A RELATION BETWEEN PERFECT SEPARABILITY, COMPLETENESS, AND NORMALITY IN SEMI-METRIC SPACES

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1. Introduction. This paper proves that a regular semi-metric[1] topological space S may have such properties as hereditary separability, collectionwise normality [1], paracompactness [10], and weak completeness without being either a developable space [1] or a metric space. However, if S is strongly complete, then hereditary separability implies perfect separability [12] and consequently metrizability. It has been proved [1; 12] that a regular developable topological space (Moore space) is metrizable provided that it is perfectly separable. Thus, a regular semi-metric topological space may be far removed from a Moore space contrary to a result announced by C. W. Vickery [11]. The notion of p-separability due to Frechet is generalized and a question raised by W. A. Wilson [14, p. 336] is answered in the affirmative. Throughout this paper, S denotes a regular semi-metric topological space.

2. Weak and strong completeness.

DEFINITION 2.1. A space S is said to be $\begin{Bmatrix}\text{weakly complete}\\ \text{strongly complete}\end{Bmatrix}$ provided there exists a distance function d such that (1) the topology of S is invariant with respect to d and (2) if $\{M_i\}$ is a monotonic decending sequence of closed subsets of S such that, for each i, there exists a $1/i$-neighborhood of a point p_i $\begin{Bmatrix}\text{in } M_i\\ \text{in } S\end{Bmatrix}$ which contains M_i, then ΠM_i contains a point.

It is now shown that strong completness is sufficient to bridge a gap between a hereditarily separable space S and a developable space.

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[1] A topological space S is said to be a semi-metric topological space provided there is a distance function d defined for S such that (1) if each of the letters x and y denotes a point of S, then $d(x, y)=d(y, x)$ denotes a non-negative number, (2) $d(x, y)=0$ if and only if $x=y$, and (3) the topology of S is invariant with respect to the distance function d, that is, if p is a limit point of a subset M of S, then p is a distance limit point of M and conversely. As usual, S is said to be regular provided that if R is an open set containing a point p of S, then there exists an open set D such that $R\supset\bar{D}\supset p$. A topological space (T_1) is defined as in [9].

THEOREM 2.2. *Every hereditarily separable and strongly complete space S is perfectly separable.*

Proof. Let d denote a semi-metric for the space S. For each pair of natural numbers h and k, let M_{hk} denote the set of all points p such that for some open set R, the spherical neighborhoods $U_{1/h}(p)$ and $U_{1/k}(p)$ satisfy $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$. It should be noted that the spherical neighborhoods defined by d may fail to be open sets. Since S is hereditarily separable, there exists a countable dense subset N_{hk} of M_{hk}. Let G_{hk} denote a countable collection of open sets such that for each point p in N_{hk}, there exists an open set R in G_{hk} such that $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$. Clearly, G_{hk} covers M_{hk}. Furthermore, each point of S lies in M_{hk} for some h and k.

Let G denote a countable collection of open sets covering S such that (1) the intersection of two elements of G is an element of G and (2) if Q is an element of G_{hk} for some h and k, then $Q \in G$. The collection G is a basis for S. For, suppose that there exists an open set R containing a point p such that there exists no element of G that contains p and lies in R. Then, for each i, there exists an integer k_i and an element R_i of G_{ik_i} which contains p such that $\prod_{j=1}^{i} R_j$ fails to lie in R. Now, there exists a point p_i such that $U_{1/i}(p_i) \supset \bar{R}_i \supset \prod_{j=1}^{i} \bar{R}_j \cdot (S-R) = M_i$. Since M is strongly complete, ΠM_i contains a point $q \neq p$. Thus, $d(p_i, q) < 1/i$ and $d(p_i, p) < 1/i$ for each i. This is impossible. Hence, S is perfectly separable.

It is an interesting fact that Cauchy completeness, when defined in a natural way for a space S (see [9] and footnote 2), is equivalent to weak completeness in S.

