ON DISTORTION IN PSEUDO-CONFORMAL MAPPING

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1. Introduction ; the method of the minimum integral. One aim of the theory of functions of several complex variables is to reformulate methods of the theory of conformal mappings in such a way that these methods can be successfully applied to obtain results in the theory of pseudo-conformal mappings, that is, in mappings of domains of the $(z_1, \cdots, z_n)$-space by n analytic functions of the n complex variables $z_1, \cdots, z_n$.[1] The determination of bounds for the distortion of Euclidean measures under pseudo-conformal transformation is one of the main topics of this branch of the theory.

An important tool in investigations of this kind is Bergman's method of the minimum integral [3, p. 48]. The basic idea is as follows. After an invariant[2] (non-Euclidean) metric is introduced in a domain B, *the ratios of the non-Euclidean and the Euclidean measures of geometric objects in B are expressed in terms of quantities λ_B which are solutions of the minimum problems*:

$$(1.1) \qquad \lambda_B = \min \int_B |f|^2 d\omega \ .$$

Here f is an analytic function, regular in B and is subjected to certain auxiliary conditions[3], and $d\omega$ is the element of volume (the element of area in the case of one complex variable). Because of the specific choice of the auxiliary conditions, these λ_B possess the property that they *are monotone functions of the domain B*, that is if $B_1 \supset B$ then $\lambda_{B_1} \geqq \lambda_B$. As a rule λ_B can be expressed in terms of Bergman's kernel functions of B and its derivatives and thus can be calculated for special domains. These λ_B are of much interest because they can be easily applied to obtain distortion theorems ; for instance, if $I \subset B \subset A$, where I and A are domains for which the kernel functions $K^I(z, \bar{z})$ and $K^A(z, \bar{z})$ can be expressed in a closed form[4], then $\lambda_I \leqq \lambda_B \leqq \lambda_A$ and λ_I, λ_A are known quantities. With

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[1] In the present paper we consider only the case of two complex variables, $n=2$. However, it should be stressed that the methods used here can be easily generalized to the case of n variables, $n>2$. The additional difficulties which arise are of a purely technical nature.

[2] Invariant with respect to pseudo-conformal transformation.

[3] By varying the auxiliary conditions, one obtains different λ_B's. As a rule upper indices on λ_B indicate the auxiliary conditions. For details see § 2.

[4] In such case we refer to I and A as "domains of comparison".

the aid of such inequalities one can obtain bouuds for the ratio R of a Euclidean and a non-Euclidean measure of an object j located in B (since R is a known function of λ_B). If B is mapped onto a domain B^* by a pseudo-conformal transformation and there exist domains of comparison I^{**}, A^{**}, such that $I^{**} \subset B^* \subset A^{**}$ then one can obtain also bounds for R^*—the ratio of the Euclidean and non-Euclidean measures of the object j^* which is the image of j. It is clear that the bounds which one obtains for R^*/R are actually bounds for the ratio of the Euclidean measures of j and j^* because the non-Euclidean measure is, by definition, invariant under pseudo-conformal transformation. See [3, pp. 49, 56] and [4, p. 140], where also special results are described in detail.

The more information one has about the various λ_B, the more distortion theorems one can obtain. In § 2 we derive relations between the various λ_B. These relations involve sometimes the volume of the domain B. In many cases it is even of interest to obtain bounds for the λ_B in one direction; in § 3 we derive such bounds in terms of the volume of B and the domains of comparison I and A; $I \subset B \subset A$. We apply these results (§ 4) to obtain bounds for the ratios of the Euclidean and non-Euclidean measures of objects such as arc-length and analytic angle, from which distortions under pseudo-conformal mappings of the Euclidean measures follow.

The function

$$J_B(z_1, z_2) = K^{(B)}(z_1, z_2; \bar{z}_1, \bar{z}_2)/D_B \ , \tag{1.2}$$

where $K^{(B)} = K^{(B)}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is the Bergman kernel of B, and

$$D_B = T_{11}^{(B)} T_{22}^{(B)} - |T_{12}^{(B)}|^2, \ T_{\mu v}^{(B)} = [\partial^2 \log K^{(B)}/\partial z_\mu \partial \bar{z}_v] \ , \tag{1.3}$$

is known to be invariant under pseudo-conformal transformation [2, p. 55.]. Bounds for $J_B(z_1, z_2)$ were applied to obtain various distortion theorems [5]. We conclude the paper in deriving bounds for this function.

As was mentioned before, the λ_B are minimal values of (1.1) for different families of analytic functions. The fact that there exist relations which connect these λ_B (see Theorem 1) is of interest because it throws some light on the interconnection between the various families under consideration. This, in turn, yields application to obtain distortion theorems.

2. Relations between some minima λ_B. Let B be a domain in the z_1, z_2-space, and t a fixed point, $t \in B$.

We shall consider certain minimum values defined as follows: Denote by (1)–(8) the auxiliary conditions

(1) $f(t)=1$ (2) $f(t)=0$ (3) $f_{z_1}(t)=1$ (4) $f_{z_1}(t)=0$

(5) $f_{z_2}(t)=1$ (6) $f_{z_2}(t)=0$ (7) $\int_B f d\omega=0$

(8) $u_1(\partial f/\partial z_1)_t+u_2(\partial f/\partial z_2)_t=1$, u_1, u_2 complex numbers;

and let

(a) λ_B^1 (b) λ_B^{01} (c) λ_B^{001} (d) λ_B^{*1}

(e) λ_B^{**1} (f) λ_B^{0*1} (g) λ_B^{*01} (h) λ_B^{1*0}

(i) λ_B^{010} (j) λ_B^{*10} (k) λ_B^{10} (l) $\lambda_B^{(2)}$

(m) $\lambda_B^{(4)}$; $(\lambda_B\equiv\lambda_B(t), t\in B)$,

be the minima of integral:

$$\int_B |f|^2 d\omega \equiv \iiiint_B |f(z_1, z_2)|^2 dx_1 dy_1 dx_2 dy_2\,, \qquad z_k=x_k+iy_k \quad k=1, 2. \tag{2.1}$$

for functions $f\in\mathscr{L}^2(B)$[5] which are normalized at $t\in B$ by the respective auxiliary conditions

(a) (1); (b) (2) and (3); (c) (2), (4) and (5);

(d) (3); (e) (5); (f) (2) and (5);

(g) (4) and (5); (h) (1) and (6); (i) (2), (3) and (6);

(j) (3) and (6); (k) (1) and (4); (l) (2) and (8);

(m) (8).

Let G be a domain containing a domain B, $B\subset G$. We denote by

(n) λ_{GB}^1 (o) λ_{GB}^{01} (p) λ_{GB}^{001} (q) λ_{GB}^{0*1}

(r) λ_{GB}^{010} (s) $\lambda_{GB}^{(2)}$; $(\lambda_B\equiv\lambda_B(t), t\in B)$,

the minima of the integral

$$\int_G |f|^2 d\omega\,, \qquad B\subset G \tag{2.2}$$

for functions $f\in\mathscr{L}^2(G)$ and normalized at $t\in B$ by the conditions

(n) (1) and (7); (o) (2), (3) and (7);

(p) (2), (4), (5) and (7); (q) (2), (5) and (7);

(r) (2), (3), (6) and (7); (s) (2), (7) and (8).

[5] That is, functions f such that (2.1) is finite. All integrations in this paper are in the Lebesgue sense.

