NOTE ON A THEOREM OF HADWIGER

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Throughout this paper, H denotes a Hilbert space over the real or complex numbers and (x, y) denotes the inner product of the vectors x, y of H. The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if $\{u_\alpha\}$ is an orthonormal basis of H, then the Parseval relation

$$(1) \qquad (x, y)=\Sigma(x, u_\alpha)(u_\alpha, y)$$

is valid for each pair of vectors x, y of H. It is easy to see that (1) is also valid if $\{u_\alpha\}$ is the projection of an orthonormal basis $\{w_\alpha\}$ and if we restrict x and y to the range of the projection. Indeed, if E is the projection, so that $w_\alpha E=u_\alpha$ for each α, then

$$(x, y)=\Sigma(x, w_\alpha)(w_\alpha, y)=\Sigma(xE, w_\alpha)(w_\alpha, yE)=\Sigma(x, w_\alpha E)(w_\alpha E, y)$$
$$=\Sigma(x, u_\alpha)(u_\alpha, y) .$$

The theorem referred to in the title deals with this result and also with the converse question:

THEOREM 1. *If the Parseval relation* (1) *is valid for each pair of vectors x and y of H, then the set $\{u_\alpha\}$ is the projection of an orthonormal basis of a superspace K of H.*

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

Proof of Theorem 1. We choose as K coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set $\{u_\alpha\}$. We see from (1), with $x=u_\beta$, $y=u_\gamma$, that the matrix $U=((u_\alpha, u_\beta))$ is idempotent. Since U is also Hermitian, it may be interpreted as a projection acting on K. We now imbed H in K by making correspond to x in H the (row) coordinate vector $x'=\{(x, u_\alpha)\}$ in K. In particular, to the vector u_β there corresponds the βth row of U which is manifestly the image, under the projection U, of the βth coordinate basis vector. Finally, if $x'=\{(x, u_\alpha)\}$ and $y'=\{(y, u_\alpha)\}$, then $(x', y')=\Sigma(x, u_\alpha)\overline{(y, u_\alpha)}=\Sigma(x, u_\alpha)(u_\alpha, y)$ $=(x, y)$; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

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THEOREM 2. *If the Parseval relation* (1) *is valid relative to the set* $\{u_\alpha\}$ *of* H, *and, if no* u_α *is in the closed subspace spanned by the others, then* $\{u_\alpha\}$ *is an orthonormal basis.*

Proof. The second assumption implies the existence of a dual set $\{v_\alpha\}$ in H such that $(u_\alpha, v_\beta)=\delta_{\alpha\beta}$ [see 3, p. 264]. Then, using (1), we get $\delta_{\alpha\beta}=(u_\alpha, v_\beta)=\Sigma_\gamma(u_\alpha, u_\gamma)(u_\gamma, v_\beta)=\Sigma_\gamma(u_\alpha, u_\gamma)\delta_{\gamma\beta}=(u_\alpha, u_\beta)$.

We remark at this point that the methods of proof of Theorems 1 and 2 can be used to give proofs of the corresponding results about projections of biorthonormal bases of vectors $\{u_\alpha; v_\alpha\}$ for which $(u_\alpha, v_\beta)=\delta_{\alpha\beta}$. These methods are also used in our next proof [see 2, (b)].

THEOREM 3. *A necessary and sufficient condition that a set of vectors* $\{u_\alpha\}$ *of* H *be the projection of an orthonormal set* (*not necessarily a basis*) *in some superspace* K *is that, for each* x *in* H,

$$\Sigma|(x, u_\alpha)|^2 \leqq (x, x) . \tag{2}$$

Proof. By the remarks preceding Theorem 1, the necessity is clear. In proving sufficiency, we may suppose $\{u_\alpha\}$ is complete in H, since, otherwise, by adding to $\{u_\alpha\}$ an orthonormal basis of the orthogonal complement of $\{u_\alpha\}$ in H, we get a larger set which is complete, and for which the condition (2) is still valid. Next we show that, if U is the matrix $((u_\alpha, u_\beta))$, then $0\leqq U\leqq 1$, in the sense that both U and $1-U$ are nonnegative [4, p. 213]. Let ξ_α be any set of scalars of which all but a finite number are zero. Then, using Schwarz' inequality and (2), we get

