ON EMBEDDING UNIFORM AND TOPOLOGICAL SPACES

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In this note we prove the following.

THEOREM. *Every space with separated uniform structure can be embedded as a closed subset of a separated convex linear space.*

Every metric space can be isometrically embedded as a closed subset of a normed linear space.

These statements follow at once from the theorem of § 3. Such an embedding is known for any *complete* metric space; and it is also known that any metric space is isometric which a relatively closed subset of a convex subset of a Banach space.

We also describe an embedding of an arbitrary T_1 space as a closed subset of a special homogeneous space.

1. Preliminaries.

(A) A *semi-metric* on a set X is a real non-negative function ρ on $X \times X$ such that $\rho(x, x)=0$, $\rho(x, y)=\rho(y, x)$, and $\rho(x, y) \leqq \rho(x, z)+\rho(z, y)$ for all $x, y, z \in X$. A semi-metric is a metric if and only if $\rho(x, y)=0$ implies $x=y$.

A collection of semi-metrics $(\rho_\alpha)_{\alpha \in A}$ on X indexed by a set A defines a uniform structure (and a topology) on X, generated by sets $U_{a\alpha}=\{(x, y): \rho_\alpha(x, y)<a\}$, where $a>0$ and $\alpha \in A$. Conversely, every uniform structure can be defined by a family of semi-metrics; see Bourbaki [1]. We will say that the uniform structure is *separated* if for every pair $x, y \in X$ there is a ρ_α such that $\rho_\alpha(x, y) \neq 0$.

(B) If X is a real linear space, a *semi-norm* on X is a real non-negative function s on X such that $s(\lambda x)=|\lambda| s(x)$ and $s(x+y) \leqq s(x)+s(y)$ for all $x, y \in X$ and for all real numbers λ. A semi-norm is a norm if and only if $s(x)=0$ implies $x=0$.

A collection of semi-norms $(s_\alpha)_{\alpha \in A}$ on X indexed by a set A defines a (locally) convex topology (and a uniform structure) compatible with the algebraic operations in X. Conversely, every convex topology can be described by a family of semi-norms; see Bourbaki [2]. We will say that the convex topology is *separated* if for every $x \neq 0$ in X there is an s_α such that $s_\alpha(x) \neq 0$.

(C) REMARK. Let X and X' be two sets with uniform structures

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given by the semi-metrics $(\rho_\alpha)_{\alpha\in A}$ and $(\rho'_\alpha)_{\alpha\in A}$ indexed by the same set A. If ϕ: $X\to X'$ is a one-one correspondence such that for all $\alpha\in A$ we have $\rho_\alpha(x, y)=\rho'_\alpha(\phi(x), \phi(y))$, then ϕ preserves the uniform structure and topology.

2. The space of molecules.

(A) A *molecule* of a set X is a real-valued function m on X which is zero except (perhaps) at a finite number of points $x_1, \cdots, x_k$ of X and which satisfies $\sum_{i=1}^{k} m(x_i)=0$. Setting $\lambda_i=m(x_i)$, we will represent m as a linear combination $m=\sum_i \lambda_i x_i$ with $\sum_i \lambda_i=0$. The totality of molecules forms a real linear space $M(X)$ with algebraic operations defined pointwise.

(B) Suppose that X has a uniform structure defined by the semi-metrics $(\rho_\alpha)_{\alpha\in A}$. Then for each $\alpha\in A$ we define the semi-norms s_α on $M(X)$ by

$$(1) \qquad s_\alpha(m)=\inf\{\sum_j |\mu_j|\,\rho_\alpha(y_j, z_j)\}\ ,$$

the infimum being taken over all representations of $m=\sum_i \lambda_i x_i$ as $m=\sum_j \mu_j(y_j-z_j)$; the condition $\sum_i \lambda_i=0$ insures that such representations of m do exist.

It follows from the definition (1) that for all $x, y\in X$ and for any $\alpha\in A$,

$$(2) \qquad s_\alpha(x-y)\leqq \rho_\alpha(x, y)\ .$$

In fact, it is easily seen that s_α is the largest semi-norm on $M(X)$ satisfying (2); that is, given any such semi-norm s, we have $s(m)\leqq s_\alpha(m)$ for all $m\in M(X)$.

(C) Let us fix a "base point" $x_0\in X$. We then note that the set of all elements of the form $x-x_0$ with $x_0\neq x\in X$ forms a *base* for the linear space $M(X)$. Also, any linear functional F on $M(X)$ defines a real function f on X by

$$(3) \qquad f(x)=F(x-x_0)\ ;$$

conversely, any real function f on X such that $f(x_0)=0$ defines a linear functional F on $M(X)$ by $F(m)=\sum_i \lambda_i f(x_i)$, and the relation (3) holds.

