SOME INTEGRAL FORMULAS FOR CLOSED HYPERSURFACES IN RIEMANNIAN SPACE

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Introduction. Let V^n be a hypersurface twice differentiably imbedded in a Riemannian space R^{n+1} of $n+1$ ($n \geqq 2$) dimensions, and $\kappa_1, \cdots, \kappa_n$ the n principal curvatures at a point P of the hypersurface V^n. It is known that the ith mean curvature M_i of the hypersurface V^n at the point P is defined by

$$C_{n,i} M_i = \sum \kappa_1 \kappa_2 \cdots \kappa_i \qquad (i=1, \cdots, n), \tag{0.1}$$

where the expression on the right side is the ith elementary symmetric function of $\kappa_1, \cdots, \kappa_n$, and $C_{n,i}$ denotes the number of combinations of n different things taken i at a time. Let dA be the area element of the hypersurface V^n at the point P, and p the scalar product of the unit normal vector of the hypersurface V^n at the point P and the position vector of the point P with respect to any orthogonal frame in the space R^{n+1}.

The purpose of this paper is to prove the following four theorems concerning closed hypersurfaces by first showing that:

a) If V^n is an orientable hypersurface, with a closed boundary V^{n-1} of dimension $n-1$ ($n \geqq 2$), which is twice differentiably imbedded in an $(n+1)$-dimensional Riemannian space R^{n+1}, then the integral $\int_{V^n} (1+M_1 p) dA$ can be expressed as an integral over the boundary V^{n-1}.

b) If in addition V^n is of class C^3 and the space R^{n+1} is of constant Riemannian curvature, then the integral $\int_{V^n} (M_{n-1} + M_n p) dA$ can also be expressed as an integral over V^{n-1}.

These results have been obtained in a previous paper [2] by the author for an orientable hypersurface V^n twice differentiably imbedded in a Euclidean space E^{n+1} of $n+1$ ($n \geqq 2$) dimensions.

THEOREM 1. *Let V^n be a closed orientable hypersurface twice differentiably imbedded in a Riemannian space R^{n+1} of $n+1$ ($n \geqq 2$) dimensions, then*

$$A + \int_{V^n} M_1 p dA = 0 \, . \tag{I}$$

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THEOREM 2. *Let V^n be a closed orientable hypersurface of class C^3 imbedded in an $(n+1)$-dimensional $(n\geqq 2)$ Riemannian space R^{n+1} of constant Riemannian curvature K, then*

$$\text{(II)} \qquad \int_{V^n} M_{n-1}dA+\int_{V^n} M_n p dA=0 .$$

THEOREM 3. *Let V^n be a hypersurface satisfying the conditions of Theorem 2. Suppose that the principal curvatures $\kappa_1, \cdots, \kappa_n$ at each point of the hypersurface V^n are positive and that in the space R^{n+1} there exists a point O for which either $p\leqq -1/M_1$ or $p\geqq -1/M_1$ at all points of the hypersurface V^n. Then every point of the hypersurface V^n is umbilic.*

THEOREM 4. *Let V^n be a hypersurface satisfying the conditions of Theorem 2. Suppose that the principal curvatures $\kappa_1, \cdots, \kappa_n$ at each point of the hypersurface V^n are positive and M_{n-1} is constant, and that in the space R^{n+1} there exists a point O for which the function p is of the same sign at all points of the hypersurface V^n. Then every point of the hypersurface V^n is umbilic.*

1. Preliminaries. In a Riemannian space R^{n+1} of dimension $n+1$ $(n\geqq 2)$ with a positive definite fundamental form we consider a fixed orthogonal frame $Oe_1\cdots e_{n+1}$, where $e_1, \cdots, e_{n+1}$ form an ordered set of $n+1$ mutually orthogonal contravariant unit vectors at a point O in R^{n+1}. With respect to this orthogonal frame let y^α $(\alpha=1, \cdots, n+1)$ be[1] the coordinates of a point in R^{n+1} and $a_{\alpha\beta}dy^\alpha dy^\beta$ the fundamental form for R^{n+1}, where $a_{\alpha\beta}=a_{\beta\alpha}$ and the matrix $\|a_{\alpha\beta}\|$ is positive definite so that the determinant $a=|a_{\alpha\beta}|>0$.

