SIMPLE FAMILIES OF LINES

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1. Introduction. Planar families of lines are studied by P. C. Hammer and the author in [2], and families of lines in the plane and in ordinary space by the author in [6]. Families of lines in vector spaces E_3 and E_n are mentioned in connection with convex bodies in [1]. The present paper gives a classification of simple types of families in $(n+1)$ dimensional real vector space E_{n+1}. Theorems are obtained on relations between the type of the family F, and the properties which F may possess, of containing exactly one line in every direction, and of simply or multiply covering the points of E_{n+1}.

2. Notation and definitions. With respect to an n dimensional vector subspace E_n of $(n+1)$ dimensional real vector space E_{n+1} a line L in E_{n+1} will be called *horizontal* if it is parallel to E_n. Any family F of non-horizontal lines in E_{n+1}, for which there is a hyperplane H parallel to E_n such that each point of H is covered exactly once by F, determines a single valued function $y=f(x)$ on H to any parallel hyperplane K: x, y are the points in which the line L of F which covers x intersects H, K. Corresponding to any basis in E_n, and choice of origins in H, K, the function $f(x)$ will be represented by real valued functions $y_i=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$. (For definiteness, let E_{n+1} be Euclidean, and choose the origins in H, K to be their points of intersection with the line through the common origin of E_{n+1}, E_n, which is orthogonal to E_n.)

A family F will be said to be *composed* of two lower dimensional *associated* families, F_p and F_{n-p}, if there is a choice of basis such that the n real functions have the form $y_i=f_i(x_1, \cdots, x_p)$, $i=1, \cdots, p$; $y_j=f_j(x_{p+1}, \cdots, x_n)$, $j=p+1, \cdots, n$. (The dimension of an associated family of course is one greater than the subscript; thus for example a three dimensional family may be composed of two associated two dimensional families.)

A family F is *primary* if it contains exactly one line in every non-horizontal direction, *representative* if it contains exactly one line in every direction. We say that a family F of lines is *simple* if every point of E_{n+1} is covered exactly once by the family; *outwardly simple* if every point exterior to some sphere S_n has the same property in relation to the family. If the distances from the origin of the lines of an outwardly simple family are bounded, then for a sufficiently large sphere S_n, if P, $g(P)$ are the points in which the line L of F covering P pierces S_n,

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the transformation g is an involutory mapping of S_n into itself which has no fixed point. By the theorem proved in [4], if g is continuous, such an outwardly simple family covers the interior of S_n and therefore covers all of E_{n+1}. Note the difference in the present usage of the term *outwardly simple*, and the usage in [2], [1] (where in order that F be called *outwardly simple* the additional requirements are made that F is representative, and that the corresponding involutory transformation g of S_n into itself is continuous and has no fixed point.)

3. **Stacks and sheafs.** In case the lines of F are all contained in the *p-sheaf* of all p-flats in E_{n+1} parallel to a fixed p dimensional vector subspace E_p, F will be called a *p-stack*, $1 \leqq p \leqq n$. If $p=1$, the 1-stack or 1-sheaf F is a simple sheaf of parallel lines in E_{n+1}. A p-stack F may be such that lines of the sub-family, for each of the parallel p-flats R_p, are contained in $(p-1)$-flats of a $(p-1)$-sheaf in R_p; such a family may be called a *p, (p−1)-stack*. A family F is a p, $(p-1)$, $\cdots$, *q-stack* if it divides successively into sub-families contained in parallel p-, $(p-1)$-, $\cdots$, q-, sheafs, where not all of the sub-families in the flats of lowest dimension q are stacks. Evidently a q-stack is a p, $\cdots$, q-stack for all p in $q < p \leqq n$. A k-stack, for any $k \leqq n$, cannot be a primary family, since the directions of its lines are confined to the directions contained in a k dimensional subspace E_k.

A family F of non-horizontal lines is an n, $\cdots$, $(n-p)$-stack if, with respect to some basis in E_n, the last $(p+1)$ equations for the family are of the form

$$y_n = x_n + u_n,\ y_{n-1} = x_{n-1} + u_{n-1}(x_n),\ \cdots,$$
$$y_{n-p} = x_{n-p} + u_{n-p}(x_n,\ \cdots,\ x_{n-p+1})\,.$$

This follows since $y_k = x_k + c_k$, c_k constant, is the equation of a k-sheaf in a $(k+1)$-flat, $k=1, \cdots, n$.