THEOREM 2.3. *A necessary and sufficient condition*[2] *that a semi-metric space S be weakly complete is that every Cauchy sequence*[3] *of points of S have a limit point in S.*

Proof. The condition is necessary. Suppose that there exists a Cauchy sequence $\{p_i\}$ of points of S which has no limit point in S. Thus, there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that for each i,

[2] This theorem was proved independently by my classmate Wyman Richardson in one of F. B. Jones' classes.

[3] A Cauchy sequence $\{p_i\}$ of points is said to have a limit point p provided that there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ which converges to p. There exists a Cauchy sequence of points in a space S which has a limit point but which has no sequential limit point.

$U_{1/i}(p_{n_i}) \supset p_{n_j}$ for $j \geqq i$. Let $M_i = \sum_{j=i}^{\infty} p_{n_j}$. Since $\Pi M_i = 0$, there is a contradiction to the hypothesis that S is weakly complete.

The condition is sufficient. Suppose that $\{M_i\}$ denotes a monotonic descending sequence of closed subsets of S such that for each i, there exists a point p_i such that $p_i \in M_i \subset U_{1/i}(p_i)$. Since $\{p_i\}$ is a Cauchy sequence, the set ΠM_i contains a point p.

3. Non-equivalence of regular semi-metric topological spaces and regular developable (Moore) spaces. Many theorems which are true for Moore spaces have analogues which hold for regular semi-metric topological spaces[4]. However, the fact that a regular semi-metric topological space S is far removed from a Moore space is stressed by the following examples and theorems. From these, it follows that the condition of either separability or screenability for the metrization of a normal Moore space due to Jones [5] and Bing [1], respectively, has no analogue which holds in a normal space S.

Consider the following example of a regular semi-metric space which is not a Moore space. Some additional properties of this space are given in Theorem 3.2.

EXAMPLE 3.1. Let X denote the x-axis of the Cartesian plane E^2. A semi-metric $D(p, q)$ will be defined for E^2 in the following way. Suppose that each of the letters p and q denotes a point of E^2. If X contains both or neither of the points p and q, then define $D(p, q)$ to be the Cartesian distance $d(p, q)$. If $p \in X$ and $q \notin X$, then define $D(p, q)$ to be $d(p, q) + \alpha$ where α is a non-obtuse angle (measured in radians) between X and the line L determined by p and q. If $p \notin X$ and $q \in X$, define $D(p, q)$ to be $D(q, p)$. Clearly, D is a semi-metric for E^2. For each positive integer n and each point p, $U_{1/n}(p)$ is defined to be an open set provided that either $p \in X$ or $U_{1/n}(p)$ lies in one of the two components of $E^2 - X$. Considering the open sets defined in this way as the elements of basis for a topology, E^2 becomes a regular connected and locally connected semi-metric topological space S which is not a Moore space. It should be noted that S is hereditarily separable since it is the sum of two hereditarily separable sets $S - X$ and X.

THEOREM 3.2. *There exists a connected and locally connected regular semi-metric topological space S which is hereditarily separable, weakly complete, strongly screenable* [1], *collectionwise normal, completely normal, and paracompact but which is neither perfectly separable nor a Moore space nor metrizable.*

[4] This is included in unpublished work of F. B. Jones.

Proof. Let S be the space E^2 with the topology defined in Example 3.1. The space S is not metrizable since it is not a Moore space.

Suppose that S is perfectly separable. Then there exists a countable collection H of spherical neighborhoods in S that defines the topology of S. For each number $e>0$ and each point p in X, $U_e(p)$ contains an element $h(e, p)$ of H. By the definition of $U_e(p)$, it follows that the center of the spherical neighborhood $h(e, p)$ is p. This is impossible since H is countable and X is uncountable.