Let $A_{i|}$ $(i=1, \cdots, n)$ be n vectors at a point in the space R^{n+1} whose contravariant components with respect to the frame $Oe_1\cdots e_{n+1}$ are $A^\alpha_{i|}$ $(\alpha=1, \cdots, n+1)$. First we define the vector product of the n vectors $A_{i|}$ $(i=1, \cdots, n)$ to be a vector in R^{n+1}, denoted by $A_{1|}\times\cdots\times A_{n|}$, whose contravariant components are given by

$$\text{(1.1)} \qquad A_{1|}\times\cdots\times A_{n|}=(-1)^n\begin{vmatrix} e_1 & e_2 & \cdots & e_{n+1} \\ a_{\alpha 1}A^\alpha_{1|} & a_{\alpha 2}A^\alpha_{1|} & \cdots & a_{\alpha, n+1}A^\alpha_{1|} \\ a_{\alpha 1}A^\alpha_{2|} & a_{\alpha 2}A^\alpha_{2|} & \cdots & a_{\alpha, n+1}A^\alpha_{2|} \\ \cdot & \cdot & \cdots & \cdot \\ a_{\alpha 1}A^\alpha_{n|} & a_{\alpha 2}A^\alpha_{n|} & \cdots & A_{\alpha, n+1}A^\alpha_{n|} \end{vmatrix} .$$

From the definition of the scalar product of any two vectors $A_{i|}$ and $A_{j|}$,

[1] Throughout this paper Greek indices take the values 1 to $n+1$, and Latin indices the values 1 to n unless stated otherwise. We use the convention that repeated indices imply summation.

namely, $A_{i|}\cdot A_{j|}=a_{\alpha\beta}A^{\alpha}_{i|}A^{\beta}_{j|}$, it follows immediately that $A_{1|}\times\cdots\times A_{n|}$ is orthogonal to $A_{i|}$ $(i=1,\cdots,n)$.

Now we consider a hypersurface V^n twice differentiably imbedded in the space R^{n+1}. With respect to the orthogonal frame $Oe_1\cdots e_{n+1}$ the hypersurface V^n can be given by the parametric equations

$$y^{\alpha}=f^{\alpha}(x^1,\cdots,x^n) \qquad (\alpha=1,\cdots,n+1), \tag{1.2}$$

or the vector equation

$$Y=F(x^1,\cdots,x^n)\ , \tag{1.3}$$

where y^{α} and f^{α} are respectively the contravariant components of the two vectors Y and F, the parameters $x^1,\cdots,x^n$ take values in a simply connected domain D of the n-dimensional real number space, and $f^{\alpha}(x^1,\cdots,x^n)$ is of rank n at all points of D. Let the first fundamental form of the hypersurface V^n at a point P be

$$ds^2=g_{ij}dx^idx^j\ , \tag{1.4}$$

where the matrix $\|g_{ij}\|$ is positive definite so that the determinant $g=|g_{ij}|>0$, and

$$g_{ij}=a_{\alpha\beta}y^{\alpha}_{,i}y^{\beta}_{,j}\ , \tag{1.5}$$

$$y^{\alpha}_{,i}=\partial y^{\alpha}/\partial x^i\ . \tag{1.6}$$

Let $A^{\alpha}_{\beta i}$ be a mixed tensor of the second order in the y's, and a covariant vector in the x's, as indicated by the Greek and Latin indices. Then following Tucker [3], the generalized covariant derivative of $A^{\alpha}_{\beta i}$ with respect to the x's is defined as

$$A^{\alpha}_{\beta i;j}=\frac{\partial A^{\alpha}_{\beta i}}{\partial x^j}+\left\{{\alpha \atop \gamma\,\delta}\right\}A^{\gamma}_{\beta i}y^{\delta}_{,j}-\left\{{\gamma \atop \beta\,\delta}\right\}A^{\alpha}_{\gamma i}y^{\delta}_{,j}-\left\{{k \atop ij}\right\}A^{\alpha}_{\beta k}\ , \tag{1.7}$$