THEOREM 1. *Let B be any domain with finite Euclidean volume, vol $B<\infty$. Then*

$$(1/\lambda_B^1)=(1/\lambda_{BB}^1)+(1/\mathrm{vol}\,B) \tag{2.3}$$

$$(1/\lambda_B^{01})=(1/\lambda_{BB}^{01})+(\lambda_B^1/\mathrm{vol}\,B)\cdot\{(1/\lambda_B^{*1})-(1/\lambda_{BB}^{01})\} \tag{2.4}$$

$$(1/\lambda_B^{001})=(1/\lambda_{BB}^{001})+(\lambda_B^{10}/\mathrm{vol}\,B)\cdot\{(1/\lambda_B^{*01})-(1/\lambda_{BB}^{001})\} \tag{2.5}$$

$$(1/\lambda_B^{0*1})=(1/\lambda_{BB}^{0*1})+(\lambda_B^1/\mathrm{vol}\,B)\cdot\{(1/\lambda_B^{**1})-(1/\lambda_{BB}^{0*1})\} \tag{2.6}$$

$$(1/\lambda_B^{010})=(1/\lambda_{BB}^{010})+(\lambda_B^{1*0}/\mathrm{vol}\,B)\cdot\{(1/\lambda_B^{*10})-(1/\lambda_{BB}^{010})\} \tag{2.7}$$

$$(1/\lambda_B^{(2)})=(1/\lambda_{BB}^{(2)})+(\lambda_B^1/\mathrm{vol}\,B)\cdot\{(1/\lambda_B^{(4)})-(1/\lambda_{BB}^{(2)})\}\ . \tag{2.8}$$

(See [2, p. 30])

Proof. Since $(1/\lambda_B^1)=K^{(B)}$, (2.3) is given in [1]. To establish (2.4)-(2.8) we first consider the following general minimum problem. Let $\{\varphi^{(\nu)}(z)\}$, $\nu=1, 2, \cdots$, be a system of functions orthonormal in B[6] and complete for $\mathscr{L}^2(B)$. Let α_{qp}, $p=1, 2, \cdots$, $q=1, 2, \cdots, n$, be a system of complex numbers such that $\sum_{\nu=1}^{\infty}|\alpha_{q\nu}|^2<\infty$ for $q=1, 2, \cdots, n$, and let $X_1, \cdots, X_n$ be complex numbers. Finally, let λ represent the minimum of the integral

$$\int_B |f|^2 d\omega=\sum_{\nu=1}^{\infty} A_\nu \overline{A}_\nu\ , \qquad A_\nu=\int_B f\cdot\varphi^{(\nu)} d\omega\ , \tag{2.9}$$

for functions $f\in\mathscr{L}^2(B)$ and satisfying

$$\sum_{\nu=m}^{\infty} A_\nu \alpha_{q\nu}=X_q\ , \qquad q=1, 2, \cdots, n. \tag{2.10}$$

To obtain the A_ν which render (2.9) minimum, we set equal to zero the derivative of

$$\sum_{\nu=1}^{\infty} A_\nu\overline{A}_\nu-\sum_{k=1}^{n}\left[L_k\left(\sum_{\nu=m}^{\infty}A_\nu\alpha_{k\nu}-X_k\right)+\overline{L}_k\left(\sum_{\nu=m}^{\infty}\overline{A}_\nu\overline{\alpha}_k-\overline{X}_k\right)\right] \tag{2.11}$$

with respect to A_ν, $\nu=1, 2, \cdots$, obtaining

$$A_\nu=0,\ \nu<m;\ A_\nu=\sum_{k=1}^{n}\overline{L}_k\overline{\alpha}_{k\nu},\ \nu\geqq m\ . \tag{2.12}$$

The L_k are evaluated by substituting (2.12) into (2.10), and we obtain (Cf. [2, pp. 41-43])

[6] By "$\varphi^{(\mu)}(z)$ and $\varphi^{(\nu)}(z)$ are orthonormal in B" we mean that $\int_B \varphi^{(\mu)}(z)\overline{\varphi^{(\nu)}(z)}d\omega=\delta_{\mu\nu}$, where $\delta_{\mu\nu}=0$ for $\mu\neq\nu$, $\delta_{\nu\nu}=1$.

$$\lambda = -\begin{vmatrix} 0 & (\bar{X})' \\ (X) & (D) \end{vmatrix} \Big/ |(D)| \tag{2.13}$$

where (X) is the column matrix of n rows having X_r as element in rth row, $(\bar{X})'$ is the transpose of (X) conjugated, (D) is the square matrix of n rows having $\sum_{\nu=m}^{\infty} \alpha_{r\nu}\bar{\alpha}_{s\nu}$ as element in rth row, sth column, and $|(D)|$ is the determinant of (D).

In the special case that λ is the minimum of (2.9) for functions $f \in \mathscr{L}^2(B)$ and satisfying at $t \in B$ the auxiliary conditions $f(t) = \sum_{\nu=1}^{\infty} A_\nu \varphi^{(\nu)}(t) = 0$, $f_{z_1}(t) = \sum_{\nu=1}^{\infty} A_\nu \varphi_{z_1}^{(\nu)}(t) = 1$, and $\int_B f d\omega = 0$, we have $\lambda = \lambda_{GB}^{01}$ with $G = B$. Taking $\varphi^{(1)} = (\text{vol } B)^{-1/2}$ the last auxiliary condition implies $A_1 = 0$, and the auxiliary conditions may be written $\sum_{\nu=2}^{\infty} A_\nu \varphi^{(\nu)}(t) = 0$, $\sum_{\nu=2}^{\infty} A_\nu \varphi_{z_1}^{(\nu)}(t) = 1$. Thus λ_{BB}^{01} is given by formula (2.13) in taking $m=2$, $n=2$, $X_1=0$, $X_2=1$, $\alpha_{1\nu} = \varphi^{(\nu)}(t)$, and $\alpha_{2\nu} = \varphi_{z_1}^{(\nu)}(t)$. Likewise λ_B^1, λ_B^{*1}, λ_B^{01} are computed from (2.13) by taking $m=1$, $n=1$, $X_1=1$, $\alpha_{1\nu} = \varphi^{(\nu)}(t)$ for λ_B^1; $m=1$, $n=1$, $X_1=1$, $\alpha_{1\nu} = \varphi_{z_1}^{(\nu)}(t)$ for λ_B^{*1}; and $m=1$, $n=2$, $X_1=0$, $X_2=1$, $\alpha_{1\nu} = \varphi^{(\nu)}(t)$, $\alpha_{2\nu} = \varphi_{z_1}^{(\nu)}(t)$ for λ_B^{01}. Equation (2.4) now follows by eliminating sums involving $\varphi^{(\nu)}(t)$ and $\varphi_{z_1}^{(\nu)}(t)$ from the expressions for λ_B^1, λ_B^{*1}, λ_B^{01}, λ_{BB}^{01} as given by (2.13).

