

ASYMPTOTIC RELATIONS BETWEEN SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. Introduction. A. Wintner [1], N. Levinson [2], H. Weyl [3] and others have obtained interesting asymptotic relations between the solutions of a given system of differential equations and those of an approximate system. In their investigations the solutions of the approximate system of differential equations are assumed to be bounded. In this paper we consider the asymptotic problems of the solutions from a more general point of view, given only certain order relations satisfied by the solutions of the approximate system. The method we shall use is to study an associated system of integral equations which yields the asymptotic relations between the solutions of the perturbed and unperturbed equations. With the aid of the Phragmén-Lindelöf Theorems [4], our results can be easily extended to the complex domain.

2. Asymptotic relations in the real domain. Consider the system of differential equations written in the vector form

$$(2.1) \quad \frac{dy}{dx} = A(x)y + f(x, y),$$

where $A(x) = \|a_{ij}(x)\|$ is a $n \times n$ matrix and y and $f(x, y)$ are respectively column vectors with components $y^{(i)}$ and $f^{(i)}(x, y)$, $i=1, 2, \dots, n$. Defining the norms $\|y\|$, $\|A\|$ of vectors y and matrices A by

$$\|y\| = \sum_{i=1}^n |y^{(i)}|, \quad \|A\| = \sum_{i,j=1}^n |a_{ij}|,$$

it is easy to verify that

$$\begin{aligned} \|y_1 + y_2\| &\leq \|y_1\| + \|y_2\|, & \|A_1 + A_2\| &\leq \|A_1\| + \|A_2\|, \\ \|A_1 A_2\| &\leq \|A_1\| \|A_2\|, & \|A y\| &\leq \|A\| \|y\|. \end{aligned}$$

In this section we assume that $a_{ij}(x)$ and $f^{(i)}(x, y)$ (for each fixed complex y) are complex-valued functions of the real variable x belonging to $L(0, R)$ for every positive R . Furthermore we assume that for each $x \geq 0$, $f^{(i)}(x, y)$ is a continuous function of y for all complex y and $f(x, y)$ satisfies the Lipschitz condition

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$$(2.2) \quad \|f(x, y_1) - f(x, y_2)\| \leq g(x) \|y_1 - y_2\|$$

for all complex y_1 and y_2 , $g(x)$ being of the class $L(0, R)$ for every positive R . (If all the functions $a_{ij}(x)$ and $f^{(i)}(x, y)$ are real and only real-valued solutions are considered, it is enough to assume that $f^{(i)}(x, y)$ is continuous for all real y and (2.2) holds for all real y_1 and y_2 .)

By a solution of (2.1) we mean a vector function $y(x)$ which is absolutely continuous and satisfies the equation (2.1) almost everywhere. For the existence and uniqueness of solutions, see [5, Sections 68.3 and 68.5].

In this section we shall establish two asymptotic relations between the solutions of (2.1) and those of the approximate equation

$$(2.3) \quad \frac{dy}{dx} = A(x)y.$$

Equation (2.3) has n linearly independent vector solutions y_i , $i=1, 2, \dots, n$. Let Y be the $n \times n$ matrix whose columns are the n vector solutions y_i , y_i being so chosen to satisfy the initial condition $Y(0)=I$. Y is non-singular and has an inverse Y^{-1} . Since each column of Y is a solution of (2.3) it is clear that Y satisfies

$$(2.4) \quad \frac{dY}{dx} = A(x)Y, \quad Y(0)=I.$$

If $\text{tr } A$ denotes the sum of the diagonal elements of A , it is well-known [6] that

$$\det Y(x) = \exp\left(\int_0^x \text{tr } A(t) dt\right).$$

For convenience we first establish the following results.

LEMMA 1. *Let*

(a) $y(x)$ be a solution of (2.1), $y(0)=c$,

(b) $Y(x)$ be the matrix solution of (2.4).

Then $y(x)$ satisfies the vector integral equation

$$(2.5) \quad y(x) = Y(x)c + \int_0^x Y(x)Y^{-1}(t)f(t, y(t))dt.$$

Proof. Set $y(x)=Y(x)z(x)$. Then a substitution into (2.1) gives

$$(2.6) \quad \frac{dYz}{dx} + Y \frac{dz}{dx} = AYz + f.$$

Using (2.4), we have

$$\frac{dz}{dx} = Y^{-1}f$$

or

$$(2.7) \quad z(x) = c + \int_0^x Y^{-1}(t)f(t, y(t))dt .$$

Multiplication of (2.7) by $Y(x)$ gives the result (2.5).

The following lemma is due to R. Bellman.