In order to show that S is weakly complete, a distance function E different from that given in Example 3.1 will be introduced. Let L_1 and L_2 be two distinct lines parallel to and at a unit distance from X, and denote by $C(X)$ the component of $S-(L_1+L_2)$ that contains X. For any pair of points p and q of $C(X)-X$, define $E(p, q)$ to be $d(p, q)/d(X, p)d(X, q)$, where d is the ordinary Cartesian distance function. If either of two points p and q fails to lie in $C(X)-X$, then define $E(p, q)$ to be $D(p, q)$ as given in Example 3.1. It follows that the topology of S is unchanged by E. In the remainder of this paragraph, the spherical neighborhoods considered will be those defined by E. Now, suppose that $\{M_i\}$ is a monotonic descending sequence of closed point sets and $\{p_i\}$ is a sequence of points such that for each i, $p_i \in M_i \subset U_{1/i}(p_i)$. If there exists a subscript n such that $X\cdot M_n=0$, then $X\cdot M_i=0$ for $i>n$. From this it follows that there exists $m>n$ such that $U_{1/m}(p_m)\cdot X=0$. For, suppose that this is not the case. Then there must exist a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that $\{d(X, p_{n_i})\}$ converges to 0. Consequently, by the definition of E, the sequence $\{E(p_n, p_{n_i})\}$ of real numbers is unbounded. This is contrary to the assumption that $U_{1/n}(p_n)\supset M_n$. Thus, the existence of the required integer m is established. It follows that $\Pi M_i\neq 0$ in this case. For the remaining case, suppose that for each i, $X\cdot M_i\neq 0$. Since $\{X\cdot M_i\}$ is a bounded monotonic descending sequence of non-empty closed subsets of X, it follows that $\Pi M_i \neq 0$. Hence, S is weakly complete.

The space S is strongly screenable. Consider the metric subspaces $S-X$ and X of S. These are strongly screenable by theorems due to Bing [**1**]. Let G denote an open covering of S. Denote by H and K open coverings of X and $S-X$, respectively, such that for g in G, $g\cdot X \in H$ and $g\cdot(S-X)\in K$. There exists a sequence $\{H_i\}$ of discrete collections [1] of open intervals of X such that ΣH_i covers X and for each i, H_i is a refinement of H. Let I deote an interval in H_i for some i. Since $\bar{I}$ contains no point of the closure of $(H_i-I)^*$ [the logical sum of the elements of H_i-I] and I lies in some element g of G, it follows that there exist discrete collections P and Q of $1/n$-neighborhoods of points in X such that (1) each element of P and each element of Q lies in g, (2) the closure of no element of either P or Q intersects the closure of

$(H_i - I)^*$, and (3) $P+Q$ covers I. It follows that there exists a sequence $\{X_i\}$ of discrete collections each of which is a refinement of G and such that $\Sigma X_i \supset X$. Similarly, there exists a sequence $\{K_i\}$ of discrete collections each of which is a refinement of K and such that $\Sigma K_i \supset S-X$. For each natural number i, let $G_{2i}=X_i$ and $G_{2i-1}=K_i$. Thus, $\{G_i\}$ is a sequence of discrete collections of open subsets of S such that ΣG_i covers S and G_i refines G for each i. Hence, S is strongly screenable.

Now S, being a regular strongly screenable topological space, is collectionwise normal [1]. It also follows that S is paracompact by a theorem due to Ernest Michael [6].

To complete the proof of theorem 3.2, it must be shown that S is completely normal. It has been proved by F. B. Jones [5] that every normal Moore space is completely normal[5]. A simple modification of his argument shows that every normal semi-metric topological space is completely normal. This completes the proof.

Mary E. Estill [3] has considered complete Moore spaces in any one of three definitions of completeness. Perhaps intuition would lead one to suspect that a complete Moore space, in one of these senses, would be strongly complete. The following example and theorem shows that this is not the case. As a matter of fact, in a Moore space, the concept of strong completeness is more restrictive than that of completeness.

EXAMPLE 3.3. Let X denote the x-axis of the Cartesian plane E^2. A semi-metric $D(p, q)$ will be defined for E^2 in the following way. Suppose that p and q are two distinct points of E^2. If neither p nor q lies in X, then define $D(p, q)$ to be $d(p, q)$ where d is the ordinary Cartesian metric. If $p \in X$, then let $D(p, q)=d(p, q)+\alpha$ where α is an angle (measured in radians) between a line L_1 containing $p+q$ and a vertical line L_2 containing p such that $0 \leqq \alpha \leqq \pi/2$. If $D(q, p)$ is not defined above, then let $D(q, p)=D(p, q)$. For p in X, let $D(p, p)=0$. Clearly, D is a semi-metric for E^2. For each point p in E^2 and each natural number n, $U_{1/n}(p)$ is defined to be an open set. With this definition of open sets, E^2 becomes a regular connected and locally connected semi-metric topological space S. It should be noted that S is separable but not hereditarily separable.