With that identification of functionals, we have the following.

PROPOSITION. *The linear functionals F on $M(X)$ which are continuous in the topology of the semi-norms $(s_\alpha)_{\alpha\in A}$ correspond to those real* *functions f on X vanishing at x_0 and satisfying*

$$|f(x)-f(y)|\leqq K\rho_\alpha(x, y) \tag{4}$$

for some constant K and semi-norm ρ_α, both depending on f. If X is a metric space then the continuous linear functionals correspond to the Lipschitz functions on X vanishing at x_0.

Note that the functions f are uniformly continuous on X.

Proof. The functional F is continuous on $M(X)$ if and only if it is bounded (by some K) for some semi-norm s_α; thus if F is continuous and defines f as in (3), $|f(x)-f(y)|=|F(x-y)|\leqq Ks_\alpha(x-y)\leqq K\rho_\alpha(x, y)$ by (2). Conversely, if f is a function such that $f(x_0)=0$ and which satisfies (4) for some K and ρ_α, then for any $m\in M(X)$ and $\varepsilon>0$ we can choose a representation $m=\sum_j \mu_j(y_j-z_j)$ such that $\sum_j |\mu_j|\rho_\alpha(y_j, z_j)\leqq s_\alpha(m)+\varepsilon$. Then

$$|F(m)|\leqq\sum_j|\mu_j|\,|f(y_j)-f(z_j)|\leqq K\sum_j|\mu_j|\,\rho_\alpha(y_j, z_j)\leqq K[s_\alpha(m)+\varepsilon]\,.$$

Since that is true for all $\varepsilon>0$, we have $F(m)\leqq Ks_\alpha(m)$, whence F is continuous on $M(X)$.

Relation (2) is, in fact, always an equality:

PROPOSITION. *For any $x, y\in X$* and *$\alpha\in A$ we have*

$$s_\alpha(x-y)=\rho_\alpha(x, y)\,. \tag{5}$$

Proof. The function $f(z)=\rho_\alpha(z, y)$ clearly satisfies $f(y)=0$ and also (4) with $K=1$; let F be the corresponding linear functional with x_0 replaced by y. Given any representation of the molecule $x-y=\sum_j\mu_j(y_j-z_j)$, we have $\rho_\alpha(x, y)=f(x)=F(x-y)=\sum_j \mu_j F(y_j-z_j)$, whence $\rho_\alpha(x, y)\leqq \sum_j|\mu_j|\,|\rho_\alpha(y_j, y)-\rho_\alpha(z_j, y)|\leqq\sum_j|\mu_j|\,\rho_\alpha(y_j, z_j)$. Taking the infimum over all such representations of $x-y$, we have $\rho_\alpha(x, y)\leqq s_\alpha(x-y)$, proving (5).

The following two statements (and their converses) are easy consequences of (5).

PROPOSITION. *If the uniform structure on X is separated, then so is the induced convex topology on $M(X)$.*

If the uniform structure on X is given by a single metric (or is metrizable), then the induced convex topology on $M(X)$ is normed (is normable).

(D) REMARK. There are many interesting variants of the semi-norms (1). For instance, suppose we let $\tilde{M}(X)$ denote the linear space of all $m=\sum_i\lambda_i x_i$, with no additional conditions on the λ_i; then by choosing a base point $x_0 \in X$ we can define the semi-norm $\tilde{s}_\alpha$ corresponding to the semi-metric ρ_α by

$$(6)\qquad \tilde{s}_\alpha(m)=\inf\{\sum_k |\nu_k|\,\rho_\alpha(w_k, x_0)+\sum_j |\mu_j|\,\rho_\alpha(y_j, z_j)\},$$

the infimum being taken over all representations of m as a sum $m=m_1+m_2$, where $m_1=\sum_k \nu_k w_k$ and $m_2=\sum_j \mu_j(y_j-z_j)$. It can be shown that for all $\alpha \in A$ the semi-norm (6) is equal to the semi-norm (1) on the subspace $M(X)$ of $\tilde{M}(X)$.

Semi-norms related to those of type (6) have been studied (in quite a different connection) by H. Whitney; see [4, p. 249].

3. Embedding a uniform space. Take a base point $x_0 \in X$, and then define the transformation ϕ: $X \to M(X)$ by $\phi(x)=x-x_0$. Then ϕ is clearly one-one, and by (5) we have $s_\alpha(\phi(x))=\rho_\alpha(x, x_0)$.