LEMMA 2. *Let $u(x)$ and $v(x)$ be real-valued functions defined for $x \geq a$, both being non-negative. If*

(a) *$v(x)$ and $u(x)v(x)$ belong to $L(a, R)$ for every $R \geq a$.*

(b) *$u(x) \leq M + \int_a^x u(t)v(t)dt$, $M > 0$, ($x \geq a$),*

then

$$u(x) \leq Me^{\int_a^x v(t)dt}, \quad (x \geq a).$$

Proof. Multiplication of (b) by $v(x)$ gives

$$(2.8) \quad u(x)v(x) \leq v(x) \left(M + \int_a^x u(t)v(t)dt \right).$$

Divide both sides of (2.8) by $M + \int_a^x u(t)v(t)dt$ and integrate the result from a to x , obtaining

$$(2.9) \quad \log \left(M + \int_a^x u(t)v(t)dt \right) \leq \log M + \int_a^x v(t)dt .$$

Then lemma follows from (b) and (2.9).

We first prove the following asymptotic relation. $\Re a$ denotes the real part of a complex number a .

THEOREM 1. *Let*

(a) *$\|Y(x)\| = O(h(x))$ as $x \rightarrow \infty$, $h(x)$ being measurable on $0 \leq x < \infty$,*

(b) *$\|f(x, 0)\| [h(x)]^{n-1} \exp \left[- \Re \int_0^x \text{tr } A(t) dt \right]$ and*

$$g(x)[h(x)]^n \exp \left[- \Re \int_0^x \text{tr } A(t) dt \right] \text{ belong to } L(0, \infty) .$$

Then, for each solution $y(x)$ of (2.1), $\|y(x)\| = O(h(x))$ as $x \rightarrow \infty$ and there is a solution $Y(x)c$ of (2.3) such that

$$(2.10) \quad y(x) = Y(x)(c + \varepsilon(x)),$$

where $\|\varepsilon(x)\|$ tends to zero as $x \rightarrow \infty$.

Proof. According to Lemma 1, $y(x)$ satisfies the integral equation

$$(2.11) \quad y(x) = Y(x)c + \int_0^x Y(x)Y^{-1}(t)f(t, y(t))dt.$$

It follows that

$$(2.12) \quad \|y(x)\| \leq \|Y(x)\| \|c\| + \int_0^x \|Y(x)\| \|Y^{-1}(t)\| \|f(t, y(t))\| dt.$$

Since (2.2) implies that

$$(2.13) \quad \|f(x, y)\| \leq g(x)\|y\| + \|f(x, 0)\|$$

and $\|Y(x)\|$ does not vanish, one obtains from (2.12)

$$(2.14) \quad \frac{\|y(x)\|}{\|Y(x)\|} \leq \|c\| + \int_0^x \|Y^{-1}(t)\| (g(t)\|y(t)\| + \|f(t, 0)\|) dt.$$

Since each component of $Y^{-1}(x)$ is the cofactor of the corresponding component in $\det Y(x)_t$, divided by $\det Y(x)$, $Y(x)_t$ being the transpose of $Y(x)$, it is clear that as $x \rightarrow \infty$

$$(2.15) \quad \|Y^{-1}(x)\| = O\left([h(x)]^{n-1} \exp\left[-\int_0^x \operatorname{tr} A(t) dt\right]\right).$$

From (b), it follows that as $x \rightarrow \infty$

$$(2.16) \quad \int_0^x \|Y^{-1}(t)\| \|f(t, 0)\| dt = O(1).$$

From (2.14) we then have

$$(2.17) \quad \frac{\|y(x)\|}{\|Y(x)\|} \leq M + \int_0^x g(t) \|Y^{-1}(t)\| \|Y(t)\| \frac{\|y(t)\|}{\|Y(t)\|} dt$$

for some constant M . An appeal to Lemma 2 yields

$$(2.18) \quad \frac{\|y(x)\|}{\|Y(x)\|} \leq M \exp\left(\int_0^x g(t) \|Y^{-1}(t)\| \|Y(t)\| dt\right).$$

Conditions (a), (b) and (2.15) ensure that the integral in (2.18) is bounded for $x \geq 0$ and consequently as $x \rightarrow \infty$

$$(2.19) \quad \|y(x)\| = O(h(x)).$$

Since

$$\int_0^x \|Y^{-1}(t)\| \|f(t, y(t))\| dt \leq \int_0^x \|Y^{-1}(t)\| (g(t)\|y(t)\| + \|f(t, 0)\|) dt$$

and the integral on the right hand side is convergent (by virtue of conditions (a), (b), (2.15) and (2.19)), the integral

$$\int_0^\infty Y^{-1}(t) f(t, y(t)) dt$$

exists. Splitting the integral in (2.11) into two, one integrating from 0 to x and the other from x to ∞ , we have

$$(2.20) \quad y(x) = Y(x)(c + \varepsilon(x))$$

for an appropriate constant c , where

$$(2.21) \quad \varepsilon(x) = - \int_x^\infty Y^{-1}(t) f(t, y(t)) dt .$$

This completes the proof of Theorem 1.