where the Christoffel symbols $\left\{{\alpha \atop \beta\gamma}\right\}$ with Greek indices are formed with respect to the $a_{\alpha\beta}$ and the y's, and those $\left\{{k \atop ij}\right\}$ with Latin indices with respect to the g_{ij} and the x's. It should be noted that the definition of generalized covariant differentiation can be applied to any tensor in the x's and y's and that the generalized covariant differentiation of sums and products obeys the ordinary rules. If a tensor is one with respect to the x's only, so that only Latin indices appear, its generalized covariant derivative is the same as its covariant derivative with respect to the x's. Moreover, in generalized covariant differentiation the fundamental tensors $a_{\alpha\beta}$ and g_{ij} can be treated as constants. Since y^{α} is an invariant for transformation of the x's, its generalized covariant derivative is the same as its covariant derivative with respect to the x's; so that

$$y^{\alpha}_{;i}=y^{\alpha}_{,i}=\partial y^{\alpha}/\partial x^{i}\ . \tag{1.8}$$

By (1.7) the generalized covariant derivative of $y^{\alpha}_{;i}$ is

$$y^{\alpha}_{;ij}=\frac{\partial^2 y^{\alpha}}{\partial x^i \partial x^j}-\left\{ {h \atop ij} \right\} y^{\alpha}_{,h}+\left\{ {\alpha \atop \beta\gamma} \right\} y^{\beta}_{,i} y^{\gamma}_{,j}\ , \tag{1.9}$$

which is symmetric in the indices i and j.

Let N be the unit normal vector at a point P of the hypersurface V^n, then

$$a_{\alpha\beta}N^{\alpha}N^{\beta}=1\ , \tag{1.10}$$

$$a_{\alpha\beta}N_{\alpha}y^{\beta}_{,i}=0 \qquad (i=1,\cdots,n). \tag{1.11}$$

We can easily obtain (see, for instance, [4, Chap. VIII]):

$$y^{\alpha}_{;ij}=\Omega_{ij}N^{\alpha}\ , \tag{1.12}$$

$$\Omega_{ij}=y^{\alpha}_{;ij}a_{\alpha\beta}N^{\beta}\ , \tag{1.13}$$

$$N^{\alpha}_{;i}=-\Omega_{ij}g^{jk}y^{\alpha}_{,k}\ , \tag{1.14}$$

where $\Omega_{ij}=\Omega_{ji}$ are the coefficients of the second fundamental form of the hypersurface V^n at the point P, and g^{ij} denotes the cofactor of g_{ij} in g divided by g so that

$$g^{ij}g_{jk}=\delta^{i}_{k}\ , \tag{1.15}$$

δ^i_k being the Kronecker delta. Moreover, we have

$$R_{lijk}=(\Omega_{lj}\Omega_{ik}-\Omega_{lk}\Omega_{ij})+\bar{R}_{\beta\gamma\delta\varepsilon}y^{\beta}_{,l}y^{\gamma}_{,i}y^{\delta}_{,j}y^{\varepsilon}_{,k}\ , \tag{1.16}$$

$$\Omega_{ij,k}-\Omega_{ik,j}=-\bar{R}_{\beta\gamma\delta\varepsilon}N^{\beta}y^{\gamma}_{,i}y^{\delta}_{,j}y^{\varepsilon}_{,k}\ , \tag{1.17}$$

where R_{lijk} and $\bar{R}_{\beta\gamma\delta\varepsilon}$ are Riemann symbols formed with the tensors g_{ij} and $a_{\alpha\beta}$ respectively. In particular, if the space R^{n+1} is of constant Riemannian curvature K, it follows from the definition of Riemannian curvatures of the space R^{n+1} that

$$\bar{R}_{\beta\gamma\delta\varepsilon}=K(a_{\beta\delta}a_{\gamma\varepsilon}-a_{\beta\varepsilon}a_{\gamma\delta})\ , \tag{1.18}$$

