OSCILLATION CRITERIA FOR LINEAR DIFFERENTIAL SYSTEMS WITH COMPLEX COEFFICIENTS

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1. **Introduction.** The basic oscillation and comparison theorems of the Sturmian theory for a self-adjoint second order linear differential equation with real coefficients have been extended to self-adjoint differential systems with real coefficients through the work of various authors. In this connection the reader is referred to the works of Morse [4], Birkhoff and Hestenes [1] and Reid [5], [6; Part II] listed in the bibliography at the end of this paper, and also to references to other literature on the subject cited by these authors.

The results of the present paper center around oscillation criteria for a self-adjoint linear differential system with complex-valued coefficients as developed in §§ 2 and 3. As a self-adjoint system with complex coefficients involving complex-valued dependent functions $u_1(x), \cdots, u_n(x)$ is equivalent to a self-adjoint system with real coefficients involving real-valued dependent functions $y_1(x), \cdots, y_{2n}(x)$, one might feel that all worthwhile criteria for a system with complex coefficients would be immediate consequences of known criteria for systems with real coefficients. Such is not the case, however, as appears in the treatment of §§ 2 and 3. For those portions of the theory of systems with complex coefficients that parallel closely the theory of systems with real coefficients the treatment is limited to a concise statement of results. Here no attempt is made to discuss for self-adjoint systems with complex coefficients the analogues of the general comparison and separation theorems obtained by Morse [4; Chapter IV] for self-adjoint systems with real coefficients. Also, no attention is given to systems with complex coefficients that are direct generalizations of the accessory equations for a variational problem of Bolza type, although many of our results have direct extensions to such systems. Certain aspects of these topics will appear in a subsequent paper on a problem related to that herein discussed.

Section 4 of this paper is devoted to specific criteria of oscillation and non-oscillation for self-adjoint systems. There are given certain criteria that are direct generalizations of results of Wintner [12] for a single equation of the second order, and there is stated without proof a theorem on a necessary and sufficient condition for non-oscillation near infinity that extends a result of Sternberg [7]. There is established also a sufficient condition for oscillation near infinity that extends

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a result of Wintner [11], even in the case of a single equation of the second order. Finally, there is proved a sufficient condition for non-oscillation on a compact interval that is the analogue of a result of Liapounoff for a second order differential equation.

Section 5 is concerned with the application of the results of the earlier sections on self-adjoint systems to the derivation of sufficient conditions for non-oscillation in the case of a general second order linear homogeneous vector differential equation.

For the sake of generality the assumptions on the coefficients of the system are of weak character so that a "solution" of a linear vector differential system is a vector with a.c. (absolutely continuous) components such that the given equation holds a.e. (almost everywhere) on the interval of consideration.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector $\eta=(\eta_\alpha)$, $(\alpha=1,\cdots,n)$, the norm $|\eta|$ is given by $(|\eta_1|^2+\cdots+|\eta_n|^2)^{1/2}$. The transpose of a matrix M is indicated by $M^\blacktriangle$, and the conjugate transpose by M^*; the symbol 0 is used indiscriminately for the zero matrix of any dimensions. The notation $M\geqq N$ $(M>N)$ is used to mean that M and N are hermitian matrices of the same dimensions and $M-N$ is a nonnegative (positive) hermitian matrix. If the elements of a matrix $M(x)$ are a.c. on an interval ab then $M'(x)$ signifies the matrix of derivatives at values x for which these derivatives exist, and the zero matrix elsewhere. Correspondingly, if the elements of $M(x)$ are integrable on ab then $\int_a^b M(x)dx$ denotes the matrix of integrals of respective elements of $M(x)$. If matrices $M(x)$ and $N(x)$ are equal a.e. on their domain of definition we write $M(x)\simeq N(x)$. In the totality of finite dimensional rectangular matrices with elements defined on a given interval ab we denote by $\mathfrak{L}$ the set of all matrices whose elements are (Lebesgue) integrable on ab, by $\mathfrak{L}_2$ the set of all matrices $M(x)$ whose elements $M_{\alpha\beta}(x)$ are measurable and $|M_{\alpha\beta}(x)|^2\in\mathfrak{L}$, and by $\mathfrak{L}_\infty$ the set of all matrices with elements measurable and essentially bounded on ab. For brevity, a matrix is termed a.c., etc. when each element of the matrix possesses the specified property.