Given stronger conditions, we can show that $\|Y(x)\varepsilon(x)\|$ tends to zero as x tends to infinity.

THEOREM 2. *Let*

(a) $\|Y(x)\| = O(h(x))$ as $x \rightarrow \infty$, $h(x)$ being non-decreasing on $0 \leq x < \infty$,

(b) $\|f(x, 0)\| [h(x)]^n \exp \left[- \mathcal{R} \int_0^x \operatorname{tr} A(t) dt \right]$ and

$g(x) [h(x)]^{n+1} \exp \left[- \mathcal{R} \int_0^x \operatorname{tr} A(t) dt \right]$ belong to $L(0, \infty)$.

Then for each solution $y(x)$ of (2.1) there is a solution $Y(x)c$ of (2.3) such that

$$(2.22) \quad y(x) = Y(x)c + \varepsilon(x) ,$$

where $\|\varepsilon(x)\|$ tends to zero as x tends to infinity.

Proof. Since the conditions of Theorem 1 are satisfied, we obtain at once the relation (2.22) with

$$\varepsilon(x) = - Y(x) \int_x^\infty Y^{-1}(t) f(t, y(t)) dt .$$

Clearly

$$\begin{aligned} \|\varepsilon(x)\| &\leq \int_x^\infty \|Y(x)\| \|Y^{-1}(t)\| \|f(t, y(t))\| dt \\ &\leq M \int_x^\infty h(t) \|Y^{-1}(t)\| (g(t)\|y(t)\| + \|f(t, 0)\|) dt \end{aligned}$$

for some constant M , the existence of the last integral being guaranteed by the conditions (b). It follows that $\|\varepsilon(x)\|$ tends to zero as x tends to infinity. This completes the proof.

Remark. Under the conditions of Theorem 2, the singular integral equation

$$(2.23) \quad y(x) = Y(x)c - Y(x) \int_x^\infty Y^{-1}(t)f(t, y(t))dt$$

defines actually a one-to-one correspondence between the solutions of (2.1) and (2.3) such that the corresponding solutions are asymptotically equivalent in the sense of Theorem 2. The previous statement remains true if "Theorem 2" is replaced by "Theorem 1" provided $h(x)$ is bounded on $0 \leq x \leq x_0$ for every positive x_0 . To see this it is sufficient to prove the existence and uniqueness of the solution of the integral equation (2.23) for an arbitrary but fixed c , the result then follows from the fact that a solution of (2.23) is a solution of (2.1).

Define $y_0(x) = 0$ and

$$(2.24) \quad y_n(x) = Y(x)c - Y(x) \int_x^\infty Y^{-1}(t)f(t, y_{n-1}(t))dt, \quad n = 1, 2, 3, \dots$$

If $\|y_{n-1}(x)\| = O(h(x))$, the integral in (2.24) exists. It follows that $\|y_n(x)\| = O(h(x))$ and the integral in (2.24) exists for all $y_{n-1}(x)$. If

$$(2.25) \quad H(x) = \int_x^\infty [h(t)]^n g(t) \exp \left[-\mathcal{L} \int_0^t \text{tr} A(u) du \right] dt,$$

it is easy to verify that

$$(2.26) \quad \|y_{n+1}(x) - y_n(x)\| \leq M^{2n+1} h(x) \frac{[H(x)]^n}{n!} \quad (n = 0, 1, 2, \dots)$$

for an appropriate constant M . It follows that $y_n(x)$ converges to a limit $y(x)$ uniformly on $0 \leq x \leq x_0$ for every positive x_0 . Clearly $\|y(x)\| = O(h(x))$, $\|y_n(x)\| = O(h(x))$ uniformly in n and $y(x)$ satisfies (2.23). Suppose that (2.23) has another solution $z(x)$, then $\|z(x)\| = O(h(x))$ and

$$(2.27) \quad \|y(x) - z(x)\| \leq \|Y(x)\| \int_x^\infty \|Y^{-1}(t)\| g(t) \|y(t) - z(t)\| dt$$

or

$$(2.28) \quad \|y(x) - z(x)\| \leq M^3 h(x) H(x)$$

for some constant M . Substituting (2.28) into (2.27) and repeating the process, we have

$$(2.29) \quad \|y(x) - z(x)\| \leq M^{2n+1} h(x) \frac{[H(x)]^n}{n!}$$

for every n . (2.29) implies that $y(x)$ and $z(x)$ are equal and hence the solution of (2.23) is unique.