THEOREM 3.4. *There exists a complete Moore space S which is not strongly complete.*

Proof. Let S be the space E^2 with the topology defined in Example

[5] A space S is said to be completely normal provided that for two mutually separate subsets H and K of S there exists matually exclusive open coverings of H and K. See [5].

3.3. It will first be shown that S is not strongly complete.

Suppose that S is strongly complete. Then there exists a semimetric E defined for S such that (1) the topology of S is unchanged by E and (2) if $\{M_i\}$ is a monotonic descending sequence of closed subsets of S such that for each i and some point p_i in S, $U_{1/i}(p_i)\supset M_i$, then $\Pi M_i\neq 0$. It should be noted that the spherical neighborhoods defined by E may fail to be open sets.

Consider an interval A of X. For each pair of natural numbers h and k, let M_{hk} denote the subset of A of all points p such that for some open set R, $U_{1/h}(p)\supset R\supset U_{1/k}(p)$. For some natural number h_1, the set M_{1h_1} is uncountable. Now, M_{1h_1} contains an uncountable subset N_{1h_1} such that

(1) there exists a line L_1 parallel to X where $d(L_1, X)\leqq 1$ and

(2) for each point p in N_{1h_1}, there exists an open set $R(p)$ where $U_1(p)\supset R(p)\supset U_{1/h_1}(p)$ such that $R(p)$ contains an interval I of L_1 whose length (in the Cartesian sense) is greater than a positive number e_1 and which has as its center a point q whose projection on X is p.

Now there exists an integer $h_2>h_1$ such that N_{1h_1} contains an uncountable subset N_{2h_2} such that

(1) there exists a line L_2 parallel to X where $d(L_2, X)\leqq 1/2$ and

(2) for each point p in N_{2h_2}, there exists an open set $R(p)$ where $U_{1/2}(p)\supset R(p)\supset U_{1/h_2}(p)$ such that $R(p)$ contains an interval I of L_2 whose length is greater than a positive number e_2 and which has as its center a point q whose projection on X is p.

If follows that there exists a monotonic descending sequence $\{N_{ih_i}\}$ of subsets of A and a sequence $\{L_i\}$ of lines parallel to X and converging to it such that for each i, if p_i is a point of N_{ih_i}, there exists an open set $R(p_i)$ where $U_{1/i}(p_i)\supset R(p_i)\supset U_{1/h_i}(p_i)$ such that $R(p_i)$ contains an interval I_i of L_i whose length is greater than a positive number e_i and which has as its center a point q_i whose projection on X is p_i. Since A is a compact subset of E^2 there exists a monotone sequence $\{p_i\}$ of points converging to a point p in A such that for each i, $p_i\in N_{ih_i}$. Let L be a vertical line containing p, and for each i, define $x_i=L\cdot L_i$. It follows that there exists a monotonic increasing sequence $\{k_i\}$ of natural numbers such that for each i, $E(x_i, p_j)<1/i$ for all $j>k_i$. The set $M_i=\sum_{k=k_i}^{\infty}p_k$ is closed in S for each i and $U_{1/i}(x_i)\supset M_i$. It follows that $\Pi M_i=0$.

This is contrary to the assumption that S is strongly complete.

It now remains to be shown that S is a complete Moore space. For a point p in X, there exists a sequence $\{R_i\}$ of open sets closing down[6]

[6] A sequence of open sets $\{R_i\}$ is said to close down on a point p if for each i, $R_i\supset\overline{R}_{i+1}$ and $\Pi R_i=p$.

on p. On the other hand, if p denotes a point of $S-X$, there exists a sequence $\{R_i\}$ of open sets closing down on p such that for each i, $R_i \cdot X=0$. With each point p of S, associate exactly one such sequence $\{R_i\}$. For each i, let G_i denote the collection of all open sets R such that for some point p of S, R is the jth member of the sequence associated with p, and $j \geqq i$. It follows that S is a complete Moore space.