2. A self-adjoint system with complex coefficients.

For x on the compact interval ab: $a\leqq x\leqq b$ let $\omega(x,\eta,\pi)$ denote the hermitian form

$$\omega(x,\eta,\pi)\equiv\pi^*R(x)\pi+\pi^*Q(x)\eta+\eta^*Q^*(x)\pi+\eta^*P(x)\eta \tag{2.1}$$

in the $2n$ variables $\eta,\pi=(\eta_1,\cdots,\eta_n,\pi_1,\cdots,\pi_n)$, where $R(x)$, $Q(x)$, $P(x)$ are $n\times n$ matrices with complex elements satisfying on ab the following hypotheses: (H_1) *$P(x)$, $Q(x)$ and $R(x)$ belong to $\mathfrak{L}$ and $R(x)$, $P(x)$ are* *hermitian*; (H_2) *there is an hermitian matrix* $R^{-1}(x)\in\mathfrak{L}$ *such that* $R(x)R^{-1}(x)\simeq I$; (H_3) *the matrices* $A=-R^{-1}Q$, $B=R^{-1}$, $C=P-Q^*R^{-1}Q$ *belong to* $\mathfrak{L}$. The symbol $\Gamma(\omega)$ will denote the totality of a.c. n-dimensional vectors $\eta(x)$ such that the (Lebesgue) integral

$$I[\eta]\equiv\int_a^b\omega(x,\eta,\eta')dx \tag{2.2}$$

exists and is finite; clearly $\Gamma(\omega)$ contains all n-dimensional vectors with Lipschitzian components on ab. In particular, if (H_1) and (H_2) hold and $Q(x)\in\mathfrak{L}_2$, $R(x)\in\mathfrak{L}_\infty$, $R^{-1}(x)\in\mathfrak{L}_\infty$, then (H_3) holds also and $\Gamma(\omega)$ includes all a.c. vectors $\eta(x)$ with $\eta'(x)\in\mathfrak{L}_2$.

The symbol $\mathfrak{L}(\omega)$ will denote the totality of a.c. n-rowed matrices $U(x)$ for which there is a corresponding a.c. matrix $V(x)$ of the same dimensions as $U(x)$ and such that

$$V(x)\simeq R(x)U'(x)+Q(x)U(x)\,. \tag{2.3}$$

If $U(x)\in\mathfrak{L}(\omega)$ then $U'(x)\simeq A(x)U(x)+B(x)V(x)$, $Q^*(x)U'(x)+P(x)U(x)\simeq C(x)U(x)-A^*(x)V(x)$, and $L[U]\equiv V'(x)-C(x)U(x)+A^*(x)V(x)\in\mathfrak{L}$; occasionally we shall write $[RU'+QU]'-[Q^*U'+PU]$ for $L[U]$ instead of the more precise expression in terms of U and V. For (2.2) the vector Euler equation is

$$L[u]\simeq 0\,, \tag{2.4}$$

which may be written in terms of the canonical variables $u(x)$, $v(x)\simeq R(x)u'(x)+Q(x)u(x)$ as

$$u'\simeq A(x)u+B(x)v\,,\qquad v'\simeq C(x)u-A^*(x)v\,. \tag{2.5}$$

As the coefficient matrices of (2.5) belong to $\mathfrak{L}$ by (H_3), if $a\leqq x_0\leqq b$ and η_0, ζ_0 are given n-dimensional vectors then by well-known existence theorems there is a unique pair of vectors a.c. on ab, and satisfying (2.5) with the initial conditions $u(x_0)=\eta_0$, $v(x_0)=\zeta_0$. By a solution $u(x)$ of (2.4) will be understood an a.c. vector $u(x)$ that belongs to an a.c. pair $u(x)$, $v(x)$ satisfying (2.5).