By use of (2.13), the other minima of the lemma are expressed in terms of u_1, u_2, and sums involving $\varphi^{(\nu)}(z)$ and its derivatives at $z=t$, by taking for m, n, X_j, $\alpha_{j\nu}$, $j=1,2,3$, the values indicated by the following table.

	m	n	X_1	X_2	X_3	$\alpha_{1\nu}$	$\alpha_{2\nu}$	$\alpha_{3\nu}$
λ_{BB}^{001}	2	3	0	0	1	$\varphi^{(\nu)}$	$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$
λ_B^{001}	1	3	0	0	1	$\varphi^{(\nu)}$	$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$
λ_B^{10}	1	2	1	0		$\varphi^{(\nu)}$	$\varphi_{z_1}^{(\nu)}$	
λ_B^{*01}	1	2	0	1		$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$	
λ_{BB}^{0*1}	2	2	0	1		$\varphi^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$	
λ_B^{0*1}	1	2	0	1		$\varphi^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$	
λ_B^{**1}	1	1	1			$\varphi_{z_2}^{(\nu)}$		
λ_{BB}^{010}	2	3	0	1	0	$\varphi^{(\nu)}$	$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$
λ_B^{010}	1	3	0	1	0	$\varphi^{(\nu)}$	$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$
λ_B^{1*0}	1	2	1	0		$\varphi^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$	
λ_B^{*10}	1	2	1	0		$\varphi_{z_1}^{(\nu)}$	$\varphi_{z_2}^{(\nu)}$	
$\lambda_{BB}^{(2)}$	2	2	0	1		$\varphi^{(\nu)}$	$u_1\varphi_{z_1}^{(\nu)} + u_2\varphi_{z_2}^{(\nu)}$	
$\lambda_B^{(2)}$	1	2	0	1		$\varphi^{(\nu)}$	$u_1\varphi_{z_1}^{(\nu)} + u_2\varphi_{z_2}^{(\nu)}$	
$\lambda_B^{(4)}$	1	1	1				$u_1\varphi_{z_1}^{(\nu)} + u_2\varphi_{z_2}^{(\nu)}$	

Equations (2.5)–(2.8) follow by combining the expressions just obtained so as to eliminate u_1, u_2, and sums involving $\varphi^{(\nu)}$ and its derivatives.

3. Upper bounds for the $\lambda_B(t)$ in terms of the volume of B and domains of comparison.

THEOREM 2. *Let B be any domain with finite Euclidean volume, in the (z_1, z_2)-space such that $\operatorname{vol} B \leqq V < \infty$. Then if I and A are any domains $I \subset B \subset A$, we have*

$$(3.1) \qquad (1/\lambda_B^1) \geqq (1/\lambda_{AB}^1) + (1/V)$$

$$(3.2) \qquad (1/\lambda_B^{01}) \geqq \{1-(\lambda_1^1/V)\} \cdot (1/\lambda_{AB}^{01}) + (\lambda_1^1/V)(1/\lambda_A^{*1})$$

$$(3.3) \qquad (1/\lambda_B^{001}) \geqq \{1-(\lambda_I^{10}/V)\} \cdot (1/\lambda_{AB}^{001}) + (\lambda_I^{10}/V) \cdot (1/\lambda_A^{*01})$$

$$(3.4) \qquad (1/\lambda_B^{0*1}) \geqq \{1-(\lambda_I^1/V)\} \cdot (1/\lambda_{AB}^{0*1}) + (\lambda_I^1/V) \cdot (1/\lambda_A^{**1})$$

$$(3.5) \qquad (1/\lambda_B^{010}) \geqq \{1-(\lambda_I^{1*0}/V)\} \cdot (1/\lambda_{AB}^{010}) + (\lambda_I^{1*0}/V)(1/\lambda_A^{*10})$$

$$(3.6)^7 \qquad (1/\lambda_B^{(2)}) \geqq \{1-(1-(\lambda_I^1/V)\} \cdot (1/\lambda_{AB}^{(2)}) + (\lambda_I^1/V) \cdot (1/\lambda_A^{(4)})$$

at $t \equiv (t_1, t_2) \in I$.

Proof. Here we use the monotone properties:

$$(3.7a) \qquad \lambda_I \leqq \lambda_B \leqq \lambda_A \qquad \text{for } I \subset B \subset A$$

and

$$(3.7b) \qquad \lambda_{AB} \geqq \lambda_{BB}\,, \qquad A \supset B.$$

To establish (3.7b) observe that the functions competing to give λ_{AB} are also admissible to the competition to give λ_{BB}.

Since the integrand in (2.1) is non-negative, and since $A \supset B$, (3.7) follows. (3.1) follows from (2.3), $V \geqq \operatorname{vol} B$, and (3.7).

To establish (3.2)–(3.6) we proceed as follows. Each equation (2.4)–(2.8) is of the form[8]

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[7] It is interesting to note that inequality (3.6) is a relation between two quadratic forms in the complex quantities ζ_1, ζ_2:

$$(1/\lambda_G^{(2)}) = K^{(G)} \cdot H(\zeta, \bar{\zeta}), \quad H(\zeta, \bar{\zeta}) \equiv \sum_{m,n=1}^{2} T_{\underline{mn}}^{(G)} \zeta_m \bar{\zeta}_n\,, \quad (z=t)\,,$$

[**2**, p. 46] where $H(\zeta, \bar{\zeta})$ is invariant under pseudo-conformal transformations of G. It can be shown also that

$$(1/\lambda_G^{(4)}) \equiv \sum_{m,n=1}^{2} U_{\underline{mn}}^{(G)} \zeta_m \bar{\zeta}_n\,, \qquad U_{\underline{mn}}^{(G)} \equiv [\partial^2 K^{(G)}/\partial z_m \bar{z}_n]\,, \quad (z=t)\,,$$

which is invariant under pseudo-conformal transformations with Jacobian having absolute value one at each point of G.

[8] To obtain (2.4)–(2.8) from (3.8) we take for $(\lambda_B\,, \lambda_{BB}\,, \lambda_B^{(+)}\,, \lambda_B^{(*)})$ the values $(\lambda_B^{01}, \lambda_{BB}^{01}, \lambda_B^1\,, \lambda_B^{*1})$, $(\lambda_B^{001}\,, \lambda_{BB}^{001}\,, \lambda_B^{10}\,, \lambda_B^{*01})$, $(\lambda_B^{0*1}, \lambda_{BB}^{0*1}, \lambda_B^1, \lambda_B^{**1})$, $(\lambda_B^{010}\,, \lambda_{BB}^{010}\,, \lambda_B^{1*0}\,, \lambda^{*10})$ and $(\lambda_B^{(2)}\,, \lambda_{BB}^{(2)}, \lambda_B^1\,, \lambda_B^{(4)})$, respectively.

(3.8) $$(1/\lambda_B)=(1/\lambda_{BB})+(\lambda_B^+/\text{vol}\,B)\cdot\{(1/\lambda_B^{(*)})-(1/\lambda_{BB})\}$$

where the auxiliary conditions associated with $\lambda_B^{(*)}$ are among the auxiliary conditions associated with λ_{BB}, and the first factor in the last term is $\leqq 1$. Hence $\lambda_{BB}\geqq\lambda_B^{(*)}$, and the brace in (3.8) is non-negative. Using in addition $V\geqq\text{vol}\,B$, and (3.7), from (3.8) we obtain

(3.9) $$\begin{aligned}(1/\lambda_B)&\geqq(1/\lambda_{BB})+(\lambda_I^{(+)}/V)\{(1/\lambda_B^{(*)})-(1/\lambda_{BB})\}\\&=\{1-(\lambda_I^{(+)}/V)\}\cdot(1/\lambda_{BB})+(\lambda_I^+/V)(1/\lambda_B^{(*)})\ ,\\(1/\lambda_B)&\geqq\{1-(\lambda_I^{(+)}/V)\}\cdot(1/\lambda_{AB})+(\lambda_I^+/V)(1/\lambda_A^{(*)})\end{aligned}$$

which yields (3.2)–(3.6).