and therefore (1.16), (1.17) reduce to

$$R_{lijk}=(\Omega_{lj}\Omega_{ik}-\Omega_{lk}\Omega_{ij})+K(g_{lj}g_{ik}-g_{lk}g_{ij})\ , \tag{1.19}$$

$$\Omega_{ij,k}-\Omega_{ik,j}=0\ . \tag{1.20}$$

Taking the generalized covariant derivative of each side of (1.14) and making use of (1.12), (1.19), (1.20) we thus obtain

$$N^{\alpha}_{;ji}-N^{\alpha}_{;ij}=N^{\alpha}g^{lk}(R_{jikl}-\overline{R}_{\beta\gamma\delta\varepsilon}y^{\beta}_{,j}y^{\gamma}_{,i}y^{\delta}_{,k}y^{\varepsilon}_{,l}) \; . \tag{1.21}$$

The n principal curvatures $\kappa_1, \cdots, \kappa_n$ of the hypersurface V^n at the point P are the roots of the determinant equation

$$|\Omega_{ij}-\kappa g_{ij}|=0 \; . \tag{1.22}$$

From (0.1) and (1.22) it follows immediately that

$$M_n=\Omega/g \; , \quad nM_1=\Omega_{ij}g^{ij} \; , \quad nM_{n-1}=g_{ij}\Omega^{ij}/g \; , \tag{1.23}$$

where $\Omega=|\Omega_{ij}|$ and Ω^{ij} is the cofactor of Ω_{ij} in Ω.

Consider the two matrices

$$\phi=\|\phi^i_\gamma\| \; , \qquad \psi=\|\psi^\gamma_i\| \; ; \tag{1.24}$$

where

$$\phi^i_\gamma=a_{\beta\gamma}y^{\beta}_{,i} \; , \qquad \psi^\gamma_i=y^\gamma_{,i} \qquad (i=1, \cdots, n \, ; \; \gamma=1, \cdots, n+1), \tag{1.25}$$

the superscript of the element ϕ^i_γ or ψ^γ_i indicating the row to which the element belongs and the subscript indicating the column. Solving (1.11) for N^α, we obtain

$$N^\alpha=(-1)^{n-\alpha+1}cA^\alpha \qquad (\alpha=1, \cdots, n+1), \tag{1.26}$$

where c is a constant and A^α the determinant of nth order obtained by deleting the αth column from the matrix ϕ. Substitution of (1.26) in (1.10) gives

$$c^2=\frac{1}{aA} \; , \tag{1.27}$$

where

$$A=\begin{vmatrix} A^1 & -A^2 & \cdots & (-1)^nA^{n+1} \\ y^1_{,1} & y^2_{,1} & \cdots & y^{n+1}_{,1} \\ \cdot & \cdot & \cdots & \cdot \\ y^1_{,n} & y^2_{,n} & \cdots & y^{n+1}_{,n} \end{vmatrix} \; , \tag{1.28}$$

which is equal to the sum of the products of the corresponding determinants of nth order of the two matrices (1.24). By an elementary theorem on determinants (see, for instance, [1, p. 102]), from (1.5) it follows immediately that

$$A=|\phi^i_\gamma\psi^\gamma_j|=g \; . \tag{1.29}$$

Now we choose the direction of the unit normal vector N in such a way that the two frames $PY_{,1}\cdots Y_{,n}N$ and $Oe_1\cdots e_{n+1}$ have the same orientation. Then from (1.10), (1.26), (1.27), (1.29) we obtain

$$\sqrt{ga}N=Y_{,1}\times\cdots\times Y_{,n}\,, \tag{1.30}$$

$$|Y_{,1},\cdots,Y_{,n},N|=\sqrt{g/a}\,. \tag{1.31}$$

The area element of the hypersurface V^n at the point P is given by

$$dA=\sqrt{g}\,dx^1\cdots dx^n\,. \tag{1.32}$$

Let $A_{i|}$ $(i=1,\cdots,n)$ be n vectors at a point in the space R^{n+1}, whose contravariant components with respect to the frame $Oe_1\cdots e_{n+1}$ are differentiable functions of $x^1,\cdots,x^n$, then by (1.1) and the differentiation of determinants

$$(A_{1|}\times\cdots\times A_{n|})_{;i}=\sum_j(A_{1|}\times\cdots\times A_{j-1|}\times A_{j|;i}\times A_{j+1|}\times\cdots\times A_{n|})\,. \tag{1.33}$$

