > 0$ in the order in $S$.

It now follows from Lemmas 7, 11 and the fact that the sum of positive elements of $S$ is positive that $\mu(E) \geqq 0$ for $E \in \mathscr{D}$.

3. **On Generating a Ring.** The set of intervals $I_p$, having the properties of Lemmas 2, 4 is an example of a class $\mathscr{C}$ of sets satisying the following conditions:

(i) $0 \in \mathscr{C}$.

(ii) If $A, B \in \mathscr{C}$ then $A \cap B \in \mathscr{C}$.

(iii) If $A_1, \cdots, A_m \in \mathscr{C}$ there are disjoint $B_1, \cdots, B_n \in \mathscr{C}$ such that

$$\bigcup_{i=1}^{m} A_i = \bigcup_{j=1}^{n} B_j\,.$$

Let $\mathscr{D}$ be the class of sets $E$ such that

(iv) $E = A - \bigcup_{i=1}^{m} A_i$, $A, A_i \in \mathscr{C}$, $A_i$ disjoint, $A_i \subset A$.

Let $\mathscr{R}$ be the class of sets $M$ such that

(v) $M = \bigcup_{i=1}^{m} E_i$, $E_i \in \mathscr{D}$, $E_i$ disjoint.

We note that $\mathscr{C} \subset \mathscr{D} \subset \mathscr{R}$. It will be shown that $\mathscr{R}$ is a ring.

LEMMA 13. *If $E, F \in \mathscr{D}$ then $E \cap F \in \mathscr{D}$.*

*Proof.* There are sets $A$, $A_i$, $B$, $B_j$ satisfying (iv) such that

$$E = A - \bigcup_{i=1}^{m} A_i\,, \qquad F = B - \bigcup_{j=1}^{n} B_j\,.$$

Now

$$E \cap F = A \cap B - \Big( \bigcup_{j=1}^{n} (A \cap B_j) \Big) \cup \Big( \bigcup_{i=1}^{m} (A_j \cap B) \Big)\,.$$

By (ii), $A \cap B$, $A \cap B_j$, $A_i \cap B$ are in $\mathscr{C}$. It follows from (iii) that there are disjoint $C_1, \cdots, C_s \in \mathscr{C}$ such that

$$\left(\bigcup_{j=1}^{n}(A\cap B_j)\right)\cup\left(\bigcup_{i=1}^{m}(A_i\cap B)\right)=\bigcup_{k=1}^{s}C_k\ .$$

Since $C_k\subset A\cap B$ and

$$E\cap F=A\cap B-\bigcup_{k=1}^{s}C_k\ ,$$

we have $E\cap F\in\mathscr{D}$.

LEMMA 14. *$E, F\in\mathscr{D}$ there are disjoint $E_0,\cdots,E_s\in\mathscr{D}$ such that*

$$E-F=\bigcup_{k=0}^{s}E_k\ .$$

*Proof.* There are $A$, $A_i$, $B$, $B_j\in\mathscr{C}$ satisfying (iv) such that

$$E=A-\bigcup_{i=1}^{m}A_i\ ,\qquad F=B-\bigcup_{j=1}^{n}B_j\ .$$

Let

$$E_0=(A-A\cap B)\cap E\ ,\qquad E_j=B_j\cap E\ ,\qquad j=1,\cdots,n.$$

Now $A-A\cap B\in\mathscr{D}$ and it follows from Lemma 13 that $E_j\in\mathscr{D}$, $j=0,\cdots,n$. Since $E_0\cap B=0$, $E_j\subset B_j\subset B$ and the $B_j$, $j=1,\cdots,n$, are disjoint, $E_0, E_1,\cdots,E_n$ are disjoint. From

$$\bigcup_{j=0}^{n}E_j\subset E$$

and

$$E_0\cap F\subset(A-A\cap B)\cap B=0\ ,\qquad E_j\cap F\subset B_j\cap F=0\ ,\quad j=1,\cdots,n$$

follows

$$\bigcup_{j=0}^{n}E_j\subset E-F\ .$$

On the other hand

$$E-F\subset\left(A-\bigcup_{i=1}^{m}A_i\right)-\left(B-\bigcup_{j=1}^{n}B_j\right)\subset(A-A\cap B)\cap E\cup\left(\bigcup_{j=1}^{n}(B_j\cap E)\right)$$

$$=\bigcup_{j=0}^{n}E_j\ .$$

Hence

$$E-F=\bigcup_{j=0}^{n}E_j\ ,\qquad E_j\in\mathscr{D}\ ,\qquad E_j \text{ disjoint.}$$