3. Asymptotic relations in the complex domain. In this section the results of § 2 are extended to the complex domain for a system of linear differential equations. In § 2 the success of the method depends on the Lipschitz condition (2.2). But if $f(x, y)$ is an analytic function of the complex variables x and y for x in a region R and all y and satisfies the Lipschitz condition

$$\|f(x, y_1) - f(x, y_2)\| \leq g(x) \|y_1 - y_2\|$$

for all y_1 and y_2 , it is necessary that $f(x, y)$ is a polynomial of degree 1 in y or containing no y . If $f(x, y)$ is not analytic for all y , it is not clear to the author how this problem may be attacked in the complex domain.

For convenience we use z as the independent variable and write $z = x + iy = r \exp(i\theta)$, where x, y, r and θ are real and r is non-negative.

We need the following Phragmén-Lindelöf theorems.

Let R_1 be the strip of the complex plane

$$(3.1) \quad x \geq 0, \quad y_1 \leq y \leq y_2, \quad y_2 - y_1 = a^{-1}\pi,$$

and R_2 the sector

$$(3.2) \quad r \geq 0, \quad \theta_1 \leq \theta \leq \theta_2, \quad \theta_2 - \theta_1 = a^{-1}\pi.$$

LEMMA 3. *Let $u(z)$ be an analytic function in R_1 . If*

- (a) $|u(z)| \leq M$ in R_1 for $x=0, y=y_1$ and $y=y_2$,
- (b) $u(z) = O(e^{Ne^{bx}})$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$ for some constants b and $N, b < a$,

then $|u(x)| \leq M$ in R_1 .

LEMMA 4. *Let $u(z)$ be analytic in R_1 . If*

- (a) $u(x + iy_k)$ tends to a limit a_k as $x \rightarrow \infty$, $k=1, 2$,
- (b) $u(z)$ is bounded in R_1 ,

then as $z \rightarrow \infty$ $u(z)$ tends to a limit a uniformly in R_1 and $a = a_1 = a_2$.

LEMMA 5. *Let $u(z)$ be analytic in R_2 . If*

- (a) $|u(z)| \leq M$ in R_2 for $\theta = \theta_1$ and $\theta = \theta_2$,
- (b) $u(z) = O(e^{Nr^b})$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$ for some constants b and $N, b < a$,

then $|u(z)| \leq M$ in R_2 .

LEMMA 6. Let $u(z)$ be analytic in R_2 . If

- (a) $u(re^{i\theta_k})$ tends to a limit a_k as $r \rightarrow \infty$, $k=1, 2,$
- (b) $u(z)$ is bounded in R_2 ,

then as $z \rightarrow \infty$ $u(z)$ tends to a limit a uniformly in R_2 and $a = a_1 = a_2$.

For proof of these theorems, see [4].

Consider the system of linear differential equations written in the vector form

$$(3.3) \quad \frac{dw}{dz} = (A(z) + B(z))w + f(z),$$

where $A(z) = \|a_{ij}(z)\|$ and $B(z) = \|b_{ij}(z)\|$ are $n \times n$ matrices and w and $f(z)$ column vectors with components $w^{(i)}$ and $f^{(i)}(z)$ respectively, $i=1, 2, \dots, n$. We assume that $a_{ij}(z)$, $b_{ij}(z)$ and $f^{(i)}(z)$ are single-valued analytic functions in the regions with which we shall be concerned in this section.

Let $W(z)$ be a matrix solution of

$$(3.4) \quad \frac{dW}{dz} = A(z)W, \quad W(z_0) = I.$$

We shall establish asymptotic relations between the solutions of (3.3) and those of the approximate system

$$(3.5) \quad \frac{dw}{dz} = A(z)w.$$

Write

$$(3.6) \quad H(x, y) = H(x, y, x_0) = \int_{x_0}^x [h(x, y)]^n \|B(z)\| \exp \left[- \mathcal{R} \int_{z_0}^z \text{tr} A(u) du \right] dx,$$

$$(3.7) \quad K(x, y) = K(x, y, x_0) = \int_{x_0}^x [h(x, y)]^{n-1} \|f(z)\| \exp \left[- \mathcal{R} \int_{z_0}^z \text{tr} A(u) du \right] dx,$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, z_0 being in R_1 .

THEOREM 3. Let

- (a) $\|W(z)\| = O(h(x, y))$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$, $\|W(z)\| = O(h(x, y)e^{bx})$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$, where $b < a$, $y_2 - y_1 = a^{-1}\pi$, $h(x, y) = |h(z)|$ and $h(z)$ is analytic in R_1 ,
- (b) $H(x, y) = O(1)$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$, $H(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$,
- (c) $K(x, y) = O(1)$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$, $K(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$.