If $u(x)\in\mathfrak{L}(\omega)$ then for arbitrary a.c. $\eta(x)$ the integral

$$I[\eta,u]\equiv\int_a^b[\eta^{*\prime}(Ru'+Qu)+\eta^*(Q^*u+Pu)]dx$$

exists and is equal to

$$\eta^*v\Big|_a^b-\int_a^b\eta^*L[u]dx\,,$$

from which it follows that if $u(x)\in\mathfrak{L}(\omega)$ then $u(x)$ is a solution of (2.4) if and only if $I[\eta,u]=0$ for arbitrary a.c. $\eta(x)$ satisfying $\eta(a)=0=\eta(b)$.

In particular, if $u(x)$ is a solution of (2.4) and $u(x_1)=0=u(x_2)$, $a\leqq x_1<x_2\leqq b$, then

$$I[u\,;\,x_1,\,x_2]\equiv\int_{x_1}^{x_2}\omega(x,\,u,\,u')dx=0\ .$$

In the following discussion if $u(x)$, $U(x)$, $U_1(x)$, etc. are matrices in $\mathfrak{L}(\omega)$ then without further comment the corresponding symbol $v(x)$, $V(x)$, $V_1(x)$, etc. will be employed for the associated a.c. matrix which satisfies (2.3). Moreover, if U_1 and U_2 are matrices of $\mathfrak{L}(\omega)$ of dimensions $n\times r$ and $n\times s$, respectively, then $\{U_1,\,U_2\}$ will signify the $r\times s$ a.c. matrix

$$\{U_1,\,U_2\}=U_1^*V_2-V_1^*U_2\ . \tag{2.6}$$

Clearly $\{U_1,\,U_2\}^*=-\{U_2,\,U_1\}$, $\{U_1M_1,\,U_2M_2\}=M_1^*\{U_1,\,U_2\}M_2$ for constant matrices M_1 and M_2 of r and s rows, respectively, and $\{U_1,\,U_2+U_3\}=\{U_1,\,U_2\}+\{U_1,\,U_3\}$ if U_2 and U_3 are both $n\times s$ matrices.

Let $\mathscr{J}\equiv\|\mathscr{J}_{\mu\nu}\|$, $(\mu,\,\nu=1,\,\cdots,\,2n)$, be the real skew symmetric matrix with $\mathscr{J}_{\alpha\beta}=0=\mathscr{J}_{n+\alpha\,n+\beta}$, $\mathscr{J}_{n+\alpha\,\beta}=\delta_{\alpha\beta}=-\mathscr{J}_{\alpha\,n+\beta}$, $(\alpha,\,\beta=1,\,\cdots,\,n)$. For a given $n\times r$ matrix $U(x)\equiv\|U_{\alpha j}(x)\|$, $(\alpha=1,\,\cdots,\,n\,;\,j=1,\,\cdots,\,r)$, the corresponding boldface German letter $\mathfrak{U}(x)$ will denote the $2n\times r$ matrix $\|\mathfrak{U}_{\mu j}(x)\|$ with $\mathfrak{U}_{\alpha j}(x)=U_{\alpha j}(x)$, $\mathfrak{U}_{\alpha+n\,j}(x)=V_{\alpha j}(x)$. The relation (2.6) may then be written as

$$\{U_1,\,U_2\}=-\mathfrak{U}_1^*\mathscr{J}\mathfrak{U}_2\ . \tag{2.6'}$$

The following preliminary results are immediate; in particular, the second relation of (2.7) embodies the self-adjoint character of (2.4).

LEMMA 2.1. *If* $U_\gamma(x)\in\mathfrak{L}(\omega)$, $(\gamma=1,\,2)$, then

$$\begin{aligned} U_1^*L[U_2]&\simeq[U_1^*V_2]'-[V_1^*BV_2+U_1^*CU_2]\ ,\\ U_1^*L[U_2]-(U_2^*L[U_1])^*&\simeq\{U_1,\,U_2\}'\ ; \end{aligned} \tag{2.7}$$

in particular, if $L[U_\gamma]\simeq 0$, $(\gamma=1,\,2)$, *then* $\{U_1,\,U_2\}$ *is constant on* ab.