2. **Proof of the formula** (I). First we observe that the vector $Y_{,1}\times\cdots\times Y_{,i-1}\times N\times Y_{,i+1}\times\cdots\times Y_{,n}$ is orthogonal to the normal vector N and can therefore be written in the form

$$Y_{,1}\times\cdots\times Y_{,i-1}\times N\times Y_{,i+1}\times\cdots\times Y_{,n}=c^{ij}Y_{,j}\quad(i=1,\cdots,n). \tag{2.1}$$

Taking the scalar products of both sides of (2.1) with the vector $Y_{,k}$ and making use of (1.2), (1.5), (1.31), we obtain

$$c^{ij}g_{jk}=-\sqrt{ga}\,\delta^i_k\qquad(i,k=1,\cdots,n). \tag{2.2}$$

Solving (2.2) for c^{ij} for each fixed i and substituting the results in (2.1), we are led to

$$Y_{,1}\times\cdots\times Y_{,i-1}\times N\times Y_{,i+1}\times\cdots\times Y_{,n}=-\sqrt{ga}\,g^{ij}Y_{,j}\quad(i=1,\cdots,n). \tag{2.3}$$

Making use of the relation $Y_{;ij}=Y_{;ji}$ and (1.14), (1.23), (1.30), (1.33), it is easily seen that

$$\begin{aligned}\sum_i(Y_{,1}\times\cdots\times Y_{,i-1}\times N\times Y_{,i+1}\times\cdots\times Y_{,n})_{;i}\\ =\sum_i Y_{,1}\times\cdots\times Y_{,i-1}\times N_{;i}\times Y_{,i+1}\times\cdots\times Y_{,n}\\ =-n\sqrt{ga}M_1N\,.\end{aligned} \tag{2.4}$$

Thus, from (2.3) and (2.4),

$$n\sqrt{g}\,M_1N=(\sqrt{g}\,g^{ij}Y_{,i})_{;j}\,. \tag{2.5}$$

Taking the scalar products of both sides of (2.5) with the vector Y, we obtain in consequence of (1.5) and (1.15)

(2.6) $$nM_1p\sqrt{g}=(\sqrt{g}\,g^{ij}\eta_i)_{;j}-n\sqrt{g}\ ,$$

where we have put

(2.7) $$p=Y\cdot N\ ,\qquad \eta_i=Y\cdot Y_{,i}\qquad\qquad (i=1,\cdots,n).$$

Now let us consider a hypersurface V^n, with a closed boundary V^{n-1} of dimension $n-1$ $(n\geqq 2)$, twice differentiably imbedded in a Riemannian space R^{n+1} of dimension $n+1$. Integrating (2.6) with respect to $x^1,\cdots,x^n$ over this hypersurface V^n and applying the general theorem of Stokes to the first term on the right side of (2.6), we obtain

(2.8) $$A+\int_{V^n}M_1pdA=\frac{1}{n}\int_{V^{n-1}}\sum_j(-1)^{j-1}\sqrt{g}\,g^{ij}\eta_i dx^1\cdots dx^{j-1}dx^{j+1}\cdots dx^n\ .$$

In particular, when the hypersurface V^n is closed and orientable, the integral on the right side of (2.8) vanishes and hence we obtain the formula (I).

3. **Proof of the formula** (II). For the same reason as in the preceding section, the vector $N_{;1}\times\cdots\times N_{;i-1}\times N\times N_{;i+1}\times\cdots\times N_{;n}$ is orthogonal to the normal vector N and can therefore be written in the form

(3.1) $$N_{;1}\times\cdots\times N_{;i-1}\times N\times N_{;i+1}\times\cdots\times N_{;n}=c^{ij}Y_{,j}\qquad (i=1,\cdots,n).$$