In order to obtain for the $\lambda_B(t)$ upper bounds which are smaller than the $\lambda_A(t)$, we make the following

(3.10) *Assumption*: If A is the exterior domain of comparison for B, there exist domains V_ν, $\nu=1, 2, \cdots, N$, such that $V_\nu\subset A$, $V_\nu\cap B=0$ for $\nu=1, 2, \cdots, N$. The volume of $\sum_{\nu=1}^{N} V_\nu$ is known and is different from zero. With this assumption we can take $V=\text{vol}\,A-\text{vol}\left(\sum_{\nu=1}^{N} V_\nu\right)$ in theorem 1, so that we have

(3.11) $$\text{vol}\,A> V\geqq\text{vol}\,B\ .$$

4. Distortion theorems using assumption (3.10). There are domains B for which the information about B contained in assumption (3.10) can be used with advantage in deriving distortion theorems. Preparatory to proving this, we make some remarks about distortion of arc length.

Let B be a domain in the (z_1, z_2)-space, $I\subset B\subset A$, where I and A are suitably chosen domains of comparison for B at $t\in B$. Denote by dS the element of Euclidean measure and by ds the element of non-Euclidean measure in B of arc length in the direction (u_1, u_2) at t, defined as follows:

(4.0) $$ds^2=\sum_{\mu,\nu=1}^{2} T_{\mu\bar\nu}^{(B)}u_\mu\bar u_\nu\ ,\qquad dS^2=\sum_{\nu=1}^{2}|u_\nu|^2\ .$$

It was shown by S. Bergman [3, p. 56] that

(4.1) $$l_1\leqq ds^2/dS^2\leqq l_2$$

where

$$l_{\substack{1\\2}}=\frac{1}{2}(T_{1\bar 1}^{(B)}+T_{2\bar 2}^{(B)})(1\mp\sqrt{1-p^2})\ ,$$

$$p=2\sqrt{D}/(T_{1\bar{1}}^{(B)}+T_{2\bar{2}}^{(B)}), \qquad D\equiv T_{1\bar{1}}^{(B)}\cdot T_{2\bar{2}}^{(B)}-|T_{1\bar{2}}^{(B)}|^2 .$$

We make use of the fact that at $z=t$

$$(4.2) \qquad T_{1\bar{1}}=\lambda_B^1/\lambda_B^{01}, \qquad T_{2\bar{2}}=\lambda_B^1/\lambda_B^{0*1}, \qquad D=(\lambda_B^1)^2/(\lambda_B^{01}\cdot\lambda^{001}),$$

[2, p. 45; 3, p. 56].

It follows that

$$p=\frac{2\sqrt{(\lambda_B^1)^2/(\lambda_B^{01}\cdot\lambda_B^{001})}}{(\lambda_B^1/\lambda_B^{01})+(\lambda_B^1/\lambda_B^{0*1})}$$

or

$$p=2\{\sqrt{\lambda_B^{01}\cdot\lambda_B^{001}}[(1/\lambda_B^{01})+(1/\lambda_B^{0*1})]\}^{-1}$$

The quantity $D\equiv T_{1\bar{1}}^{(B)}\cdot T_{2\bar{2}}^{(B)}-|T_{1\bar{2}}^{(B)}|^2$ is invariant under change of variables, so that in addition to $D=(\lambda_B^1)^2/(\lambda_B^{01}\cdot\lambda_B^{001})$, we also have (when replacing z_1, z_2 by z_2, z_1, respectively,

$$(4.3) \qquad D=(\lambda_B^1)^2/(\lambda_B^{0*1}\cdot\lambda_B^{010}) .$$

Hence $\lambda_B^{01}\cdot\lambda_B^{001}=\lambda_B^{0*1}\cdot\lambda_B^{010}$, and

$$l_{\frac{1}{2}}=\frac{1}{2}\lambda_B^1(1/\lambda_B^{01}+1/\lambda_B^{0*1})(1\mp\sqrt{1-p^2}),$$

where

$$p=2\{\sqrt{\lambda_B^{001}/\lambda_B^{01}}+\sqrt{\lambda_B^{010}/\lambda_B^{0*1}}\}^{-1} .$$

Using the monotonicity of the λ's, we obtain from (4.1) the inequality

$$(4.4) \qquad q_1\leqq ds^2/dS^2\leqq q_2$$

where

$$q_1=\frac{1}{2}\lambda_I^1(1/\lambda_A^{01}+1/\lambda_A^{0*1})(1-\sqrt{1-p_0^2}),$$

$$q_2=\frac{1}{2}\lambda_A^1(1/\lambda_I^{01}+1/\lambda_I^{0*1})(1+\sqrt{1-p_0^2}),$$

$$p_0=2\{\sqrt{\lambda_A^{001}/\lambda_I^{01}}+\sqrt{\lambda_A^{010}/\lambda_I^{0*1}}\}^{-1} .$$

Using hyperspheres centered at the point under consideration as domains of comparison, (4.4) gives the distortion theorem (23) of [3, p. 57].

Another way to find bounds for ds^2/dS^2 is to make use of the relation

$$(4.5) \qquad ds^2\equiv\sum_{\mu,\nu=1}^{2}T_{\mu\bar{\nu}}^{(B)}u_\mu\bar{u}_\nu=\lambda_B^1/\lambda_B^{(2)}$$

[2, p. 53; 4, p. 142]. Indeed,

$$(\lambda_B^1/\lambda_A^1)(\lambda_A^1/\lambda_A^{(2)}) \leqq \lambda_B^1/\lambda_A^{(2)} \leqq ds^2 = \lambda_B^1/\lambda_B^{(2)} \leqq \lambda_B^1/\lambda_I^{(2)} = (\lambda_B^1/\lambda_I^1)(\lambda_I^1/\lambda_I^{(2)})$$

or

$$(\lambda_B^1/\lambda_A^1)\cdot \sum_{\mu,\nu=1}^{2} T_{\mu\bar{\nu}}^{(A)} u_\mu \bar{u}_\nu \leqq ds^2 \leqq (\lambda_B^1/\lambda_I^1)\cdot \sum_{\mu,\nu=1}^{2} T_{\mu\bar{\nu}}^{(I)} u_\mu \bar{u}_\nu$$

hence

$$(\lambda_I^1/\lambda_A^1)\cdot \frac{1}{2}(T_{1\bar{1}}^{(A)} + T_{2\bar{2}}^{(A)})(1-\sqrt{1-p_A^2}) \leqq ds^2/dS^2 \tag{4.6}$$

$$\leqq (\lambda_A^1/\lambda_I^1)\cdot \frac{1}{2}(T_{1\bar{1}}^{(I)} + T_{2\bar{2}}^{(I)})(1+\sqrt{1-p_I^2})$$

where

$$p_A = 2\sqrt{D_A}/(T_{1\bar{1}}^{(A)} + T_{2\bar{2}}^{(A)}) , \qquad D_A = T_{1\bar{1}}^{(A)} T_{2\bar{2}}^{(A)} - |T_{1\bar{2}}^{(A)}|^2 ,$$

$$p_I = 2\sqrt{D_I}/(T_{1\bar{1}}^{(I)} + T_{2\bar{2}}^{(I)}) , \qquad D_I = T_{1\bar{1}}^{(I)} T_{2\bar{2}}^{(I)} - |T_{1\bar{2}}^{(I)}|^2 .$$

Using relations (4.2) and the monotonicity of the λ's, (4.6) becomes

$$r_1 \leqq ds^2/dS^2 \leqq r_2 \tag{4.7}$$

$$r_1 = \frac{1}{2}\lambda_I^1[(1/\lambda_A^{01}) + (1/\lambda_A^{0*1})](1-\sqrt{1-p_A^2}) ,$$

$$r_2 = \frac{1}{2}\lambda_A^1[(1/\lambda_I^{01}) + (1/\lambda_I^{0*1})](1+\sqrt{1-p_I^2}) ,$$

$$p_A = 2\{\sqrt{\lambda_A^{001}/\lambda_A^{01}} + \sqrt{\lambda_A^{010}/\lambda_A^{0*1}}\}^{-1} , \qquad p_I = 2\{\sqrt{\lambda_I^{001}/\lambda_I^{01}} + \sqrt{\lambda_I^{010}/\lambda_I^{0*1}}\}^{-1} .$$

Since $\lambda_I < \lambda_A$ for $I \subset A$, $I \not\equiv A$, it is clear that (4.7) is a better inequality than (4.4).