4. **Linear transformation corresponding to a pencil.** Choose a basis in E_{n+1} so that the equations of H, K are respectively $x_{n+1}=a$, $y_{n+1}=b$. Then points in H may be denoted by $(x\,; a)$, in K by $(y\,; b)$, and any point in E_{n+1} by $(z\,; z_{n+1})$, where x, y, z are in E_n.

We determine the transformation $y=f(x)$ which corresponds to the pencil of lines through a point $(w\,; w_{n+1})$, w in E_n, of E_{n+1}. Any non-horizontal line of the pencil has equations

$$\frac{z_1-w_1}{m_1} = \cdots = \frac{z_n-w_n}{m_n} = \frac{z_{n+1}-w_{n+1}}{m_{n+1}}\,,$$

where $(m_1, \cdots, m_{n+1})$ is a non-horizontal $(m_{n+1} \neq 0)$ unit vector of E_{n+1}.

(Let it be understood that if $m_k=0$, $1\leqq k\leqq n$, the presence of the ratio $(z_k-w_k)/0$, in this form of the equations for the line, means that $z_k=w_k$ is one of the equations.) The coordinates of the points of intersection x, y of this line with H, K therefore satisfy the equations

$$\frac{y_1-w_1}{m_1}=\cdots=\frac{y_n-w_n}{m_n}=\frac{b-w_{n+1}}{m_{n+1}},$$

$$\frac{x_1-w_1}{m_1}=\cdots=\frac{x_n-w_n}{m_n}=\frac{a-w_{n+1}}{m_{n+1}},$$

or

$$\frac{y_1-w_1}{x_1-w_1}=\cdots=\frac{y_n-w_n}{x_n-w_n}=\frac{b-w_{n+1}}{a-w_{n+1}},\ y_j-w_j=\frac{b-w_{n+1}}{a-w_{n+1}}(x_j-w_j),$$

$$j=1,\ \cdots,\ n.$$

Thus the transformation corresponding to the pencil, in vector or matrix form, is

$$(y-w)=cI(x-w),\ c=\frac{b-w_{n+1}}{a-w_{n+1}}, \tag{4.1}$$

where I is the identity matrix. Solving for w_{n+1} in terms of c, we obtain

$$w_{n+1}=\frac{ca-b}{c-1}=a-\frac{b-a}{c-1}.$$

5. Affine families. Equation (4.1) for a pencil suggests consideration of the families corresponding to any linear transformation $(y-w)=T(x-w)$, or to any affine transformation $y=Tx+u$, where T may be regarded as the matrix of the transformation, w, u, y, x as column matrices of the coordinates of the corresponding points or vectors in E_n. Let the family corresponding to $y=Tx+u$ be called an *affine* family. It is shown below that, in case T is singular, hyperplane K may be replaced by a parallel hyperplane such that the matrix T for the family F, referred to H and the new hyperplane, is non-singular. In our consideration of affine families, let it be assumed, if necessary, that such a new choice for K always is made.

Let M be a hyperplane parallel to H, K. Then for any non-horizontal line, if x, y, z are its points of intersection with H. K, M, we have

$$z-y=d(y-x),$$

or

$$z=(1+d)y-dx=[(1+d)T-dI]x,$$

for some real d uniquely determined by the position of M. Referred to H, M instead of to H, K, the family of lines is represented by matrix $[(1+d)T-dI]$ instead of by matrix T. The eigenvalues of $[(1+d)T-dI]$ are all of the form $\lambda-d/(1+d)$, where λ is an eigenvalue of T. Since T has only a finite number of different eigenvalues, d may be chosen so that the eigenvalues of $[(1+d)T-dI]$ are all different from zero. That is, in case T is singular, d may be chosen so that the new matrix $[(1+d)T-dI]$ is non-singular.

In the equation (4.1) for a pencil of lines, the multiplier c is never 1, since $a\neq b$. If $c=0$, the center of the pencil is in K; $c=\infty$ corresponds to the center being in H. Thus the eigenvalues of matrix cI are all real, equal to c, and different from 1.