Taking the scalar products of both sides of (3.1) with the vector $Y_{,k}$ and making use of (1.1), (1.14), (1.31), we obtain

$$c^{ij}g_{jk}=(-1)^{n+i}a|N,Y_{,1},\cdots,Y_{,n}|$$

$$\cdot\begin{vmatrix} 1 & 0 & \cdots & & & & & 0\\ 0 & \Omega_{1j}g^{j1} & \cdots & \Omega_{i-1,j}g^{j1} & \Omega_{i+1,j}g^{j1} & \cdots & \Omega_{nj}g^{j1} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \Omega_{1j}g^{j,k-1} & \cdots & \Omega_{i-1,j}g^{j,k-1} & \Omega_{i+1,j}g^{j,k-1} & \cdots & \Omega_{nj}g^{j,k-1} & 0\\ 0 & \Omega_{1j}g^{jk} & \cdots & \Omega_{i-1,j}g^{jk} & \Omega_{i+1,j}g^{jk} & \cdots & \Omega_{nj}g^{jk} & 1\\ 0 & \Omega_{1j}g^{j,k+1} & \cdots & \Omega_{i-1,j}g^{j,k+1} & \Omega_{i+1,j}g^{j,k+1} & \cdots & \Omega_{nj}g^{j,k+1} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \Omega_{1j}g^{jn} & \cdots & \Omega_{i-1,j}g^{jn} & \Omega_{i+1,j}g^{jn} & \cdots & \Omega_{nj}g^{jn} & 0 \end{vmatrix}$$

$$=(-1)^{n+i+k}\sqrt{g}a\begin{vmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n}\\ \cdots & \cdots & \cdots & \cdots\\ \Omega_{i-1,1} & \Omega_{i-1,2} & \cdots & \Omega_{i-1,n}\\ \Omega_{i+1,1} & \Omega_{i+1,2} & \cdots & \Omega_{i+1,n}\\ \cdots & \cdots & \cdots & \cdots\\ \Omega_{n1} & \Omega_{n2} & \cdots & \Omega_{nn}\\ g_{k1} & g_{k2} & \cdots & g_{kn} \end{vmatrix}\begin{vmatrix} g^{11} & \cdots & g^{1,k-1} & g^{1,k+1} & \cdots & g^{1n} & g^{1k}\\ g^{21} & \cdots & g^{2,k-1} & g^{2,k+1} & \cdots & g^{2n} & g^{2k}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ g^{n1} & \cdots & g^{n,k-1} & g^{n,k+1} & \cdots & g^{nn} & g^{nk} \end{vmatrix},$$

and therefore

$$c^{ij}g_{jk}=(-1)^n\sqrt{a/g}\,g_{kj}\Omega^{ij} \qquad (i, k=1, \cdots, n). \tag{3.2}$$

Solving (3.2) for c^{ij} for each fixed i and substituting the results in (3.1), we find

$$N_{;1}\times\cdots\times N_{;i-1}\times N\times N_{;i+1}\times\cdots\times N_{;n}=(-1)^n\sqrt{a/g}\,\Omega^{ij}Y_{,j}\ . \tag{3.3}$$

Making use of (1.1), (1.14), (1.21), (1.23), (1.30), (1.33), it is easily seen that

$$\begin{aligned}
&\sum_i (N_{;1}\times\cdots\times N_{;i-1}\times N\times N_{;i+1}\times\cdots\times N_{;n})_{;i}\\
&\quad=\sum_i (N_{;1}\times\cdots\times N_{;i-1}\times N_{;i}\times N_{;i+1}\times\cdots N_{;n})\\
&\quad=n\begin{vmatrix} e_1 & a_{\alpha 1}y^{\alpha}_{,1} & \cdots & a_{\alpha 1}y^{\alpha}_{,n}\\ e_2 & a_{\alpha 2}y^{\alpha}_{,1} & \cdots & a_{\alpha 2}y^{\alpha}_{,n}\\ \cdots & \cdots & \cdots & \cdots\\ e_{n+1} & a_{\alpha,n+1}y^{\alpha}_{,1} & \cdots & a_{\alpha,n+1}y^{\alpha}_{,n}\end{vmatrix}\cdot\begin{vmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & \Omega_{1j}g^{j1} & \Omega_{2j}g^{j1} & \cdots & \Omega_{nj}g^{j1}\\ 0 & \Omega_{1j}g^{j2} & \Omega_{2j}g^{j2} & \cdots & \Omega_{nj}g^{j2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \Omega_{1j}g^{jn} & \Omega_{2j}g^{jn} & \cdots & \Omega_{nj}g^{jn}\end{vmatrix}\\
&\quad=(-1)^n n\sqrt{g}aM_nN\ .
\end{aligned}$$