THEOREM 1. *$\mathscr{R}$ is a ring.*

*Proof.* For $M, N \in \mathscr{R}$ there are disjoint sets $E_i \in \mathscr{D}$ and disjoint sets $F_j \in \mathscr{D}$ such that

$$M = \bigcup_{i=1}^{m} E_i , \qquad N = \bigcup_{j=1}^{n} F_j .$$

The sets $E_i \cap F_j$ are disjoint and, by Lemma 13, belong to $\mathscr{D}$. Hence

$$(1) \qquad M \cap N = \left(\bigcup_{i=1}^{m} E_i\right) \cap \left(\bigcup_{j=1}^{n} F_j\right) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (E_i \cap F_j) \in \mathscr{R} .$$

Now

$$M - M \cap N = \bigcup_{i=1}^{m} E_i - \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (E_i \cap F_j) = \bigcup_{i=1}^{m} \left(E_i - E_i \cap \left(\bigcup_{j=1}^{n} F_j\right)\right)$$

$$= \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} (E_i - E_i \cap F_j) .$$

By Lemma 14, $M_{ij} = E_i - E_i \cap F_j$ is the union of a finite number of disjoint sets in $\mathscr{D}$ and so $M_{ij} \in \mathscr{R}$. It follows from (1) that

$$M_i = \bigcap_{j=1}^{n} M_{ij} \in \mathscr{R} , \qquad i = 1, \cdots, m.$$

Since each $M_i \subset E_i$ and the $E_i$ are disjoint, the $M_i$ are disjoint. Each $M_i$ is the union of a finite number of disjoint sets in $\mathscr{D}$. Hence

$$(2) \qquad M - M \cap N = \bigcup_{i=1}^{m} M_i \in \mathscr{R} .$$

Finally,

$$M \cup N = (M - M \cap N) \cup (M \cap N) \cup (N - M \cap N) .$$

It follows from (1), (2) that each summand is in $\mathscr{R}$. Since the summands are disjoint and are the unions of disjoint sets in $\mathscr{D}$,

$$(3) \qquad M \cup N \in \mathscr{R} .$$

That $\mathscr{R}$ is a ring follows from (1), (2), (3).

**4. The Measure Function on $\mathscr{R}$ to $S$.** The function $\mu(I_p)$ on the class $\mathscr{C}$ of intervals $I_p$ to $S$ is extended to a function on $\mathscr{D}$ to $S$ which is additive and nonnegative in the sense of the corollary to Lemma 11 and Lemma 12. If $M$ is in the ring $\mathscr{R}$ of unions of disjoint sets in $\mathscr{D}$ then

$$M = \bigcup_{i=1}^{m} E_i$$

where the $E_i$ are disjoint sets in $\mathscr{D}$. We define

$$\mu(M)=\sum_{i=1}^{m}\mu(E_i)\ .$$

THEOREM 2. *$\mu(M)$ is a single valued function on $\mathscr{R}$ to $S$ such that $\mu(M)\geqq 0$ and*

$$\mu(M)=\sum_{i=1}^{m}\mu(M_i)\ \textit{if}\ \ M=\bigcup_{i=1}^{m}M_i,\ M_i\in\mathscr{R}\ , \qquad M_i\ \textit{disjoint}.$$