If one or several eigenvalues of T are equal to 1, by suitable choice of basis, T may be put in the form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, where the eigenvalues of sub-matrix U are all different from 1, and V is superdiagonal with all diagonal elements equal to 1. (See [5].) Thus the corresponding family is composed of two associated families, one corresponding to a transformation U which has eigenvalues different from 1, the other being an $s, \cdots, 1$-stack, where s, the multiplicity of the eigenvalue 1 of T, is the dimension of V. The family F, by the last paragraph of § 3, accordingly is an $n, \cdots, (n-s+1)$-stack, and is not representative or primary. Consideration of stacks reduces to consideration of lower dimensional families which are not stacks. For affine families which are not stacks, the eigenvalues of T are different from 1.

To put the equation for an affine family in the form $(y-w)=T(x-w)$, we must have $u=-Tw+w$, or $(T-I)w=-u$. This is possible with $w=0$ if $u=0$, or for any u if $|T-I|\neq 0$. In the latter case, 1 is not an eigenvalue of T, and a unique solution for w exists for any u. This means that for any affine family, not a stack, the vertical line $x=y=w=-(T-I)^{-1}u$ is a central line of symmetry of the family, as in the case of a pencil.

In case of an affine family, not a stack, the eigenvalues of T are all different from zero and from one. If T further is such that its eigenvalues are all real, and corresponding eigenvectors span E_n, then if the eigenvectors are chosen as the basis, T has diagonal form, and evidently the corresponding family of lines is composed of associated lower dimensional pencils with centers on $x=y=w$, there being one associated pencil for each distinct eigenvalue t_j, and the heights of the centers are given by $w_{n+1,j}=(at_j-b)/(t_j-1)$. The dimension of the space of each associated pencil is one greater than the multiplicity of the corresponding eigenvalue t_j. Such a family will be called a *quasi-pencil*, with *centers* $\{(w;\, w_{n+1,j})\}$.

For example, in E_3 the family F given by the equations

$$y_1=t_1x_1,\ y_2=t_2x_2,\ t_1\neq t_2,\ t_i\neq 0,\ \neq 1,\ i=1,\ 2\,,$$

is a quasi-pencil, and may be described as the set of all lines of intersection of planes of the pencil of planes $y_1=t_1x_1$ with planes of the pencil $y_2=t_2x_2$. The lines $z_1=0$, $z_3=a+(b-a)/(1-t_1)$; $z_2=0$, $z_3=a+(b-a)/(1-t_2)$, are infinitely covered by F; all other points in the planes $z_3=a+(b-a)/(1-t_1)$, $z_3=a+(b-a)/(1-t_2)$, are not covered by F. Every other point of E_3 is covered exactly once by F. In order to make the quasi-pencil F cover all of space, it may be extended by addition of the horizontal 1-sheafs of lines of intersection of the pencils of planes with the horizontal planes $z_3=a+(b-a)/(1-t_1)$, $z_2=a+(b-a)/(1-t_2)$, but because of the infinite covering of the two skew horizontal lines, even the extended quasi-pencil is not outwardly simple.

In case T has a single real eigenvalue t_1, $\neq 0$, $\neq 1$, let the basis be chosen so that T assumes superdiagonal form. If it is impossible to choose the basis so that all elements above the diagonal vanish, let the corresponding family F be called a *skew pencil*. It may easily be shown that a skew pencil simply covers all points of E_{n+1} except points in the hyperplane $w_{n+1}=(at_1-b)/(t_1-1)$, and that in this hyperplane, all points outside the $(n-1)$ dimensional flat R of points $(w\,;w_{n+1})$ where $w_n=0$, cannot be covered. If $t_{12}, t_{23}, \cdots, t_{n-1,n}$ are all different from zero, then the $(n-1)$ dimensional flat R is covered by all lines of F through points $(x_1, x_2, \cdots, x_n;\ a)$ in H, where $x_2, \cdots, x_n$ are uniquely determined by $w_1, \cdots, w_{n-1}$, but x_1 is arbitrary; therefore in this case R is infinitely covered by F. Otherwise a smaller dimensional flat in the hyperplane $w_{n+1}=(at_1-b)/(t_1-1)$ is infinitely covered, and the rest of the hyperplane is not covered, by F.