Then, for each solution $w(z)$ of (3.3), $\|w(z)\| = O(h(x, y))$ as $x \rightarrow \infty$,

uniformly in $y_1 \leq y \leq y_2$, and there is a solution $W(z)c$ of (3.5) such that

$$(3.8) \quad w(z) = W(z)(c + \varepsilon(z))$$

in R_1 , where $\|\varepsilon(z)\|$ tends to zero uniformly in R_1 as $z \rightarrow \infty$.

Proof. First we observe that $W(z)/h(z)$ is of $O(1)$ on the lines $y=y_1$, $y=y_2$, of $O(e^{bx})$ uniformly in R_1 . According to Lemma 3, $W(z)/h(z)$ is of $O(1)$ in R_1 . It follows that

$$\|W(z)\| = O(h(x, y)) \quad \text{as } x \rightarrow \infty,$$

uniformly in $y_1 \leq y \leq y_2$.

For a solution, $w(z)$, of (3.3), $w(z_0) = c$, we have

$$(3.9) \quad w(z) = W(z)c + \int_{z_0}^z W(z)W^{-1}(t)[B(t)w(t) + f(t)]dt,$$

where z and z_0 are in R_1 . Multiplication of (3.9) by $W^{-1}(z)$ yields

$$(3.10) \quad W^{-1}(z)w(z) = c + \int_{z_0}^z W^{-1}(t)[B(t)w(t) + f(t)]dt.$$

Since

$$\left\| \int_{x_0+iy}^{x+iy} W^{-1}(t)f(t)dt \right\| \leq MK(x, y)$$

for a suitable constant M , it follows that

$$\int_{z_0}^z W^{-1}(t)f(t)dt,$$

being analytic in R_1 , is bounded in R_1 , by virtue of the conditions in (c) and Lemma 3. Also the integral

$$\int_{x_0+iy_0}^{x_0+iy} W^{-1}(t)B(t)w(t)dt$$

is bounded in R_1 . From (3.10), we then have

$$(3.11) \quad W^{-1}(z)w(z) = E(z) + \int_{x_0+iy}^{x+iy} W^{-1}(t)B(t)w(t)dt$$

for some function $E(z)$, $E(z)$ being bounded in R_1 , and the integral taken along a straight line. Write $w(t) = W(t)W^{-1}(t)w(t)$. Then

$$(3.12) \quad \|W^{-1}(z)w(z)\| \leq M + \int_{x_0+iy}^{x+iy} \|W^{-1}(t)B(t)\| \|W^{-1}(t)w(t)\| dt$$

for some constant M , and $x \geq x_0$. An appeal to Lemma 2 yields

$$(3.13) \quad \|W^{-1}(z)w(z)\| \leq M \exp \left[\int_{x_0+iy}^{x+iy} \|W^{-1}(t)B(t)W(t)\| dt \right], \quad x \geq x_0.$$

In view of the definition of $H(x, y)$, (3.13) gives

$$(3.14) \quad \|W^{-1}(z)w(z)\| \leq M e^{N H(x, y)}$$

for some constant N . It follows that $W^{-1}(z)w(z)$, being analytic in R_1 , is bounded in R_1 , by virtue of the conditions in (b) and Lemma 3. Since

$$\|w(z)\| \leq \|W(z)\| \|W^{-1}(z)w(z)\|,$$

we have also

$$(3.15) \quad \|w(z)\| = O(h(x, y))$$

as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$. From (3.10), the analytic function

$$(3.16) \quad \int_{z_0}^z W^{-1}(t)[B(t)w(t) + f(t)] dt$$

is bounded in R_1 . By virtue of (3.6), (3.7), (b) and (c), the integral (3.16) tends to a limit c_1 as $z \rightarrow \infty$ along $y=y_1$ and to a limit c_2 along $y=y_2$, and hence, by Lemma 4, it tends to a limit c_0 uniformly in R_1 and $c_0=c_1=c_2$. Now (3.10) can be written as

$$(3.17) \quad W^{-1}(z)w(z) = c - \int_z^\infty W^{-1}(t)[B(t)w(t) + f(t)] dt$$

for some c , the integral being convergent to zero uniformly in R_1 . (3.17) is equivalent to (3.8). This proves the theorem.

THEOREM 4. *Let the conditions of Theorem 3 be satisfied. Furthermore let*

- (a) $h(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in R_1 ,
- (b) $h(x, y_i)H(\infty, y_i, x)$ and $h(x, y_i)K(\infty, y_i, x)$ tend to zero $x \rightarrow \infty$, $i=1, 2$.

Then, for each solution $w(z)$ of (3.3), there is a solution $W(z)c$ of (3.5) such that

$$(3.18) \quad w(z) = W(z)c + \varepsilon(z)$$

in R_1 , where $\|\varepsilon(z)\|$ tends to zero uniformly in R_1 as $z \rightarrow \infty$.