THEOREM. *The transformation ϕ is a uniformly bi-continuous homeomorphism of X into $M(X)$. If the uniform structure of X if separated, then ϕ* maps *X onto a closed subset of $M(X)$.*

If X is a metric space, then ϕ is an isometric map of X onto a closed subset of $M(X)$.

Proof. As we have remarked in § 1 C, such a ϕ is a uniformly continuous homeomorphism and an isometry if X is metric.

Supposing that the uniform structure of X is separated, we will now show that $\phi(X)$ is closed in $M(X)$. Given $m \in M(X)$ not belonging to $\phi(X)$, we will construct a neighborhood of m not meeting $\phi(X)$. Suppose first of all that m has the form $\lambda(y-x)$; since $m \notin \phi(X)$, we have $y \neq z$, $\lambda \neq 0$.

In case $z \neq x_0$, there is a semi-metric ρ and a constant $a>0$ such that $\rho(y, z) \geqq a$, $\rho(x_0, z) \geqq a$; in fact, ρ can be defined as the sum of two suitably chosen semi-metrics of the separating family $(\rho_\alpha)_{\alpha \in A}$. Let s_ρ be the semi-norm defined by (1) using ρ. Set $f(x)=\max\{a-\rho(x, z), 0\}$, and let F be the corresponding continuous linear functional as in § 2 C; we note that $|F(n)| \leqq s_\rho(n)$ for all $n \in M(X)$. Then for any $m_0=x-x_0$ in $\phi(X)$, we have

$$F(m_0-m)=f(x)-f(x_0)-|\lambda|f(y)+|\lambda|f(z)=f(x)+|\lambda|\,a,$$

whence $s_\rho(m_0-m) \geqq |\lambda|\,a$.

In case $z=x_0$, we have $\lambda \neq 1$ since $m \notin \phi(X)$. As before, take a semi-metric ρ such that $\rho(y, x_0)>2a$. Then for any $m_0=x-x_0$ in $\phi(X)$, either $\rho(x, x_0)>a$ or $\rho(x, y)>a$. In the former event define $f(z)=\max\{a-\rho(z, x_0), 0\}$; then $s_\rho(m_0-m) \geqq |F(m_0-m)|=||\lambda|-1|a$. In the latter event define $f(z)=\max\{a-\rho(z, y), 0\}$; then $s_\rho(m_0-m)\geqq|\lambda|$.

Thus in any case $s_\rho(m_0-m)$ exceeds some positive constant independent of m_0; thus if $m=\lambda(y-z)\notin\phi(X)$, then m has a neighborhood not meeting $\phi(X)$. In general, let $m=\sum_{i=1}^{k}\lambda_i x_i$ with $k>2$; we can suppose that the x_i are distinct and that $|\lambda_i|\geqq b>0$ for all i. As usual, take a semi-norm ρ on X such that $\rho(x_i, x_j)\geqq 2c$ for some $c>0$ and for all pairs i, j with $i\neq j$. Now suppose $m'=\sum_j \lambda'_j x'_j$ is a molecule with less than k points. Then there is an i such that $\rho(x'_j, x_i)\geqq c$ for all j. Let $f(x)=\max\{c-\rho(x, x_i), 0\}$. Then $s_\rho(m-m')\geqq|F(m)-F(m')|=|F(m)|\geqq bc$. Thus if m' satisfies $s_\rho(m-m')<bc$, then m' has at least as many points as m. Since every element of $\phi(X)$ has the form $x-x_0$, it follows that we can construct a neighborhood of $m=\sum_{i=1}^{k}\lambda_i x_i$ which does not intersect $\phi(X)$. The proof of the theorem is now complete.

4. Embedding topological spaces.

(A) M. Shimrat [3] has shown that every topological space X can be embedded in a homogeneous space X^* (a space X^* is *homogeneous* if for every two points $x, y\in X^*$ there is a homeomorphism h of X^* into itself such that $h(x)=y$); furthermore, if X is T_1, then so is X^* and the image of X is closed in X^*. In the following theorem we shall show that any T_1 space X can be embedded as a closed subset of a T_1 space X^* such that for any two points $x, y\in X^*$ there is a homeomorphism of period two interchanging the points.

However, Shimrat manages to prove that if X has stronger separation properties (for example, X is Hausdorff, regular, normal), then X^* has these same properties. No such conclusion can be drawn for our X^*. Shimrat also produces a variant construction embedding a metric space X as a closed set in a metrically homogeneous space X^*; his X^* (as he points out) is not necessarily locally connected, whereas our embedding space $X^*=M(X)$ in § 3 is (being a normed linear space).