Hence in estimating distortion of arc length it is of distinct advantage to first make use of the relation (4.5). This is true regardless of what domains are used as domains of comparison.

It is interesting to know that the inequality (4.7) ean still be improved in many cases by using the relations of § 3.

THEOREM 3. *Let B be a given domain, $I \subset B \subset A$, where I and A are domains of comparison for B at $t \in B$. Denote by dS the element of Euclidean measure and by ds the element of non-Euclidean measure in B of arc length in the direction (u_1, u_2) at t (see (4.0)). Then*

$$c_1 \leqq ds^2/dS^2 \leqq c_2 \tag{4.8}$$

where

$$c_1 = \max\left\{(\lambda_I^1 \cdot l)/\sum_{\nu=1}^{2}|u_\nu|^2,\ \frac{1}{2}\lambda_I^1(v_1+v_2)(1-\sqrt{1-p_1^2})\right\}$$

$$c_2 = \frac{1}{2}(1/h)[(1/\lambda_I^{01})+(1/\lambda_I^{0*1})](1+\sqrt{1-p_I^2})$$

$$l = \max\{(1/\lambda_A^{(2)}),\ [1-(\lambda_I^1/V_B)](1/\lambda_{AB}^{(2)})+(\lambda_I^1/V_B)(1/\lambda_A^{(4)})\} \tag{4.9}$$

$$p_1 = 2\{\sqrt{1/(w_1\cdot\lambda_I^{01})}+\sqrt{1/(w_2\cdot\lambda_I^{0*1})}\}^{-1}$$

$$p_I = 2\{\sqrt{\lambda_I^{001}/\lambda_I^{01}}+\sqrt{\lambda_I^{010}/\lambda_I^{0*1}}\}^{-1}$$

$$h = \max\{(1/\lambda_A^1),\ (1/\lambda_{AB}^1)+(1/V_B)\},\quad V_B \geqq \operatorname{vol} B$$

$$v_1 = \max\{(1/\lambda_A^{01}),\ [1-(\lambda_I^1/V_B)](1/\lambda_{AB}^{01})+(\lambda_I^1/V_B)(1/\lambda_A^{*1})\}$$

$$v_2 = \max\{(1/\lambda_A^{0*1}),\ [1-(\lambda_I^1/V_B)](1/\lambda_{AB}^{0*1})+(\lambda_I^1/V_B)(1/\lambda_A^{**1})\}$$

$$w_1 = \max\{(1/\lambda_A^{001}),\ [1-(\lambda_I^{10}/V_B)](1/\lambda_{AB}^{001})+(\lambda_I^{10}/V_B)(1/\lambda_A^{*01})\}$$

$$w_2 = \max\{(1/\lambda_A^{010}),\ [1-(\lambda_I^{1*0}/V_B)](1/\lambda_{AB}^{010})+(\lambda_I^{1*0}/V_B)(1/\lambda_A^{*10})\}\ .$$

REMARK. In particular if I and A denote respectively the hyperspheres

$$|z_1|^2+|z_2|^2<m^2$$

and

$$|z_1-\varepsilon Me^{i\theta_1}|^2+|z_2-\varepsilon Me^{i\theta_2}|^2<M^2$$

where

$$0\leqq\varepsilon^2<\frac{1}{2}\ ,\quad 0<m<M\cdot(1-\sqrt{2}\,\varepsilon)\ ,\quad 0\leqq\theta_\nu<2\pi\ ,\qquad \nu=1,2,$$

then the quantities λ_I are functions of m; the λ_A not occurring in (4.9) are functions of ε, M; $\lambda_A^{(4)}$ is a function of ε, M, θ_ν, u_ν, $\nu=1,2$; the λ_{AB} not occurring in (4.9) are functions of ε, M, θ_σ, $\varphi_0^{(\sigma)}$, $\varphi_1^{(\sigma)}$, φ_2, $\sigma=1,2$, where

$$\left\{\begin{aligned} \varphi_k^{(\sigma)} &= \int_B (\zeta-\varepsilon Me^{i\theta_\sigma})^k\cdot\Big[M-\varepsilon\cdot\sum_{\nu=1}^{2}(\varepsilon M-\zeta_\nu e^{-i\theta_\nu})\Big]^{-3-k}d\omega_\zeta\ , \\ \varphi_2 &= \int_B\int_B\Big[M^2-\sum_{\nu=1}^{2}(\zeta_\nu-Me^{i\theta_\nu})(\bar{\xi}_\nu-Me^{-i\theta_\nu})\Big]^{-3}d\omega_\zeta d\omega_\xi \end{aligned}\right.\qquad k=0,1, \tag{4.10}$$

and $\lambda_{AB}^{(2)}$ is a function of ε, M, θ_σ, u_σ, $\varphi_0^{(\sigma)}$, $\varphi_1^{(\sigma)}$, φ_2, $\sigma=1,2$.

Proof. Using (4.5) we obtain:

$$(4.11)\qquad \sum_{\mu,\nu=1}^{2} T_{\mu\bar{\nu}}^{(B)} u_\mu \bar{u}_\nu = \lambda_B^1/\lambda_B^{(2)} = ds^2 \leqq \lambda_B^1/\lambda_I^{(2)}$$

$$= (\lambda_B^1/\lambda_I^1)(\lambda_I^1/\lambda_I^{(2)}) = (\lambda_B^1/\lambda_I^1)\cdot \sum_{\mu,\nu=1}^{2} T_{\mu\bar{\nu}}^{(I)} u_\mu \bar{u}_\nu \ .$$

Applying transformations of the type

$$(4.12)\qquad \begin{aligned} z_1^* &= e^{i\gamma_1}\cos\theta\cdot z_1 + e^{i\gamma_2}\sin\theta\cdot z_2 \\ z_2^* &= e^{i\delta_1}\sin\theta\cdot z_1 + e^{i\delta_2}\cos\theta\cdot z_2 \end{aligned}$$

in B and I, and using (3.6), we obtain

$$(4.13)\qquad \max\left\{\lambda_B^1\cdot l/\sum_{\nu=1}^{2}|u_\nu|^2,\ \frac{1}{2}\lambda_B^1(1/\lambda_B^{01}+1/\lambda_B^{0*1})(1-\sqrt{1-p^2})\right\}$$

$$\leqq ds^2/dS^2 \leqq \frac{1}{2}\lambda_B^1(T_{1\bar{1}}^{(I)}+T_{2\bar{2}}^{(I)})(1+\sqrt{1-p_I^2})$$

where

$$p = 2\sqrt{D_B}/(T_{1\bar{1}}^{(B)}+T_{2\bar{2}}^{(B)})\ ,\qquad D_B = T_{1\bar{1}}^{(B)}\cdot T_{2\bar{2}}^{(B)} - |T_{1\bar{2}}^{(B)}|^2\ ,$$

$$p_I = 2\sqrt{D_I}/(T_{1\bar{1}}^{(I)}+T_{2\bar{2}}^{(I)})\ ,\qquad D_I = T_{1\bar{1}}^{(I)}\cdot T_{2\bar{2}}^{(I)} - |T_{1\bar{2}}^{(I)}|^2$$

and l was defined by (4.9). Using (4.2), (4.3), and (3.1)–(3.6), and taking into account the monotonicity of $\lambda_B^1(t)$ as a function of the domain, we deduce that (4.13) implies (4.9).