*Proof*. Suppose

$$M=\bigcup_{i=1}^{m}E_i=\bigcup_{j=1}^{n}F_j$$

where the sets $E_i$ and the sets $F_j$ are disjoint elements of $\mathscr{D}$. Then

$$E_i=\bigcup_{j=1}^{n}(E_i\cap F_j)\ , \qquad i=1,\cdots,m,$$

$$F_j=\bigcup_{i=1}^{m}(E_i\cap F_j)\ , \qquad j=1,\cdots,n,$$

and the disjoint sets $E_i\cap F_j$ are elements of $\mathscr{D}$ by Lemma 13. From Lemma 11,

$$\mu(E_i)=\sum_{j=1}^{n}\mu(E_i\cap F_j)\ ,$$

$$\mu(F_j)=\sum_{i=1}^{m}\mu(E_i\cap F_j)\ .$$

Since $S$ is an abelian group,

$$\mu(M)=\sum_{i=1}^{m}\mu(E_i)=\sum_{i=1}^{m}\sum_{j=1}^{n}\mu(E_i\cap F_j)=\sum_{j=1}^{n}\mu(F_j)\ .$$

Hence $\mu(M)$ is a single valued function on $\mathscr{R}$ to $S$.

Since $\mu(E)\geqq 0$ in $S$ for $E\in\mathscr{D}$ and the sum of nonnegative elements in $S$ is nonnegative, we have $\mu(M)\geqq 0$ in $\mathscr{R}$.

If $M=\bigcup_{i=1}^{m}M_i$ and the $M_i$ are disjoint elements in $\mathscr{R}$,

$$M_i=\bigcup_{j=1}^{n_i}E_{ij}\ , \qquad i=1,\cdots,m,$$

and

$$M=\bigcup_{i=1}^{m}\bigcup_{j=1}^{n_i}E_{ij}$$

where the $E_{ij}$ are disjoint elements in $\mathscr{D}$. Hence

$$\mu(M)=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\mu(E_{ij})=\sum_{i=1}^{m}\mu(M_i)\ .$$

THEOREM 3. *If* $M\in\mathscr{R}$, $X\in S$ *and*

$$X+M=\{X+Y|Y\in M\}$$

*then* $X+M\in\mathscr{R}$ *and* $\mu(X+M)=\mu(M)$.

*Proof.* If $I_p=I(a_1,\cdots,a_{p-1};a'_p,a''_p)$ and $X=(x_n)$ then

$$X+I_p=I(x_1+a_1,\cdots,x_{p-1}+a_{p-1},x_p+a'_p,x_p+a''_p)\in\mathscr{C}\subset\mathscr{R}$$

and

$$\mu(X+I_p)=\mu(I_p)\ . \tag{1}$$

If

$$M=E=I_p-\bigcup_{i=1}^{m}I_{p_i}\in\mathscr{D}\ ,$$

then

$$X+M=(X+I_p)-\bigcup_{i=1}^{m}(X+I_{p_i})\in\mathscr{D}\subset\mathscr{R}$$

and, by (1),

$$\mu(X+M)=\mu(I_p)-\sum_{i=1}^{m}\mu(I_{p_i})=\mu(M)\ . \tag{2}$$

If $M=\bigcup_{i=1}^{m}E_i$ and the $E_i$ are disjoint sets in $\mathscr{D}$, then $X+E_i$ are disjoint sets in $\mathscr{D}$ and, by (2), $\mu(E_i)=\mu(X+E_i)$. Since

$$X+M=\bigcup_{i=1}^{m}(X+E_i)\in\mathscr{R}\ ,$$

we have

$$\mu(X+M)=\sum_{i=1}^{m}\mu(X+E_i)=\sum_{i=1}^{m}\mu(E_i)=\mu(M)\ .$$

The following observations were suggested by O. Nikodým, to whom the author is indebted for a helpful reading of the manuscript. Given $X=(x_n)\in S$ such that all but a finite number of the $x_n$ are zero, there is a measurable $M\in\mathscr{R}$ such that $\mu(M)=X$. The results obtained here for real valued sequences (over the ordinals $n<\omega$) may be extended by the same methods to the space of real valued sequences $x_\alpha$ over any given initial section of ordinals $\alpha<\xi$.

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