COROLLARY. *If* $U(x)$ *is an* $n\times r$ *matrix of* $\mathfrak{L}(\omega)$ *for which* $\{U,\,U\}$ *is constant on* ab, *then the* $r\times r$ *matrix* $U^*L[U]$ *is hermitian on this interval.*

LEMMA. 2.2. *If* $U(x)$ *is an* $n\times r$ *matrix of* $\mathfrak{L}(\omega)$, *and* $\xi(x)$ *is an a.c.* r*-dimensional vector, then* $\eta(x)=U(x)\xi(x)$ *is a.c. and*

$$\omega(x,\,\eta,\,\eta')\simeq\xi^{*\prime}U^*RU\xi'+(\eta^*V\xi)'-\xi^*\{U,\,U\}\xi'-\xi^*U^*L[U]\xi\ . \tag{2.8}$$

Relation (2.8) is in essence the well-known Clebsch transformation of the second variation of a non-parametric simple integral variational problem, (see, for example, Bliss [2, Secs. 23, 39]). In particular, for $\eta(x)$ and $\xi(x)$ related as in Lemma 2.2 it follows that $\eta(x) \in \Gamma(\omega)$ if and only if the integral on ab of the function $\xi^{*\prime}U^*RU\xi'$ exists and is finite.

If u_1 and u_2 are vectors of $\mathfrak{L}(\omega)$ satisfying (2.4) on ab, then $\{u_1, u_2\} = -\mathfrak{u}_1^* \mathscr{J} \mathfrak{u}_2$ is a constant by Lemma 2.1; if the value of this constant is zero these two solutions of (2.4) are said to be (mutually) *conjoined.* In case the elements of the coefficient matrices of (2.1) are real-valued two solutions u_1 and u_2 of (2.4) with real components are conjoined if and only if they are conjugate in the sense introduced originally by von Escherich. In view of the discussion in the following section of a self-adjoint system with real coefficients that is equivalent to (2.4), however, it appears advisable to introduce a terminology distinct from "conjugate" to characterize solutions u_1, u_2 of (2.4) satisfying $\{u_1, u_2\} = 0$, and we have chosen to employ the synonym "conjoined." If the column vectors of an $n \times r$ matrix $U(x)$ are linearly independent solutions of (2.4) which are mutually conjoined, that is $\{U, U\} = 0$, these solutions are termed a *conjoined family of solutions of dimension* r. For such families of solutions one has the results of the following Lemma 2.3. It is to be commented that the proof of this lemma is far from trivial, although the proof of its counterpart for conjugate solutions is immediate.

LEMMA 2.3. *The maximal dimension of a conjoined family of solutions of* (2.4) *is* n; *moreover, a given conjoined family of solutions of dimension* $r < n$ *is contained in a conjoined family of dimension* n.