(B) For any set X let X^* denote the Boolean ring of all finite subsets m of X; the void set is denoted by 0, and $m+n$ is the symmetric difference of m and n (whence $\{x\}+\{x\}=0$).

We have a natural one-one transformation $\phi: X\to X^*$ defined by $\phi(x)=\{x\}$.

THEOREM. *Let X be a T_1 space. Then we can define a topology on X^* for which the additive translations are homeomorphism, and ϕ maps X homeomorphically onto a closed subset of X^*.*

We do not assert that X^* is a topological group under addition. We will show that the transformation $X^* \times X^* \rightarrow X^*$ defined by $(m, n) \rightarrow m+n$ is continuous *in each variable separately*, not that it is simultaneously continuous.

Proof. For every open cover $\mathscr{V}$ of X we define $(\mathscr{V})$ as the collection of those sets $m \in X^*$ whose points can be listed $x_1, x_2, \cdots, x_{2k-1}, x_{2k}$, where the "partners" x_{2j-1}, x_{2j} always lie in one element $V_j \in \mathscr{V}$. Then $0 \in (\mathscr{V})$, and if $\mathscr{U}$ is a common refinement of the open covers $\mathscr{V}$, $\mathscr{W}$, we have $(\mathscr{U}) \subset (\mathscr{V}) \cap (\mathscr{W})$.

We take the sets $m+(\mathscr{V})$ as a fundamental system of neighborhoods of $m \in X^*$, and will show that for any open cover $\mathscr{V}$ and any $m \in (\mathscr{V})$ there is an open cover $\mathscr{U}$ such that $n \in (\mathscr{U})$ implies $m+n \in (\mathscr{V})$. It will follow

1) that these neighborhoods define a unique topology on X^*, and
2) that translation by m is a homeomorphism.

We construct $\mathscr{U}$ as follows: For each $V \in \mathscr{V}$, let V_0 denote the set of points of V not in m; for each $x_i \in m \cap V$ such that its partner is also in V, we define $U_i = V_0 \cup \{x_i\}$. Thus each U_i is defined by removing a finite number of points from V, and since points of X are closed, it follows that U_i is open. We define the open cover $\mathscr{U}$ of X as the collection of all possible such U_i constructed from all $V \in \mathscr{V}$.

Now take any $n = \{y_1, y_2, \cdots, y_{2p-1}, y_{2p}\} \in (\mathscr{U})$, where y_{2j-1}, y_{2j} always belong to some $U \in (\mathscr{U})$; let us suppose all the y's are distinct. We will arrange $m+n$ into a set of partners which share elements of $\mathscr{V}$, thus showing that $m+n \in (\mathscr{V})$. If $y_{2j-1}, y_{2j} \in U_i \in \mathscr{U}$, then at most one of them belongs to m, and that one (if any) must be x_i; we then pair the other y with the partner of x_i, forming a pair not appearing in $m+n$. If neither y belongs to m, we can make them partners of each other. Elements of m not affected by these transactions shall remain partners. That completes the arrangement of $m+n$.

To show that this topology on X^* is itself T_1, take any $m \neq 0$, and let $\mathscr{V}$ be the set of complements of the sets $m+\{x\}$, where x varies over m. Then $\mathscr{V}$ is an open cover of X, and $(\mathscr{V})$ is a neighborhood of 0 in X^* which does not contain m.

We will now prove that the map $\phi(x) = \{x\}$ is a homeomorphism of X onto X^*. Given $x \in X$ and a neighborhood $(\mathscr{V})$ of $\{x\}$, we know there is an open set V such that $x \in V \in \mathscr{V}$; then for any $y \in V$ we have $\{y\} = \{x\} + (\mathscr{V})$, proving that ϕ is continuous. On the other hand, given $\{x\} \in X^*$ and a neighborhood V of x, take the open cover $\mathscr{V} = \{V, X + \{x\}\}$. Then for any $\{y\} \in \{x\} + (\mathscr{V})$, we have $y \in V$ since $x, y \in X + \{x\}$ is impossible; that is, the mapping ϕ^{-1} is continuous.

Finally we will show that $\phi(X)$ is closed in X^*. Take any m with more than one element, and as above let $\mathscr{V}$ be the set of complements of the sets $m+\{x\}$, where x varies over the elements of m. Then $m+(\mathscr{V})$ does not intersect $\phi(X)$, for if $\{x\}+m\in(\mathscr{V})$, then x has a partner y in m; that is impossible, for no two elements of m lie in the same $V\in\mathscr{V}$. The proof of the theorem is now complete.

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