To complete the proof of the theorem, we compute the λ's for the case when I and A are hyperspheres, using formula (2.13) as explained in the table of § 2, and a similar consideration for the λ_{AB}'s.

REMARK. In the case when assumption (3.10) is satisfied, it is sometimes better to use (4.8), instead of (4.7). To prove this, we consider the following example.

Let S_1, S_2, S_3, S_η denote the hyperspheres

$$S_1:\quad |z_1-\varepsilon_0|^2+|z_2-\varepsilon_0|^2<(2\sqrt{2}\,\varepsilon_0-1)^2$$

$$S_2:\quad |z_1|^2+|z_2|^2<(1-\sqrt{2}\,\varepsilon_0)^2$$

$$S_3:\quad |z_1-\varepsilon_0|^2+|z_2-\varepsilon_0|^2<1$$

$$S:\quad |z_1-\varepsilon_0|^2+|z_2-\varepsilon_0|^2<(1-\eta)^2$$

where $1/8<\varepsilon_0^2<1/2$, and $\eta>0$ is sufficiently small.

Let B be a domain satisfying

$$S_2 \subset B \subset S_3 - S_1\ ,\qquad S_3 - S_\eta \subset B\ ,$$

and let $\rho>1$ be such that $(\operatorname{vol} B)=\rho\cdot(\operatorname{vol} S_2)$. For such a domain B we are able to show that (4.8) is a better inequality than (4.7).

To show that such a domain B exists, we must show that $S_2\subset S_3-S_1$. It is clear that $S_2\subset S_3$. To show that $S_2\cap S_1=0$, observe that the frontiers of S_2 and S_3 have a point in common, namely the point $[(1/\sqrt{2})(\sqrt{2}\varepsilon_0-1), (1/\sqrt{2})(\sqrt{2}\varepsilon_0-1)]$, the centers of S_1 and S_3 are the same, and

$$[2\cdot(\text{radius of } S_2)]+(\text{radius of } S_1)=(\text{radius of } S_3). \tag{4.14}$$

Hence $S_2\cap S_1=0$, $S_2\subset S_3-S_1$, and B exists.

Consider distortion of arc length at the origin in the direction $u_1=1$, $u_2=0$. As exterior and interior domains of comparison at the origin we take the eccentric hyperspheres

$$A: \sum_{\mu=1}^{2}|z_\mu-\varepsilon Me^{i\theta_\mu}|^2<M^2$$

$$0\leqq\varepsilon^2<\frac{1}{2},\ M>0,\ 0\leqq\theta_k<2\pi,\ k=1, 2, \text{ and}$$

$$I: \sum_{\nu=1}^{2}|z_\nu-\delta me^{i\varphi_\nu}|^2<m^2$$

$$0\leqq\delta^2<\frac{1}{2},\ m>0,\ 0\leqq\varphi_k<2\pi,\ k=1, 2.$$

From the relation $S_3-S_\eta\subset B$, it follows that any exterior domain of comparison A contains S_3. Since the quantities λ are monotone functionals of the domain, it is clear that the best choice for A is $A=S_3$, from which it follows $\varepsilon_0=\varepsilon$.

To obtain the best bounds for distortion of arc length from (4.7), I has to be chosen in such a way that the quantities

$$\lambda_I^1=\frac{1}{2}\pi^2m^4(1-2\delta^2)^3\ , \qquad \lambda_I^{01}=\frac{1}{6}\pi^2m^6(1-2\delta^2)^5/(1-\delta^2)\ ,$$

$$\lambda_I^{0*1}=\frac{1}{6}\pi^2m^6(1-\delta^2)^5/(1-\delta^2)\ , \qquad \lambda_I^{01}/\lambda_I^{001}=(1-2\delta^2)/(1-\delta^2)^2\ ,$$

$$\lambda_I^{0*1}/\lambda_I^{010}=(1-2\delta^2)/(1-\delta^2)^2$$

are as large as possible.

From (4.14) and since $B\subset S_3-S_1$ it follows that B contains no hypersphere of radius greater than the radius of S_2. Hence $m\leqq$(radius of S_2), and if we take $I=S_2$, we have $\delta=0$ and a best choice for interior domain of comparison when estimating distortion of arc length by (4.7).

Repeating the above considerations, it is easily shown that $A=S_3$

and $I=S_2$ is also a best choice for domains of comparison when estimating distortion of arc length by using (4.8).

On comparing

$$r_1 \leqq ds^2/dS^2 \leqq r_2 \tag{4.7}$$

and

$$c_1 \leqq ds^2/dS^2 \leqq c_2 \tag{4.8}$$

we find that

$$\frac{c_1}{r_1} \geqq \frac{(\lambda_I^1 \cdot l)/\sum_{\nu=1}^{2}|u_\nu|^2}{\frac{1}{2}\lambda_I^1(1/\lambda_A^{01}+1/\lambda_A^{0*1})(1-\sqrt{1-p_A^2})} \geqq \frac{\lambda_I^1}{V_B}\cdot\frac{1}{\lambda_A^{(1)}}\cdot\frac{\lambda_A^{01}}{(-\sqrt{1-p_A^2})}$$

$p_A^2=\lambda_A^{01}/\lambda_A^{001}$ since $\lambda_A^{01}=\lambda_A^{0*1}$, $\lambda_A^{001}=\lambda_A^{010}$ in this case.

$$\lambda_A^{01}/\lambda_A^{001}=\frac{\pi^2M^6(1-2\varepsilon^2)^5}{6(1-\varepsilon^2)}\cdot\frac{6}{\pi^2M^6(1-2\varepsilon^2)^4(1-\varepsilon^2)}=\frac{(1-2\varepsilon^2)}{(1-\varepsilon^2)^2}$$

$\lambda_I^1/V_B=1/\rho$ where $V_B=\operatorname{vol}B=\rho\cdot(\operatorname{vol}I)$, $\rho>1$.