Thus, from the above equation and (3.3),

$$n\sqrt{g}\,M_nN=(\Omega^{ij}Y_{,j}/\sqrt{g}\,)_{;i}\ . \tag{3.4}$$

Taking the scalar products of both sides of (3.4) with the vector Y, we obtain in consequence of (1.23) and (2.7)

$$nM_np\sqrt{g}=(\Omega^{ij}\eta_j/\sqrt{g}\,)_{;i}-nM_{n-1}\sqrt{g}\ . \tag{3.5}$$

As in the preceding section, let us consider a hypersurface V^n, with a closed boundary V^{n-1} of dimension $n-1$ $(n\geq 2)$, differentiably of class C^3 imbedded in an $(n+1)$-dimensional Riemannian space R^{n+1} of constant Riemannian curvature K. Integrating (3.5) with respect to $x^1, \cdots, x^n$ over this hypersurface V^n and applying Stokes' theorem to the first term on the right side of (3.5), we then obtain

$$\begin{aligned}
&\int_{V^n} M_{n-1}dA+\int_{V^n} M_npdA\\
&\qquad=\frac{1}{n}\int_{V^{n-1}}\sum_j(-1)^{j-1}\frac{\Omega^{ij}\eta_j}{\sqrt{g}}dx^1\cdots dx^{j-1}dx^{j+1}\cdots dx^n\ .
\end{aligned} \tag{3.6}$$

In particular, when the hypersurface V^n is closed and orientable, the integral on the right side of (3.6) vanishes and hence the formula (II).

4. Proofs of Theorems 3 and 4. For $M_1>0$, the assumptions $p\leqq -1/M_1$ and $p\geqq -1/M_1$ are respectively equivalent to $1+M_1p\leqq 0$ and $1+M_1p\geqq 0$. From formula (I) it follows that each of the above two assumptions implies that $p=-1/M_1$. Substituting this in (II) we obtain

$$\int_{V^n}\frac{1}{M_1}(M_1M_{n-1}-M_n)dA=0\ , \tag{4.1}$$

which holds when and only when $M_1M_{n-1}-M_n=0$, since

$$\begin{aligned} M_1M_{n-1}-M_n&=\frac{1}{n^2}\Big(\sum_i \kappa_i \sum \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{n-2}}-n^2\kappa_1\kappa_2\cdots\kappa_n\Big)\\ &=\frac{1}{n^2}\sum[\kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_{n-2}}(\kappa_{i_{n-1}}-\kappa_{i_n})^2]\geqq 0\ , \end{aligned} \tag{4.2}$$

where $i_1, i_2, \cdots, i_n$ are distinct and run from 1 to n. From (4.1), (4.2) it follows that $\kappa_1=\kappa_2=\cdots=\kappa_n$ at each point of the hypersurface V^n and therefore that the quantity defined by

$$(\Omega_{ij}q^iq^j)/(g_{ij}q^iq^j)$$

at each point of the hypersurface V^n for an arbitrary direction q in the hypersurface V^n with contravariant components q^i is independent of the direction q. Hence $\Omega_{ij}=cg_{ij}$ for all i and j at each point of the hypersurface V^n, where c is a scalar invariant, so that every point of the hypersurface V^n is umbilic.

If M_{n-1} is constant, multiplying the formula (I) by M_{n-1} and subtracting the formula (II) by the resulting equation we obtain

$$\int_{V^n}p(M_1M_{n-1}-M_n)dA=0\ . \tag{4.3}$$

From this and the assumption that p is of the same sign at all points of the hypersurface V^n, Theorem 4 follows by exactly the same argument as above.

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