For example, in E_3 the family F given by the equations $y_1=t_1x_1+t_{12}x_2$, $y_2=t_1x_2$, $t_1\neq 0$, $\neq 1$, $t_{12}\neq 0$, is a skew pencil. The lines of F for fixed x_2 are the lines of the pencil $(y_1-w_1)=t_1(x_1-w_1)$, where $w_1=t_{12}x_2/(1-t_1)$, which are in the plane $y_2=t_1x_2$. The coordinates of the center of the planar pencil, for each x_2, are $(w_1,\ 0\,;w_3)$, where $w_3=(t_1a-b)/(t_1-1)$. Thus F may be described as a union of planar pencils, one in each plane of a pencil of planes through the line $z_2=0$, $z_3=w_3$, the centers of the planar pencils being located on this line at $z_1=w_1=t_{12}x_2/(1-t_1)$. Accordingly the centers move out unboundedly as x_2 increases or decreases indefinitely. This skew pencil F simply covers all points of E_3, except that points of the line of centers in the plane $z_3=w_3$ are infinitely covered, and all other points of the plane are not covered, by F.

As shown in [5], in any case when the eigenvalues of T are all real, by a suitable choice of basis, T may be put in a diagonal block form, with blocks $D_1, \cdots, D_r$ on the diagonal, the dimension of each block D_j being equal to the multiplicity p_j of the corresponding real eigenvalue t_j; D_j is in superdiagonal form with t_j's on the diagonal. More specifically, D_j may decompose into a diagonal block t_jI of dimension $s_j < (p_j - 1)$, and a block D'_j which has only one eigenvector and cannot be made diagonal. The corresponding family F to such a T is composed of associated pencils and skew-pencils, one for each block t_jI, D'_j. In case at least one D_j cannot be made diagonal, F will be called a *skew quasi-pencil.*

5.1 THEOREM. *A quasi-pencil or skew quasi-pencil F is primary, and simply covers all of E_{n+1} except the set of horizontal hyperplanes $\{z_{n+1} = (at_j - b)/(t_j - 1)\}$, where t_j, $j = 1, \cdots, r$, are the distinct real eigenvalues of T.*

Proof. If F is to contain a line in the direction of a non-horizontal unit vector $(\lambda_1, \cdots, \lambda_{n+1})$, then there must exist x, y such that $(y - x) = (T - I)x = k(\lambda_1, \cdots, \lambda_n)$, where $k\lambda_{n+1} = (b - a)$. Since F is not a stack, we have that 1 is not an eigenvalue, $|T - I| \neq 0$, and there exists a unique solution for x. Since $y = Tx + u$ is single valued for each point $(x; a)$ in the hyperplane H, H is simply covered. Any other point $(z; z_{n+1})$ in E_{n+} will be covered if there exists an x such that

$$(z - x) = k(y - x), \ (z_{n+1} - a) = k(b - a) .$$

For this we must have

$$k(T - I)(x - w) + (x - w) = [kT - (k - 1)I](x - w) = (z - w) .$$

A unique solution for $(x - w)$ exists if $(k - 1)/k$ is not an eigenvalue of T. We have

$$\frac{k - 1}{k} = \frac{(z_{n+1} - a)/(b - a) - 1}{(z_{n+1} - a)/(b - a)} = \frac{z_{n+1} - b}{z_{n+1} - a} .$$

Comparing with (4.1), we see that a unique solution for x exists for all points $(z; z_{n+1})$ not in the horizontal hyperplanes containing the centers of the associated pencils and skew pencils.

6. Complex eigenvalues. In any odd dimensional space E_{n+1}, for an affine family F such that the eigenvalues of T are all complex, we have the following theorem.

6.1 THEOREM. *Any affine family F, in $(n+1)$ dimensional space E_{n+1}, n even, such that the transformation T has no real eigenvalue, is primary and simple. That is, F contains no horizontal line, contains exactly one line in every non-horizontal direction, no pair of lines of F*

intersect, and each point of E_{n+1} *is covered by exactly one line of* F.

Proof. In the proof of Theorem 5.1, under the present hypotheses, the determinant $|kT-(k-1)I|$ vanishes for no $k\neq 0$, so there is a unique line which covers each point $(z;z_{n+1})$ not in H. Each point $(x;a)$ in H also is uniquely covered since $y=T(x-w)+w$ is single valued. As in the proof of Theorem 5.1, since the determinant $|T-I|$ is not zero, we conclude that there is exactly one line of F in every non-horizontal direction.

In any even dimensional space E_{n+1}, for an affine family F, not a stack, such that T has only one real eigenvalue, we have the following theorem.