If $U(x)$ is an $n \times r$ matrix of $\mathfrak{L}(\omega)$ such that $L[U] \simeq 0$, then the condition that the column vectors of $U(x)$ be a conjoined family of solutions of (2.4) implies that the $r \times 2n$ matrix $\mathfrak{U}^*(x)$ is of rank r, and the condition $\mathfrak{U}^* \mathscr{J} \mathfrak{U} = 0$ implies that $r \leqq 2n - r$, so that $r \leqq n$ and the first part of the lemma is proved. To establish the second part it clearly suffices to show that a given conjoined family of solutions of dimension $r < n$ is contained in a conjoined family of dimension $r+1$. Suppose that $U(x)$ is an $n \times r$ matrix whose columns form a conjoined family of solutions of (2.4) of dimension r, and denote by $U_1(x)$ an $n \times (2n - 2r)$ matrix of $\mathfrak{L}(\omega)$ such that the columns of the $n \times (2n - r)$ matrix $\| U(x) \neq U_1(x) \|$ are linearly independent solutions of (2.4), and $\{U, U_1\} = 0$; these conditions are clearly attainable by suitable choice of initial values $U_1(a)$, $V_1(a)$. For c and d arbitrary constant vectors of dimensions r and $2n - 2r$, respectively, $u(x) = U(x)c + U_1(x)d$ defines a $(2n - r)$-dimensional linear space S of solutions of (2.4) with corresponding canonical variables $v(x)=V(x)c+V_1(x)d$. If S_1 is the subspace of S on which $iu(a)+v(a)=0$, then S_1 is of dimension at least $n-r$ and for each $u\not\equiv 0$ of S_1 we have $i\{u, u\}=2u^*(a)u(a)>0$. Correspondingly, if S_2 is the subspace of S on which $iu(a)-v(a)=0$, then S_2 is of dimension at least $n-r$ and $i\{u, u\}=-2u^*(a)u(a)<0$ for arbitrary $u\not\equiv 0$ of S_2. Now for a solution $u(x)=U(x)c+U_1(x)d$ of S the conditions $\{U, U\}=0$, $\{U, U_1\}=0$ imply that $\{u, u\}=d^*\{U_1, U_1\}d$. Therefore, if $u_\gamma(x)=U(x)c_\gamma+U_1(x)d_\gamma$ is a non-identically vanishing solution in S_γ, $(\gamma=1, 2)$, then $id_1^*\{U_1, U_1\}d_1>0$ and $id_2^*\{U_1, U_1\}d_2<0$. In particular, the two $(2n-2r)$-dimensional vectors d_1, d_2 are linearly independent and for a suitable value of θ on $0<\theta<\pi/2$ the solution $u(x)=U_1(x)\,[d_1\cos\theta+d_2\sin\theta]$ is a non-identically vanishing solution of S such that $\{u, u\}=0$, $\{U, u\}=0$, and the $r+1$ solutions of (2.4) consisting of u and the column vectors of $U(x)$ form a conjoined family of dimension $r+1$.

A point x_2 is said to be *conjugate* to x_1, (with respect to the differential equation (2.4)), if there exists a non-trivial solution $u(x)$ of this equation such that $u(x_1)=0=u(x_2)$. The equation (2.4) is termed *non-oscillatory* on a given interval Δ if no two distinct points of Δ are mutually conjugate. For this concept Wintner [12] has used the terminology "disconjugate on Δ" in the case of a single equation of the second order.

Let $\Gamma_0(\omega)$ denote the set of vectors $\eta(x)$ of $\Gamma(\omega)$ satisfying $\eta(a)=0=\eta(b)$. For brevity, H_+ and H_R are used to signify the following hypotheses:

H_+. *$I[\eta]$ is positive definite on $\Gamma_0(\omega)$, that is, $I[\eta]\geqq 0$ for $\eta\in\Gamma_0(\omega)$ and the equality sign holds only if $\eta\equiv 0$.*

H_R. *$R(x)>0$ a.e. on ab.*

The following theorem is the basic result of this section, and will be used in § 4 for the derivation of specific criteria for oscillation and non-oscillation.

THEOREM 2.1. *A necessary and sufficient condition for H_+ is that H_R hold, together with one of the following conditions:*

i. *The equation (2.4) is non-oscillatory on ab.*

ii. *If $U_1(x)$ is an $n\times n$ matrix of $\mathfrak{L}(\omega)$ satisfying $L[U_1]\simeq 0$, with $U_1(a)=0$ and $V_1(a)$ non-singular, then $U_1(x)$ is non-singular on $a<x\leqq b$.*

iii. *If $U_2(x)$ is an $n\times n$ matrix of $\mathfrak{L}(\omega)$ satisfying $L[U_2]\simeq 0$, with $U_2(b)=0$ and $V_2(b)$ non-singular, then $U_2(x)$ is non-singular on $a\leqq x<b$.*

iv. *There exists an $n\times n$ non-singular matrix $U(x)$ of $\mathfrak{L}(\omega)$ whose column vectors form a conjoined family of solutions of (2.4) of dimension n.*

v. *There exists an $n\times n$ non-singular matrix $U(x)$ of $\mathfrak{L}(\omega)$ such* *that* $\{U, U\} \equiv 0$ *on* ab, *and the hermitian matrix* $U^*L[U]$ *is non-positive on this interval.*

This theorem will be established by proving the following sequence of statements: (a) iv$\rightarrow$v; (b) v, $H_R \rightarrow H_+$; (c) $H_+ \rightarrow$i, H_R; (d) i$\rightarrow$ii; (e) ii, $H_R \rightarrow$iii; (f) iii, $H_R \rightarrow$iv.