$$\begin{aligned} 0 &\leqq (\Sigma_\alpha \xi_\alpha u_\alpha, \Sigma_\beta \xi_\beta u_\beta)=\Sigma_{\alpha,\beta}\xi_\alpha\bar{\xi}_\beta(u_\alpha, u_\beta) \\ &\leqq (\Sigma_\beta|\xi_\beta|^2)^{1/2}(\Sigma_\beta|(\Sigma_\alpha\xi_\alpha u_\alpha, u_\beta)|^2)^{1/2} \leqq (\Sigma_\beta|\xi_\beta|^2)^{1/2}(\Sigma_\alpha\xi_\alpha u_\alpha, \Sigma_\gamma\xi_\gamma u_\gamma)^{1/2} . \end{aligned}$$

Thus $0\leqq\Sigma_{\alpha,\beta}\xi_\alpha\bar{\xi}_\beta(u_\alpha, u_\beta)\leqq\Sigma_\beta|\xi_\beta|^2$; so that $0\leqq U\leqq 1$, U^2 exists and $0\leqq U-U^2$ [4, p. 217]. Consider now the matrix $E=\begin{pmatrix} U & \sqrt{U-U^2} \\ \sqrt{U-U^2} & 1-U \end{pmatrix}$. [See 4, pp. 215, 224]. This is Hermitian and idempotent and hence represents a projection in coordinate Hilbert space K of the appropriate dimension. As in Theorem 1, the (row) vectors given by the upper half of E not only are the images, under E, of "half" of the coordinate basis vectors of K, but also constitute an isometric imbedding of the set $\{u_\alpha\}$ in K. Since $\{u_\alpha\}$ is complete in H, the imbedding can be extended to all of H; and the proof is complete.

At this stage, we introduce the following *definition*: *A set of vectors* $\{u_\alpha\}$ *in* H *has the property* P *if each* x *in* H *is orthogonal to all but a countable number of* u_α.

LEMMA (1) *Any orthogonal set has property P.* (2) *Property P is invariant under projection: if $\{u_\alpha\}$ has property P and E is a projection, then so does $\{u_\alpha E\}$.*

Proof. The statement (1) is a classical result [4, p. 114]. To prove (2) we select any x in H. Then $(x, u_\alpha E)=(xE, u_\alpha)$ which is zero for all but countably many u_α.

This lemma leads us to the following *conjecture*: *A necessary and sufficient condition that $\{u_\alpha\}$ be the projection of an orthogonal set (not necessarily normal) is that $\{u_\alpha\}$ has property P.*

The lemma proves necessity. We have been unable to prove sufficiency. However, we can prove the following special case:

THEOREM 4. *A necessary and sufficient condition for the set of non-zero vectors $\{u_\alpha\}$ in a separable Hilbert space H to be the projection of an orthogonal set is that the set be countable.*

Proof. Suppose first that $\{u_\alpha\}$ is the projection of an orthogonal set. Then, by the lemma, it has property P. Let $\{x_i\}$ be a (countable) basis for H. Then all but a countable number of u_α are orthogonal to each x_i and hence to their union $\{x_i\}$. That is, all but countably many u_α are 0. This proves the necessity. To prove sufficiency, we suppose that $\{u_\alpha\}$ is countable and indexed by the positive integers. We then define $v_\alpha=2^{-\alpha}u_\alpha/(u_\alpha, u_\alpha)^{1/2}$, for each α. Then, if x is any vector of H, it follows, by Schwarz' inequality, that $\Sigma|(x, v_\alpha)|^2\leqq(x, x)\Sigma(v_\alpha, v_\alpha)=(x, x)\Sigma 2^{-2\alpha}\leqq(x, x)$. Thus, by Theorem 3, $\{v_\alpha\}$ is the projection of an orthogonal set and so is $\{u_\alpha\}$.

We close with an example of a set $\{u_\alpha\}$ which is not the projection of an orthogonal set. Let $\{x_\alpha\}$ be an uncountable orthonormal set in nonseparable Hilbert space and set $u_\alpha=x_1+x_\alpha$, for each α. Then $\{u_\alpha\}$ does not have property P and hence, by the lemma, is not the projection of an orthogonal set. It is to be noted that Theorem 4 cannot be used to prove this result since every uncountable subset of $\{u_\alpha\}$ spans a nonseparable subspace of H.

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