6.2 THEOREM. *Any affine family* F, *not a stack, in* $(n+1)$ *dimensional space* E_{n+1}, n *odd, such that the transformation* T *has only one real eigenvalue* t_1, *is primary, and simply covers all of* E_{n+1} *except the hyperplane*

$$z_{n+1}=w_{n+1}=\frac{at_1-b}{t_1-1}\,.$$

An $(n-1)$ *dimensional flat* R *in the hyperplane* $z_{n+1}=w_{n+1}$ *is infinitely covered, and the rest of the hyperplane is not covered, by* F. *The family* F *is composed of an associated planar pencil, and of an associated simple family as in Theorem* 6.1, *of dimension* n.

Proof. Since F is not a stack, $|T-I|\neq 0$, and as in the proof of Theorem 5.1, we conclude that F is primary. For any point $(z;z_{n+1})$ with $z_{n+1}\neq w_{n+1}$, so that $k\neq 1/(1-t_1)$, the determinant $|kT-(k-1)I|$ does not vanish, so there is a unique line of F which covers $(z;z_{n+1})$. Let an eigenvector τ_1 corresponding to t_1 be chosen as the first vector of a basis. Then as shown in [5], the remaining basis vectors may be chosen so that T assumes the form $\begin{pmatrix} t_1 & 0 \\ 0 & V \end{pmatrix}$, where V has only complex eigenvalues. For $k=1/(1-t_1)$, the matrix $[kT-(k-1)I]$ has all zeros in it first column and first row. Accordingly $[kT-(k-1)I](x-w)=(z-w)$ has a solution only for vectors $(z-w)$ with $z_1=w_1$; for such vectors the solution for $(x_2-w_2), \cdots, (x_n-w_n)$ is unique, but (x_1-w_1) is arbitrary. Thus for each point x on the line $-\infty<x_1<\infty$, $x_2=w_2, \cdots, x_n=w_n$ in H, there is a line of F through x which covers the point $(w_1, z_2, \cdots, z_n; w_{n+1})$ of the hyperplane $z_{n+1}=w_{n+1}$. Therefore the $(n-1)$ dimensional flat R defined by $z_1=w_1$ in the hyperplane is infinitely covered, and the rest of the hyperplane is not covered at all, by F.

It has been seen that the equation for any affine family, not a stack, can be put in the form $(y-w)=T(x-w)$. The origin in E_{n+1} may be translated by a vector $(w; 0)$. With respect to the new origin, the family has equation $y=Tx$. Thus the most general affine family, $y=Tx+u$, which is not a stack, may be obtained simply by translation of the family having equation $y=Tx$. Accordingly in the remainder of this section and in the next, we take the equation for F in the homogeneous form $y=Tx$.

In case T has several real eigenvalues different than 1, and at least one pair of conjugate complex eigenvalues, then the basis may be chosen so that T has block diagonal form, with a block D_i on the diagonal for each real eigenvalue $t_i \neq 0$, and a block Q_j for each pair of conjugate complex eigenvalues $(x_j \pm iy_j)$. (See [5].) The real blocks D_i have already been described in § 5. Each complex block Q_j is of dimension $2s_j$, where s_j is the multiplicity of $(x_j \pm iy_j)$, and has s_j two dimensional blocks $\begin{pmatrix} x_j & y_j \\ -y_j & x_j \end{pmatrix}$ on its diagonal, elements of Q_j below the diagonal being zeros. The family F corresponding to T therefore is composed of associated pencils, skew pencils, and simple families as in Theorem 6.1.

In summary, the family F corresponding to a matrix T is a pencil if and only if the eigenvalues of T are all real and equal; if T has no real eigenvalue, F is primary and simple; in any other case F is primary, and simply covers all of E_{n+1} except points in the set of hyperplanes

$$\{z_{n+1}=w_{n+1,j}=(at_j-b)/(t_j-1)\}, j=1, \cdots, p,$$

where $t_1, \cdots, t_p$ are the distinct real eigenvalues of T. If the associated family of dimension (p_j+1), where p_j is the multiplicity of t_j, infinitely covers a flat of dimension (p_j-1-q_j), then in the hyperplane $z_{n+1}=w_{n+1,j}$, a flat of dimension $(n-1-q_j)$ is infinitely covered by F; the remainder of each hyperplane is not covered by F.