$$(1/\lambda_A^{(4)})=\sum_{\mu,\nu=1}^{2}\frac{\partial^2K^{(A)}}{\partial z_\mu\partial\bar{z}_\nu}u_\mu\bar{u}_\nu=K_{1010}^{(A)}=\frac{6(1+2\varepsilon^2)}{\pi^2M^6(1-2\varepsilon^2)^5}$$

$$\frac{c_1}{r_1}\geqq\frac{1}{\rho}\cdot\frac{6(1+2\varepsilon^2)}{\pi^2M^6(1-2\varepsilon^2)^5}\cdot\frac{\pi^2M^6(1-2\varepsilon^2)^5}{6(1-\varepsilon^2)}\cdot\frac{(1-\varepsilon^2)}{(1-2\varepsilon^2)}$$

or

$$\frac{c_1}{r_1}\geqq\frac{1}{\rho}\cdot\frac{(1+2\varepsilon^2)}{(1-2\varepsilon^2)}, \qquad \frac{1}{8}<\varepsilon^2<\frac{1}{2}. \tag{4.11}$$

$$\frac{r_2}{c_2}\geqq\frac{\frac{1}{2}\lambda_A^1(1/\lambda_I^{01}+1/\lambda_I^{0*1})(1+\sqrt{1-p_I^2})}{\frac{1}{2}(1/h)(1/\lambda_I^{01}+1/\lambda_I^{0*1})(1+\sqrt{1-p_I^2})},$$

$$\frac{r_2}{c_2}\geqq h\cdot\lambda_A^1\geqq(1/V_B)\cdot\lambda_A^1=\frac{1}{\rho\cdot(\operatorname{vol}I)}\cdot\frac{\pi^2(1-2\varepsilon^2)^3}{2},$$

$$\frac{r_2}{c_2}\geqq\frac{1}{\rho}\,\frac{2}{\pi^2(1-\sqrt{2}\,\varepsilon)^4}\cdot\frac{\pi^2(1-2\varepsilon^2)^3}{2}=\frac{1}{\rho}\cdot\frac{(1+\sqrt{2}\,\varepsilon)}{(1-\sqrt{2}\,\varepsilon)}, \tag{4.13}$$

$$1/(2\sqrt{2})<\varepsilon<1/\sqrt{2}.$$

Thus our assertion is proved.

The above theorem can be used to obtain new bounds for distortion of other quantities depending upon arc length, as for example the analytic angle between two vectors. For this purpose we introduced the following concepts: (see [2, p. 8])

Let $X=(X_1, X_2)$, $Y=(Y_1, Y_2)$ represent two vectors with initial points at the origin, in the (z_1, z_2)-space. If $X_k=a_k+ia_{k+2}$, $Y_k=b_k+ib_{k+2}$, $k=1, 2$ (a_ν, b_ν are real, $\nu=1, 2, 3, 4$), then the Euclidean measure F of the angle between X and Y is defined by

$$\cos F=\frac{\sum a_k b_k}{\sqrt{\sum a_k^2\cdot\sum b_k^2}}, \qquad \sum=\sum_{k=1}^{2} \tag{4.15a}$$

Using the notation $H(X, \bar{Y})\equiv H[(X_1, X_2), (Y_1, Y_2)]\equiv X_1\bar{Y}_1+X_2\bar{Y}_2$, $S(X)\equiv S(X_1X_2)\equiv\sqrt{H(X, X)}$ (4.15a) can be written in the form

$$\cos F=\frac{\mathscr{R}\{H(X, \bar{Y})\}}{S(X)\cdot S(Y)}, \quad \text{where} \quad \mathscr{R}\equiv \text{real part}. \tag{4.15b}$$

We define the non-Euclidean measure f of the angle between X and Y to be the, so-called, *analytic angle*. [2, p. 9]. This is the Euclidean angle between the two analytic planes which contain the vectors X and Y, respectively. It is known that

$$\cos f=\frac{|H(X, \bar{Y})|}{S(X)\cdot S(Y)}, \qquad \sin f=\frac{|X_1Y_2-X_2Y_1|}{S(X)\cdot S(Y)}. \tag{4.16}$$

THEOREM 4. *Let B be a domain $I\subset B\subset A$ where I and A denote the hyperspheres*

$$|z_1|^2+|z_2|^2<m^2$$

and

$$|z_1-\varepsilon Me^{i\theta_1}|^2+|z_2-\varepsilon Me^{i\theta_2}|^2<M^2,$$

$$0\leqq\varepsilon^2<\frac{1}{2},\ 0<m<M\cdot(1-\varepsilon\sqrt{2}),\ 0\leqq\theta_\nu<2\pi,\ \nu=1, 2,$$

respectively. Let F denote the Euclidean measure and f the non-Euclidean measure in B of the analytic angle between two vectors at $(0, 0)$. *Then*

$$r\cdot(m/M)^{10}\leqq\sin F/\sin f\leqq(M/m)^{10}\cdot\alpha\cdot\beta/r \tag{4.17}$$

where

$$\alpha^2=\min\{(1-2\varepsilon^2)/(1-\varepsilon^2),\ \rho_1(1-2\varepsilon^2)/(1+2\varepsilon^2)\},$$

$$\beta^2=\min\{(1-\varepsilon^2),\ \rho_J(1+2\varepsilon^2)/(1+6\varepsilon^2)\},$$

$$r=\max\{(1-2\varepsilon^2)^{-7},\ \rho_A\cdot(1-2\varepsilon^2)^{-4}\},$$

$$\rho_I=(V/\mathrm{vol}\, I)\geqq 1,\ \ \rho_A=(\mathrm{vol}\, A/V)\geqq 1,\ \ V\geqq \mathrm{vol}\, B.$$

Proof. Denote the vectors mentioned in the theorem by $X=(X_1, X_2)$, $Y=(Y_1, Y_2)$. From (4.16) and analogous considerations using Bergman's metric we have

$$\sin F=\frac{|X_1Y_2-X_2Y_1|}{S(X)\cdot S(Y)}, \qquad \sin f=\frac{\sqrt{D}\,|X_1Y_2-X_2Y_1|}{\mathscr{S}(X)\cdot\mathscr{S}(Y)} \tag{4.18}$$

where

$$\mathscr{S}^2(X)=H(X,\bar{X})\equiv\sum_{\mu,\nu=1}^{2} T_{\mu\bar{\nu}}^{(B)}X_\mu\bar{X}_\nu\ ,$$

$$T_{\mu\bar{\nu}}\equiv T_{\mu\bar{\nu}}^{(B)}\equiv\partial^2\log K^{(B)}(z,\bar{z})/\partial z_\mu\partial\bar{z}_\nu$$

and

$$D=T_{1\bar{1}}T_{1\bar{2}}-|T_{1\bar{2}}|^2\ .$$

Thus

$$\sin F/\sin f=\mathscr{S}(X)\cdot\mathscr{S}(Y)/[S(X)\cdot S(Y)\cdot\sqrt{D}\,]\ . \tag{4.19}$$

Bounds for $\mathscr{S}(X)/S(X)$ and $\mathscr{S}(Y)/S(Y)$ can be obtained as in the Theorem 3, which is a theorem on the distortion of arc length. Using (4.5) first with $u_\nu=X_\nu$ $\nu=1,2$, then with $u_\nu=Y_\nu$, $\nu=1,2$, applying to A a transformation of the type (4.12) and calculating the $T_{\mu\bar{\nu}}^{(A)}$, $T_{\mu\bar{\nu}}^{(I)}$ for A and I of this theorem, we obtain

$$\sqrt{3}\,m^2/[M^3(1-2\varepsilon^2)^2]\leqq\mathscr{S}(X)/S(X)\leqq\sqrt{6}\,\lambda_B^1/(\pi m^3) \tag{4.20}$$

and the same for $\mathscr{S}(Y)/S(Y)$.

By dropping the terms involving λ's with double subscripts in (3.1)–(3.3) and using the monotonicity of the λ's, we obtain

$$\lambda_I^1\leqq\lambda_B^1\leqq u\ ,\qquad \lambda_I^{01}\leqq\lambda_B^{01}\leqq v\ ,\qquad \lambda_I^{001}\leqq\lambda_B^{001}\leqq w \tag{4.21}$$

where

$$u=\min\{\lambda_A^1, v\}\ ,\qquad v=\min\{\lambda_A^{01}, \rho_I\cdot\lambda_A^{*1}\}\ ,$$

$$w=\min\{\lambda_A^{001}, \rho_I\cdot\lambda_A^{*01}\}\ .$$

Making use of (4.2), (4.17) follows from (4.19), (4.20), and (4.21).

