REMARK ON THE USE OF FORMS IN VARIATIONAL CALCULATIONS

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In [1] we introduced in equation (2.2) the form

$$\omega=\sum\frac{\partial F}{\partial q_i'}dq_i-\Big(\sum q_i'\frac{\partial F}{\partial q_i'}-F\Big)dt\ .$$

The purpose of this note is to explain the reason for introducing precisely this form.

We considered the integral

$$I=\int_a^b F(q_1,\ \cdots,\ q_n;\ q_1',\ \cdots,\ q_n';\ t)dt\ .$$

Now let S be the set of all forms η such that if X denotes the vector field to a curve C with equations

$$q_i=q_i(t)$$

$$q_i'=\frac{dq_i}{dt}$$

$$t=t$$

Then $\langle X,\eta\rangle=F$ or $I=\int_a^b\langle X,\eta\rangle dt$. The set S is certainly not void since $F(q_1,\ \cdots,\ q_n;\ q_1',\ \cdots,\ q_n;\ t)dt$ and ω are contained in it. We will prove the following.

THEOREM. *There exists one and only one form ω in S such that along every curve of the above type ω and $d\omega$ give rise to forms in the space $(q_1,\ \cdots,\ q_n,\ t)$.*

Proof. The hypotheses of this theorem are equivalent to the following two analytic conditions:

1. $\langle\partial/\partial q_i',\ \omega\rangle=0$
2. $\langle\partial/\partial q_i'\wedge\partial/\partial x,\ d\omega\rangle=0$ when $q_i'dt=dq_i$ where x is any of the coordinates $(q_i,\ q_i',\ t)$, $i=1,\ \cdots,\ n$.

Condition 1. implies that $\omega=\Sigma a_i dq_i+bdt$. Now since $\omega\in S$ we must have

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$$\sum a_i \frac{dq_i}{dt} + b = F \quad \text{or} \quad \omega = \sum a_i dq_i + (F - \sum a_i q_i) dt \ .$$

By computing $d\omega$ and replacing $q_i dt$ by dq_i we get

$$\frac{\partial F}{\partial q_i'} = a_i \ .$$

This proves that $\omega = \sum \frac{F}{q_i'} dq_i - \left(\sum \frac{F}{q_i'} q_i' - F \right) dt$ is the only form which satisfies the theorem.

Reference

1. Auslander, L., *The use of forms in variational calculations*, Pacific J. Math., **5** (1955), 853–859.

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