Proof. Using (a) and (b), clearly the analytic function

$$\varepsilon(z) = W(z) \int_z^\infty W^{-1}(t)[B(t)w(t) + f(t)] dt$$

converges to zero as $z \rightarrow \infty$ along the lines $y=y_1, y=y_2$ and

$$\|\varepsilon(z)\| = O(e^{bx}), \text{ uniformly in } R_1.$$

According to Lemma 3, $\|\varepsilon(z)\|$ is bounded in R_1 , and by Lemma 4, $\|\varepsilon(z)\|$ converges to zero uniformly in R_1 as $z \rightarrow \infty$. From (3.17), we have

$$(3.19) \quad w(z) = W(z)c - W(z) \int_z^\infty W^{-1}(t)[B(t)w(t) + f(t)]dt$$

which is equivalent to (3.18). This completes the proof.

We shall state two similar theorems for the region R_2 . Set

$$L(r, \theta) = L(r, \theta, r_0) = \int_{r_0}^r [h(r, \theta)]^n \|B(z)\| \exp \left[- \mathcal{R} \int_{z_0}^z \operatorname{tr} A(u) du \right] dr,$$

$$J(r, \theta) = J(r, \theta, r_0) = \int_{r_0}^r [h(r, \theta)]^{n-1} \|f(z)\| \exp \left[- \mathcal{R} \int_{z_0}^z \operatorname{tr} A(u) du \right] dr,$$

where $z = r \exp(i\theta)$, $z_0 = r_0 \exp(i\theta_0)$, z_0 being in R_2 .

THEOREM 5. *Let*

- (a) $\|W(z)\| = O(h(r, \theta))$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $\|W(z)\| = O(h(r, \theta)e^{r^b})$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$,
 where $b < a$, $\theta_2 - \theta_1 = a^{-1}\pi$, $h(r, \theta) = |h(z)|$ and $h(z)$ is analytic in R_2 ,
- (b) $L(r, \theta) = O(1)$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $L(r, \theta) = O(r^b)$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$,
- (c) $J(r, \theta) = O(1)$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $J(r, \theta) = O(e^{r^b})$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$.

Then, for each solution $w(z)$ of (3.3), $\|w(z)\| = O(h(r, \theta))$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$, and there is a solution $W(z)c$ of (3.5) such that

$$w(z) = W(z)(c + \varepsilon(z))$$

in R_2 , where $\|\varepsilon(z)\|$ tends to zero uniformly in R_2 as $z \rightarrow \infty$.

The proof is similar to that of Theorem 3.

THEOREM 6. *Let the conditions of Theorem 5 be satisfied. Furthermore let*

- (a) $h(r, \theta) = O(e^{r^b})$ as $r \rightarrow \infty$, uniformly in R_2 ,
- (b) $h(r, \theta_i)L(\infty, \theta_i, r)$ and $h(r, \theta_i)J(\infty, \theta_i, r)$ tend to zero as $r \rightarrow \infty$,
 $i = 1, 2$.

Then for each solution $w(z)$ of (3.3) there is a solution $W(z)c$ of (3.5) such that

$$w(z) = W(z)c + \varepsilon(z)$$

in R_2 , where $\|\varepsilon(z)\|$ tends to zero uniformly in R_2 as $z \rightarrow \infty$.

The proof of this theorem is similar to that of Theorem 4.

4. Second order differential equation. When the theorems in §§ 2 and 3 apply to a given second order differential equation, it is necessary to know the order relation of both the solutions and their derivatives of the approximate equation. Of course we obtain an asymptotic relation between the solutions, and their derivative of each equation. However, for second order differential equations, we are able to establish asymptotic relations just for the solutions when the order relation of the solution of the approximate equation is given. In view of the interest in second order differential equations, we state some similar results and briefly indicate the proofs.

Consider the second order differential equations

$$(4.1) \quad (r(x)y')' + p(x)y = f(x, y),$$

$$(4.2) \quad (r(x)y')' + p(x)y = 0,$$

where $1/r(x)$, $p(x)$ and $f(x, y)$ (for each fixed complex y) are complex-valued functions of the real variable x , defined for all $x \geq 0$ and belonging to $L(0, R)$ for every positive R ; furthermore for each $x \geq 0$ $f(x, y)$ is a continuous function of y for all complex y satisfying

$$(4.3) \quad |f(x, y_1) - f(x, y_2)| \leq g(x)|y_1 - y_2|$$

for all complex y_1 and y_2 , $g(x)$ being of the class $L(0, R)$ for every positive R .

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of (4.2) with

$$(4.4) \quad r(x)[y_1(x)y_2'(x) - y_1'(x)y_2(x)] = 1$$

almost everywhere.