REMARK. As in the case of arc length (Theorem 3), an example can be given which shows that an upper bound for the volume of B is of advantage in estimating distortion of analytic angle.

Another distortion theorem follows from the following considerations.

Under a pseudo-conformal mapping of a domain B^* of the (z_1^*, z_2^*)-space onto a domain B of the (z_1, z_2)-space, the Bergman kernel $K^{(B^*)}(z_1^*, z_2^*; \bar{z}_1^*, \bar{z}_2^*)$ transforms as a relative invariant; that is,

$$K^{(B)}(z_1, z_2; \bar{z}_1, \bar{z}_2) = K^{(B^*)}(z_1^*, z_2^*; \bar{z}_1^*, \bar{z}_2^*) \cdot \left| \frac{\partial(z_1^*, z_2^*)}{\partial(z_1, z_2)} \right|^2 ,$$

and we obtain an absolute pseudo-conformal invariant if we consider

$$(4.22) \qquad J_B(z_1, z_2) \equiv K^{(B)}/D_B = \lambda_B^{01} \cdot \lambda_B^{001}/(\lambda_B^1)^3 , \qquad D_B \equiv T_{1\bar{1}}^{(B)} \cdot T_{2\bar{2}}^{(B)} - |T_{1\bar{2}}^{(B)}|^2 ,$$

where $K^{(B)} \equiv K^{(B)}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is the kernel function of B, and $T_{\mu\bar{\nu}}^{(B)} \equiv [\partial^2 \log K^{(B)}/\partial z_\mu \partial \bar{z}_\nu]$ [2, pp. 51, 55]. Since J_B is a pseudo-conformal invariant, we can use the level surfaces of J_B (when J_B is not a constant) to formulate the following type of distortion theorem[9]. If a domain B^* is mapped psendo-conformally onto a domain B for which $I \subset B \subset A$, then

$$(4.23) \qquad a_1 \leqq J_{B^*} = J_B = \lambda_B^{01} \cdot \lambda_B^{001}/(\lambda_B^1)^3 \leqq a_2$$

where

$$a_1 = (\lambda_I^{01} \cdot \lambda_I^{001})/(\lambda_A^1)^3 , \qquad a_2 = (\lambda_A^{01} \cdot \lambda_A^{001})/(\lambda_I^1)^3$$

[3, p. 48; 5].

With the aid of the relations of § 3, (4.23) can be sharpened in many cases.

THEOREM 5. *Let A be an exterior domain of comparison, I an interior domain of comparison with respect to a given domain B of (z_1, z_2)-space. Then for the pseudo-conformal invariant*

$$J_B \equiv K^{(B)}/(T_{1\bar{1}}^{(B)} \cdot T_{2\bar{2}}^{(B)} - |T_{1\bar{2}}^{(B)}|^2)$$

where $T_{\mu\bar{\nu}}^{(B)} \equiv \partial^2 \log K^{(B)}/\partial z_\mu \partial \bar{z}_\nu$, we have at $t \in B$ the inequality

$$(4.24) \qquad d_1 \leqq J_B \leqq d_2$$

where

$$d_1 = \lambda_I^{01} \cdot \lambda_I^{001} \cdot h^3 , \qquad d_2 = 1/[v \cdot w \cdot (\lambda_I^1)^3] ,$$

$$h = \max\{(1/\lambda_A^1), (1/\lambda_{AB}^1) + (1/V_B)\} , \qquad V_B \geqq \operatorname{vol} B ,$$

[9] A similar procedure and an analogous theorem are valid in the theory of functions of one variable z. For, if G is a region in the z-plane, then the quantity

$$\varphi = (1/K^{(G)}) \cdot [\partial^2 \log K^{(G)}/\partial z \partial \bar{z}] , \quad K^{(G)} \equiv K^{(G)}(z, \bar{z})$$

is a conformal invariant such that -2φ is the Riemann curvature of the metric

$$ds^2 = K^{(G)}(z, \bar{z}) \cdot |dz|^2 ,$$ [4, p. 36].

$$v = \max \{(1/\lambda_A^{01}), [1-(\lambda_I^1/V_B)](1/\lambda_{AB}^{01}) + (\lambda_I^1/V_B)(1/\lambda_A^{*1})\} ,$$

$$w = \max \{(1/\lambda_A^{001}), [1-(\lambda_I^{10}/V_B)](1/\lambda_{AB}^{001}) + (\lambda_I^{10}/V_B)(1/\lambda_A^{*01})\} .$$

The proof consists in combining relations (4.22), (4.0), (3.1), (3.2) and (3.3).

In comparing

$$a_1 \leqq J_B \leqq a_2 \tag{4.23}$$

and

$$d_1 \leqq J_B \leqq d_2 \tag{4.24}$$

we can show that in many cases (4.24) is better than (4.23) for obtaining distortion theorems, as follows.

Let B, I, A, be domains as described in the remark to Theorem 3,

$$(\mathrm{vol}\, B) = \rho \cdot (\mathrm{vol}\, I) , \qquad \rho > 1 .$$

Computation yields

$$\lambda_A^1 = \frac{1}{2}\pi^2 M^4 (1-2\varepsilon^2)^3 , \qquad \lambda_A^{01} = \frac{1}{6}\pi^2 M^6 (1-2\varepsilon^2)^5/(1-\varepsilon^2) ,$$

$$\lambda_A^{001} = \frac{1}{6}\pi^2 M^6 (1-2\varepsilon^2)^4 (1-\varepsilon^2) , \qquad \lambda_A^{*1} = \frac{1}{6}\pi^2 M^6 (1-2\varepsilon^2)^5/(1+2\varepsilon^2)$$

$$\lambda_A^{*01} = \frac{1}{6}\pi^2 M^6 (1-2\varepsilon^2)^4 (1+2\varepsilon^2)/(1+6\varepsilon^2) , \qquad \lambda_I^1 = \lambda_I^{10} = \frac{1}{2}\pi^2 m^4 ,$$

where we now have $M=1$, $m=(1-\sqrt{2}\,\varepsilon)$. Thus

$$h \geqq 1/(\mathrm{vol}\, B) = 1/[\rho \cdot (\mathrm{vol}\, I)]$$

$$v \geqq [\lambda_I^1/(\mathrm{vol}\, B)](1/\lambda_A^{*1}) = 6(1+2\varepsilon^2)/[\pi^2(1-2\varepsilon^2)^5 \cdot \rho]$$

$$w \geqq [\lambda_I^{10}/(\mathrm{vol}\, B)] \cdot (1/\lambda_A^{*01}) = 6(1+6\varepsilon^2)/[\pi^2(1-2\varepsilon^2)^4(1+2\varepsilon^2) \cdot \rho]$$

so that

$$\frac{a_2}{d_2} = \lambda_A^{01} \cdot \lambda_A^{001} \cdot vw \geqq \frac{1+6\varepsilon^2}{\rho^2} , \tag{4.25}$$

$$\frac{d_1}{a_1} = (h \cdot \lambda_A^1)^3 \geqq \frac{(1+\sqrt{2}\varepsilon)^9}{\rho^3(1-\sqrt{2}\varepsilon)^3} , \qquad 1/8 < \varepsilon^2 < 1/2. \tag{4.26}$$

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