4. A question due to W. A. Wilson. An affirmative answer is given in this section to a question raised by Wilson [**14**, p. 366] in 1931. The following axioms and definitions [**14**] are listed for convenience.

A set Z is said to be a (Menger) semi-metric space provided that corresponding to each pair of points (a, b) of Z, there is a non-negative real number $d(a, b)$ satisfying the following axioms:

Axiom I. $d(a, b)=d(b, a)$.

Axiom II. $d(a, b)=0$ if and only if $a=b$.

Wilson has introduced the following additional axiom:

Axiom *W*. For each point a and each positive number k, there is a positive number r such that if b is a point for which $d(a, b) \geqq k$ and c is any point, then $d(a, c)+d(b, c) \geqq r$.

Now, let $r=f(a, k)$ denote the largest r such that $d(a, c)+d(b, c) \geqq r$ in Axiom *W*. For each point a and each positive number k, let $r=f(a, k)$, $r_1=f(a, r)$, and r_2 denote a positive number such that $r_2 < r_1$. Wilson calls the set σ of points x such that $d(a, x) < r_2$ an inner sphere, with center a, corresponding to a and k.

THEOREM 4.1. *Suppose that Z denotes a separable semi-metric space satisfying Axiom W. If d denotes a distance function defined for Z which leaves limit points invariant, then there exists a countable dense subset $E=\Sigma p_i$ of Z such that for any positive number k, each point p of Z lies in an inner sphere σ corresponding to p_i and k for some natural number i.*

Proof. By a corollary due to Wilson [**14**], Z is homeomorphic to a metric space. Since a separable metric space is hereditarily separable, it follows that Z is hereditarily separable.

Let $S_e(p)$ denote a spherial neighborhood in Z. For each pair of natural numbers h and k, let M_{hk} denote the set of all points p such that there exists an inner sphere σ corresponding to $1/h$ and p such that $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$. Since Z is hereditarily separable, M_{hk} contains a countable dense subset N_{hk}. Let K_{hk} be a countable collection of inner spheres such that if $p \in N_{hk}$, then there exists an inner sphere σ in K_{hk} corresponding to $1/h$ and p such that $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$. It follows that K_{hk} covers M_{hk}. Denote by $E=\Sigma p_i$, the countable dense subset $\sum_{h,k=1}^{\infty} N_{hk}$ of Z.

The set E satisfies the conclusion of Theorem 4.1. For, if c is any positive number, there exists a positive integer h such that $1/h<c$. Also, for p in Z, there exists k such that $p \in M_{hk}$. Since K_{hk} covers N_{hk}, there exists an inner sphere σ corresponding to p_i and $1/h$ for some i such that $\sigma \supset p$. Hence, the inner sphere σ_1 which corresponds to p_i and c contains σ and p.

Now Wilson's question referred to above is answered.

5. Generalized Frechet *p*-separability. The following definition is a natural generalization of the notion of p-separability [4]. It is proved that in a space S this notion is equivalent to hereditary separability.

DEFINITION 5.1. A regular semi-metric topological space S (or semi-metric space Z) is said to be p-separable provided that

(1) given any distance function d which leaves limit points invariant and

(2) given any collection H of subsets of S which has the property that for each number $k>0$ and each point p of S, there exists h in H such that $U_k(p) \supset h \supset U_e(p)$ for some positive number e,

then there exists a countable dense subset $E=\Sigma p_i$ such that for each positive number f, each point p of S lies in an element h of H such that $U_f(p_i) \supset h \supset p_i$ for some i.

The following theorem may be proved in a manner analogous to that used in the proof of Theorem 4.1.

THEOREM 5.2. *Every hereditarily separable semi-metric space Z is p-separable.*

THEOREM 5.3. *A necessary and sufficient condition that a regular semi-metric topological space S be hereditarily separable is that S be p-separable.*

Proof. The necessity of the condition follows from Theorem 5.2.