THEOREM 7. *Let*

(a) $|y_1(x)| + |y_2(x)| = O(h(x))$ as $x \rightarrow \infty$, $h(x)$ being measurable on $0 \leq x < \infty$,

(b) $[h(x)]^2 g(x)$ and $h(x)f(x, 0)$ belong to $L(0, \infty)$.

Then, for each solution $y(x)$ of (4.1), $y(x) = O(h(x))$ as $x \rightarrow \infty$ and there exist constants A and B such that

$$(4.5) \quad \begin{aligned} y(x) &= (A + \varepsilon_1(x))y_1(x) + (B + \varepsilon_2(x))y_2(x), \\ y'(x) &= (A + \varepsilon_1(x))y_1'(x) + (B + \varepsilon_2(x))y_2'(x), \end{aligned}$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ tend to zero as $x \rightarrow \infty$. (The second equation holds only almost everywhere.)

Proof. Let $y(x)$ be a solution of (4.1). Then, by the method of variation of parameters, $y(x)$ satisfies

$$(4.6) \quad y(x) = Ay_1(x) + By_2(x) + \int_0^x f(t, y(t)) [y_2(x)y_1(t) - y_1(x)y_2(t)] dt$$

for some constants A and B . Set $u = |y|/(|y_1| + |y_2|)$. Using (a), (b) and (4.3) one can verify that

$$u(x) \leq M + N \int_0^x g(t)h^2(t)u(t)dt$$

for some positive constants M and N and $x \geq 0$. By Lemma 2 and (b), $u(x)$ is bounded on $0 \leq x < \infty$. It follows that $y(x) = O(h(x))$ as $x \rightarrow \infty$. Write

$$(4.7) \quad \varepsilon_1(x) = \int_x^\infty f(t, y(t))y_2(t)dt, \quad \varepsilon_2(x) = - \int_x^\infty f(t, y(t))y_1(t)dt,$$

the existence of the integrals being guaranteed by (b). The first part of (4.5) follows from (4.6) by splitting the integral, the second part from differentiation of the first.

THEOREM 8. *Let*

- (a) $|y_1(x)| + |y_2(x)| = O(h(x))$ as $x \rightarrow \infty$, $h(x)$ being non-decreasing on $0 \leq x < \infty$,
 (b) $h^3(x)g(x)$ and $h^2(x)f(x, 0)$ belong to $L(0, \infty)$.

Then for each solution $y(x)$ of (4.1) there exist constants A and B such that

$$(4.8) \quad y(x) = Ay_1(x) + By_2(x) + \varepsilon(x),$$

where $\varepsilon(x)$ tends to zero as $x \rightarrow \infty$.

Proof. Under conditions (a) and (b), $\varepsilon_1(x)y_1(x)$ and $\varepsilon_2(x)y_2(x)$ tend to zero as $x \rightarrow \infty$. (4.8) then follows from the first equation of (4.5).

Remark. In some cases, with properly chosen y_1 and y_2 , condition (b) of Theorem 8 may be weakened. For instance let $r(x) = 1$, $p(x) = 0$ and take $y_1(x) = 1$, $y_2(x) = x$, $h(x) = x$; it is true that $y(x) = Ax + B + \varepsilon(x)$, $\varepsilon(x) \rightarrow 0$ as $x \rightarrow \infty$ if $h^3(x)g(x)$ and $h^2(x)f(x, 0)$ belong to $L(0, \infty)$. The discrepancy is due to the fact that in the general case we consider $y_1(x)y_2(x) = O(h^2(x))$, while in this example $y_1(x)y_2(x) = O(h(x))$.

Consider the differential equations

$$(4.9) \quad (r(z)w') + (p(z) + q(z))w = f(z),$$

$$(4.10) \quad (r(z)w') + p(z)w = 0,$$

where $r(z) \neq 0$, $p(z)$, $q(z)$ and $f(z)$ are single-valued analytic functions

in R_1 or R_2 (see (3.1) and (3.2)), as whichever is concerned. Let $w_1(z)$ and $w_2(z)$ be two linearly independent solutions of (4.10) with

$$r(z)[w_1(z)w_2'(z) - w_1'(z)w_2(z)] = 1.$$

Set

$$H(x, y) = H(x, y, x_0) = \int_{x_0}^x h^2(x, y)|q(z)|dx,$$

$$K(x, y) = K(x, y, x_0) = \int_{x_0}^x h(x, y)|f(z)|dx,$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, z_0 being in R_1 .