7. Composition of general associated families. Any family F in E_{n+1} which is the composite of associated general families (families not necessarily corresponding to a linear transformation T), F_p and F_{n-p}, in E_{p+1} and E_{n-p+1}, is primary if F_p is primary in E_{p+1} and F_{n-p} is primary in E_{n-p+1}. For by hypothesis there exists a unique $(x_1, \cdots, x_p)$ such that $(y_i-x_i)=k\lambda_i$, $i=1, \cdots, p$, and a unique $(x_{p+1}, \cdots, x_n)$ such that $(y_j-x_j)=k\lambda_j$, $j=p+1, \cdots, n$, where $k\lambda_{n+1}=(b-a)$, for any non-horizontal direction $(\lambda_1, \cdots, \lambda_{n+1})$. If further both F_p and F_{n-p} are covering and simple (like the family of Theorem 6.1), then the composite family F is covering and simple. For by hypothesis there exists a unique

$$(x_1, \cdots, x_p)$$

such that

$$(z_i - x_i) = k(y_i - x_i), \qquad i = 1, \cdots, p,$$

and a unique

$$(x_{p+1}, \cdots, x_n)$$

such that

$$(z_j - x_j) = k(y_j - x_j), \qquad j = p+1, \cdots, n,$$

where

$$k(b-a) = (z_{n+1} - a),$$

for any point $(z; z_{n+1})$ of E_{n+1}.

If however some point $(z_1, \cdots, z_p; z_{n+1})$ of E_{p+1} is multiply covered by F_p, and if F_{n-p} is covering, then since F_{n-p} covers all points $(z_{p+1}, \cdots, z_n; z_{n+1})$ where $z_{p+1}, \cdots, z_n$ are arbitary, the composite family F multiply covers all points $(z_1, \cdots z_p, z_{p+1}, \cdots, z_n; z_{n+1})$ of an $(n-p)$ dimensional flat. If F_p does not cover some point $(z_1, \cdots, z_p; z_{n+1})$, then similarly there is an $(n-p)$ dimensional flat in E_{n+1} which is not covered by F. Therefore no family F other than a pencil, which is composed of associated families which are not simple, can be outwardly simple; any outwardly simple family which is composite must be either a pencil or simple. (For completion of the justification of this statement, see the following paragraph.)

Given two representative, outwardly simple families F_p, F_{n-p}, we may compose the primary sub-families (of all non-horizontal lines of F_p, F_{n-p}), to obtain a family F which does not cover $(n-p)$ flats consisting of all points of the form $(z_1, \cdots z_p, z_{p+1}, \cdots z_n; z_{n+1})$, $(z_{p+1}, \cdots, z_n)$ arbitrary, where $(z_1, \cdots, z_p; z_{n+1})$ is a point of E_{p+1} which is covered only by an omitted horizontal line of F_p, and p flats consisting of all points of the form $(z_1, \cdots z_p, z_{p+1}, \cdots z_n; z_{n+1})$, $(z_1, \cdots, z_p)$ arbitrary, where $(z_{p+1}, \cdots, z_n; z_{n+1})$ is a point of E_{n-p+1} which is covered only by an omitted horizontal line of F_{n-p}. In case there is a one-to-one correspondence of uncovered $(n-p)$ flats and uncovered p flats, such that each corresponding pair of flats have the same values of z_{n+1}, then each such pair of corresponding flats together span a hyperplane in E_{n+1}. If n-dimensional covering line families are added in each of the hyperplanes, then the extended family F covers all of E_{n+1}. If the number of such hyperplanes is finite or denumerable, it may be possible to choose such covering horizontal families in the hyperplanes that the covering extended family F is representative. (See [6].) The extended family F can be outwardly simple, however, only in case there is just one hyperplane and the associated families F_p, F_{n-p} are pencils with common w_{n+1}, in which case F necessarily is a pencil.

8. Generalization to Banach spaces. Some of the results of the preceding sections may be carried over to Banach spaces. If $f(x)$ is any non-vanishing bounded linear functional on a Banach space B, then

$$H=[x \in B \mid f(x)=a] \text{ and } K=[y \in B \mid f(y)=b]$$

are hyperplanes which are parallel to the closed linear subspace $E=[x \in B \mid f(x)=0]$. The space B may be the Cartesian product of any Banach space E and the real number line; for such a product a bounded linear functional f always exists having E for its null subspace.