It will now be shown that the condition is sufficient. Suppose that d denotes a semi-metric for S, and that S is not hereditarily separable. Then S contains an uncountable subset N which has no limit point in S. Now, consider a semi-metric D defined in the following way. For each i, let D_i denote the set of all points x of S such that for some point p in N, x lies in an open set $R \subset u_{1/i}(p)$ where $u_{1/i}(p)$ is a spherical neighborhood defined by d. Thus, $\{D_i\}$ is a monotonic descending sequence of open sets such that $\Pi D_i = N$. For each i and each point p in $D_i - N$, associate exactly one open set $R_i(p)$ containing p and lying in D_i such that for some number e, $u_e(p) \supset \overline{R}(p)$ and $u_e(p) \cdot N = 0$. If x and y denote points of $S-N$ such that for some i, $D_i \supset x+y$ and $D_{i+1} \not\supset x+y$, then define $D(x, y)$ to be i provided that $R_i(x) \not\supset y$ and $R_i(y) \not\supset x$. For points x and y of S for which $D(x, y)$ is not defined above, let $D(x, y) = d(x, y)$. It follows that limit points are invariant with respect to D.

Next, let H denote a collection of open sets such that for each natural number i and each point p in S, there exists h in H such that $U_{1/i}(p) \supset h \supset p$ where $U_{1/i}(p)$ is a spherical neighborhood defined by D. Since S is p-separable, there exists a countable dense subset $E = \Sigma p_i$ of S such that for each positive number f, each point p of S lies in an element $h(p)$ of H such that for some i, $U_f(p_i) \supset h(p)$. There exists an uncountable subset M of $N-E$ and a natural number t such that if x is a point and $D(x, M) < 1/t$, then x lies in D_t. Let $p \in M$. Then there exist

(1) a number $e > 0$ such that $D(p, N-p) = d(p, N-p) > e$,

(2) a positive integer n such that $1/n <$ smaller $[e, 1/t]$,

(3) h in H,

(4) an integer i such that $U_{1/n}(p_i) \supset h \supset p$ [thus, $p_i \in D_t$],

(5) an integer $m \geqq t$ such that $p_i \in D_m - D_{m+1}$,

(6) an open set $R_m(p_i)$ associated with p_i and D_m such that for some number c, $u_c(p_i) \supset \overline{R}_m(p_i)$ and $u_c(p_i) \cdot N = 0$,

(7) a positive number z such that for q in $S - R_m(p_i)$, $d(p_i, q) > z$,

(8) $x \in h \cdot D_m - [R_m(p) + N]$ such that $D(p, x) < z$, and

(9) an open set $R_m(x)$ associated with x and D_m such that for some number b, $u_b(x) \supset \overline{R}_m(x)$ and $u_b(x) \cdot N = 0$.

Therefore, $b < z$. Consequently, $R_m(x) \not\supset p_i$. By definition, $D(x, p_i) = m > 1/n$. This is impossible since $U_{1/n}(p_i) \supset h \supset p + x$. Hence, S is hereditarily separable.

It follows from Theorem 3.2 that S may fail to be either perfectly separable or a metric space.

6. Conditions for semi-metric, regular developable (Moore), and metric spaces.

Consider the following three conditions on a topological space T.

A. There exists a sequence $\{H_i\}$ such that (a) for each i, H_i is a collection of open subsets of T, (b) if p is a point and R is an open set containing p, then there exists an integer n such that H_n contains exactly one element $g(p)$ associated with p such that $R \supset g(p) \supset p$ and

(c) if n is an integer and $\{g_i(p_i)\}$ is a sequence such that for each i, $g_i(p_i)$ belongs to H_n and is associated with p_i, then Σp_i has no limit point in $T-\Sigma g_i(p_i)$.

B. If p is a point and R is an open set containing p, then there exists an integer n such that for $m>n$, each element g of H_m which contains p has the property that $R\supset\bar{g}$.