THEOREM 9. *Let*

- (a) $|w_1(z)| + |w_2(z)| = O(h(x, y))$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$,
 $|w_1(z)| + |w_2(z)| = O(h(x, y)e^{bx})$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$,
 where $b < a$, $y_2 - y_1 = a^{-1}\pi$, $h(x, y) = |h(z)|$ and $h(z)$ is analytic in R_1 ,
- (b) $H(x, y) = O(1)$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$,
 $H(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in R_1 ,
- (c) $K(x, y) = O(1)$ as $x \rightarrow \infty$ for $y = y_1$ and $y = y_2$,
 $K(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in R_1 .

Then, for each solution $w(z)$ of (4.9), $w(z) = O(h(x, y))$ as $x \rightarrow \infty$, uniformly in $y_1 \leq y \leq y_2$, and there exist constants A and B such that

$$w(z) = (A + \varepsilon_1(z))w_1(z) + (B + \varepsilon_2(z))w_2(z),$$

$$w'(z) = (A + \varepsilon_1(z))w_1'(z) + (B + \varepsilon_2(z))w_2'(z),$$

in R_1 , where $\varepsilon_1(z)$ and $\varepsilon_2(z)$ tend to zero uniformly in R_1 as $z \rightarrow \infty$.

THEOREM 10. *Let the conditions of Theorem 9 be satisfied. Furthermore let*

- (a) $h(x, y) = O(e^{bx})$ as $x \rightarrow \infty$, uniformly in R_1 ,
- (b) $h(x, y_i)H(\infty, y_i, x)$ and $h(x, y_i)K(\infty, y_i, x)$ tend to zero as $x \rightarrow \infty$,
 $i = 1, 2$.

Then for each solution $w(z)$ of (4.9) there exist constants A and B such that

$$w(z) = Aw_1(z) + Bw_2(z) + \varepsilon(z)$$

in R_1 , where $\varepsilon(z)$ tends to zero uniformly in R_1 as $z \rightarrow \infty$.

Set

$$L(r, \theta) = L(r, \theta, r_0) = \int_{r_0}^r h^2(r, \theta)|q(z)|dr,$$

$$J(r, \theta) = J(r, \theta, r_0) = \int_{r_0}^r h(r, \theta) |f(z)| dr .$$

where $z = r \exp(i\theta)$, $z_0 = r_0 \exp(i\theta_0)$, z_0 being in R_2 .

THEOREM 11. *Let*

- (a) $|w_1(z)| + |w_2(z)| = O(h(r, \theta))$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $|w_1(z)| + |w_2(z)| = O(h(r, \theta)e^{r^b})$ as $r \rightarrow \infty$, uniformly in $\theta_1 \leq \theta \leq \theta_2$,
 where $b < a$, $\theta_2 - \theta_1 = a^{-1}\pi$, $h(r, \theta) = |h(z)|$ and $h(z)$ is analytic in R_2 ,
- (b) $L(r, \theta) = O(1)$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $L(r, \theta) = O(r^a)$ as $r \rightarrow \infty$, uniformly in R_2 ,
- (c) $J(r, \theta) = O(1)$ as $r \rightarrow \infty$ for $\theta = \theta_1$ and $\theta = \theta_2$,
 $J(r, \theta) = O(e^{r^b})$ as $r \rightarrow \infty$, uniformly in R_2 .

Then for each solution $w(z)$ of (4.9) there exist constants A and B such that

$$w(z) = (A + \varepsilon_1(z))w_1(z) + (B + \varepsilon_2(z))w_2(z) ,$$

$$w'(z) = (A + \varepsilon_1(z))w_1'(z) + (B + \varepsilon_2(z))w_2'(z) ,$$

in R_2 , where $\varepsilon_1(z)$ and $\varepsilon_2(z)$ tend to zero uniformly in R_2 as $z \rightarrow \infty$.

THEOREM 12. *Let the conditions of Theorem 11 be satisfied. Furthermore let*

- (a) $h(r, \theta) = O(e^{r^b})$ as $r \rightarrow \infty$, uniformly in R_2 ,
- (b) $h(r, \theta_i)L(\infty, \theta_i, r)$ and $h(r, \theta_i)J(\infty, \theta_i, r)$ tend to zero as $r \rightarrow \infty$,
 $i = 1, 2$.

Then for each solution $w(z)$ of (4.9) there exist constants A and B such that

$$w(z) = Aw_1(z) + Bw_2(z) + \varepsilon(z)$$

in R_2 , where $\varepsilon(z)$ tends to zero uniformly in R_2 as $z \rightarrow \infty$.

Combining the methods of §3 and the first part of this section, one can prove these theorems by using the integral equation

$$w(z) = Aw_1(z) + Bw_2(z) + \int_{z_0}^z [f(t) - q(t)w(t)][w_2(z)w_1(t) - w_1(z)w_2(t)] dt$$

which is satisfied by the solutions of (4.9). The details will be omitted.

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