There is an α in B such that $f(\alpha)=\|\alpha\|=1$. If P is any point of B, we have $P=f(P)\cdot\alpha+[P-f(P)\cdot\alpha]$; $[P-f(P)\cdot\alpha]$ is in the null subspace E of f. If also $P=z_f\cdot\alpha+z$, with z in E, we have $f(P)=z_f$, $0=P-f(P)\cdot\alpha-z$, or $z=P-f(P)\cdot\alpha$. Thus with respect to any fixed "vertical" vector α, any point P in B has unique coordinates $(z\,;\,z_f)$. A direction $(v\,;\,v_f)$ is "horizontal" if $f(v\,;\,v_f)=v_f=0$.

As in the finite dimensional case, the equation for any pencil of lines in B is $(y-w)=cI(x-w)$, where I is the identity transformation in E and $c\neq 1$. To show this, let the origin of B be translated from $(0;\,0)$ to $(w;\,0)$. Then the translated family of lines has equation $y=cIx$. Define

$$w_f=a-\frac{b-a}{c-1}\,.$$

Points z on the line through $(x;\,a)$ and $(y;\,b)$, where $y=cIx$, are given by $e(x;\,a)+(1-e)(y;\,b)$. There is a unique e such that $ea+(1-e)b=w_f$, namely

$$e=\frac{w_f-b}{a-b}\,,$$

and

$$ex+(1-e)y=[e+(1-e)c]x=0x=0\,.$$

Therefore all lines of the family pass through the point $(0\,;\,w_f)$. Conversely for any non-horizontal direction $(v\,;\,v_f)$, there exist a unique x and $y=cx$ in E such that

$$(y-x)=(c-1)x=kv,\ (b-a)=kv_f\,;$$

thus the family contains one line through $(0\,;\,w_f)$ in every non-horizontal direction, and is made into a pencil, with center $(0\,;\,w_f)$, by addition of all horizontal lines through $(0\,;\,w_f)$.

If the affine family $y=Tx+u$, where T is a not necessarly bounded linear transformation, is not a stack, then 1 must belong either to the resolvent set, or to the continuous or residual spectrum of T. (See [3, p. 31].) In case u is in the domain of $(T-I)^{-1}$, the corresponding family may be translated so that the equation becomes $y=Tx$. Replacement of reference hyperplane

$$K=[(z;\ z_f)\in B\,|\,z_f=b]$$

by

$$K'=[(z\,;\ z_f)\in B\,|\,z_f=b+h(b-a)]$$

does not change the family of lines, but induces replacement of T by $T'=(1+h)T-hI$. Thus an eigenvalue λ' of T' corresponds to an eigenvalue

$$\lambda=\frac{h+\lambda'}{h+1}$$

of T; in particular if 0 is an eigenvalue of T, it may be replaced by any desired value λ' except 1 by taking $h=-\lambda'$. The choice $h=-1$ is impossible since K' then would coincide with H; an eigenvalue 1 of T is preserved under this transformation.

The affine family $y=Tx$, where T is a not necessarily bounded linear transformation, if not a stack, will contain exactly one line in every non-horizontal direction $(v\,;\ v_f)$ where v is in the domain of $(T-I)^{-1}$. This follows since the system $(y-z)=(T-I)x=kv$, where $kv_f=(b-a)$, then has a unique solution for x. If U is a bounded, one-to-one linear transformation on all of E to all of E, then by a theorem of Banach, U is an isomorphism, so the affine family F corresponding to $T=U+I$ is primary; more generally F is primary for any bounded or unbounded U which is linear and one-to-one on E to all of E. If the domain D of U is not all of E, then $T=U+I$ also will be defined only on D, so that F is primary but covers only the proper subset $(D\,;\ a)$ of hyperplane H.

The affine family will simply cover a point $(z\,;\ z_f)$ in B if the system $(z-x)=k(y-x)$, or $z=[kT+(1-k)I]x$, where $k(b-a)=(z_f-a)$, has a unique solution for x. This will be the case for all $(z;\ z_f)$ in every hyperplane $z=z_f$ such that $(1-k)/k=-(b-z_f)/(a-z_f)$ is not in the point spectrum of T, and such that z is in the domain of $[kT+(1-k)I]^{-1}$.

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