C. For each i, the sum of the closures of any subcollection of H_i is closed.

THEOREM 6.1. *A necessary and sufficient condition that a topological space T be semi-metric is that T satisfy Condition A.*

Proof. It will first be shown that the condition is sufficient. It follows from Condition A that T satisfies the first axiom of countability. Consider a semi-metric d defined as follows. For two distinct points p and q of T, denote by i the least integer such that H_i contains an element $g(p)$ associated with p but not containing q. Similarly, let j denote the least integer such that H_j contains an element $g(q)$ associated with q but not containing p. Define $d(p, q)$ to be $1/\min(i, j)$. For each point p, define $d(p, p)$ to be 0.

Limit points are invariant with respect to d. For suppose that p is a limit point (defined by the open sets of T) of a subset M of T and that p is not a distance limit point of M. Then there exists a sequence $\{p_i\}$ of points of $M-p$ which converges to p such that for some integer n and each i, $d(p, p_i)>1/n$. Thus, there exists an integer m, such that, for infinitely many integers i, either (1) H_m contains $g_m(p)$ and $g_m(p)\not\supset p_i$ or (2) H_m contains $g_m(p_i)$ and $g_m(p_i)\not\supset p$. Since $\{p_i\}\to p$, (1) is impossible. By Condition A, (2) is impossible. Hence, p is a distance limit point of M. It also follows easily that a distance limit point of a subset M of T is an open set limit point of M. This completes the proof of the sufficiency.

The condition is necessary. For each point p and each pair of natural numbers h and k, let $R_{hk}(p)$ denote an open set when it exists, such that $U_{1/h}(p)\supset R_{hk}(p)\supset U_{1/k}(p)$. With h, k, and p associate exactly one such open set, and let G_{hk} denote the corresponding collection of open sets for each point p in T. There exists a sequence $\{H_i\}$ such that there is a one to one correspondence between the elements of $\{H_i\}$ and the elements of $\{G_{nm}\}$. It follows that $\{H_i\}$ satisfies Condition A.

As Example 3.1 illustrates, a regular semi-metric topological space may fail to be a Moore space.

THEOREM 6.2. *A necessary and sufficient condition that a topological space T be a Moore space is that T satisfy Conditions A and B.*

Proof. The condition is sufficient. For each positive integer i, let $G_i=\sum_{j=i}^{\infty} H_j$. If the word "region" is interpreted as "open set," then it follows that Axioms 0 and 1 (1)–(3) due to Moore [7] are satisfied.

The condition is necessary. It will be shown first that T is a semi-metric topological space. Let p and q be distinct points of T. Denote by n the least positive integer such that if $g(p)$ and $g(q)$ are regions in G_n containing p and q, respectively, then $g(p)\cdot g(q)=0$. Note that $\{G_i\}$ is given by Axiom 1 of [7]. Consequently, define $d(p, q)$ to $1/n$. It follows that d is a semi-metric distance function and that limit points are invariant with respect to d. By Theorem 6.1, T satisfies Condition A.

Now, define $\{H_i\}$ in a manner described in the proof of Theorem 6.1 with the additional requirement that $R_{hk}(p)$ lie in a region of G_h. It follows that $\{H_i\}$ satisfies Conditions A and B.

THEOREM 6.3. *A necessary and sufficient condition that a topological space T be metric is that it satisfy Conditions* A, B, *and* C.

A proof of Theorem 6.3 follows by use of Bing's Theorem 4 of [1] and Theorem 6.1 above.

Question. Is it possible to partition either Bing's Theorem 4 of [1] or Moore's metrization theorem [8; 13], stated below, into three or more parts which begins with a condition for a topological space and which ends with a condition for a metrizable space, but with necessary and sufficient conditions somewhere between these extremes for semi-metric spaces and Moore spaces?

THEOREM (Moore)[7]. *A necessary and sufficient condition that a space S satisfying Axiom* 0 *of* [7] *be metrizable is that there exist a sequence $\{K_i\}$ such that* (1) *for each natural number n, K_n is a collection of regions in S covering S and* (2) *if p is a point, q is a point distinct from p, and R is a region containing p, then there exists a natural number n such that if each of the letters h and k denotes an element of K_n, $g\supset p$, and $g\cdot h\neq 0$, then $R-q\supset h$.*

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[7] The terms "point" and "region" are undefined. Axiom 0 states that every region is a point set.

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