Statement (a) is evident since a matrix $U(x)$ which satisfies iv clearly satisfies v. To establish (b), if $U(x)$ satisfies v and $\eta(x) \in \Gamma_0(\omega)$, define $\xi(x)$ by $\eta(x) = U(x)\xi(x)$ on ab. Then $\xi(x)$ is a.c. on ab, $\xi(a)=0 =\xi(b)$, and as a consequence of Lemma 2.2 we have $I[\eta] \geqq \int_a^b \xi^{*\prime} U^*RU\xi' dx$; in view of H_R it then follows that $I[\eta] \geqq 0$, with $I[\eta]=0$ only if $U\xi' \simeq 0$, in which case $\xi \equiv 0$ and $\eta \equiv 0$ on ab.

For the proof of (c) it is to be noted first that from the discussion following equation (2.5) it follows that condition i is a consequence of H_+. The fact that H_+ implies H_R is essentially the usual Legendre condition for non-parametric simple integral variational problems. Indeed, if $\varphi(x) = 1 - |x|$ on $|x| < 1$, $\varphi(x) \equiv 0$ elsewhere, and for $a < x_0 < b$, η_0 a given n-dimensional constant vector, and $0 < \varepsilon \leqq \min(x_0 - a, b - x_0)$, we set $\eta_\varepsilon(x) = \varepsilon\eta_0\varphi((x-x_0)/\varepsilon)$, then $\eta_\varepsilon(x) \in \Gamma_0(\omega)$ and $|\eta_\varepsilon(x)| \leqq \varepsilon|\eta_0|$. Since for x_0 a.e. on $a < x_0 < b$ we have $(2\varepsilon)^{-1}\int_{x_0-\varepsilon}^{x_0+\varepsilon} \eta_0^* R(x)\eta_0 dx \rightarrow \eta_0^* R(x_0)\eta_0$ as $\varepsilon \rightarrow 0$ for arbitrary vectors η_0, the condition H_+ implies that $\eta_0^* R(x_0)\eta_0 \geqq 0$ for x_0 a.e. on ab and arbitrary vectors η_0. As $R(x)$ has a reciprocal a.e. on ab by (H_2), it then follows that $R(x) > 0$ a.e. on this interval.

The truth of (d) is immediate, since ii is equivalent to the condition that there is no point on $a < x \leqq b$ conjugate to $x = a$. For the proof of (e), it is to be noted that for $U_1(x)$, $U_2(x)$ as in ii and iii, respectively, the matrix $\{U_1, U_2\}$ is constant and $\{U_1, U_2\} = U_1^*(b)V_2(b) = -V_1^*(a)U_2(a)$. In particular, ii implies that $U_1(b)$ is non-singular, and consequently that $U_2(a)$ is non-singular also. Now for $a < x_3 < b$ the matrix $U_1(x)$ is non-singular on x_3b, and $\{U_1, U_1\} = 0$ since $U_1(a) = 0$. Therefore the matrix $U = U_1(x)$ satisfies the condition v for the interval x_3b, and the previously established statements (a), (b), (c) applied to x_3b result in the conclusion that under conditions ii and H_R the equation (2.4) is non-oscillatory on each interval x_3b with $a < x_3 < b$. Therefore $x = x_3$ is not conjugate to $x = b$, and $U_2(x_3)$ is non-singular for $a < x_3 < b$, thus completing the proof of (e).

Finally, in order to establish (f), it is to be noted that by an argument similar to that for statement (e) it follows that conditions iii and H_R imply ii. Consequently, if U_1 and U_2 are as in ii and iii, respectively, we have $\{U_1, U_1\} = 0 = \{U_2, U_2\}$ while $\{U_1, U_2\}$ is the non-singular constant matrix $M = U_1^*(b)V_2(b)$. As iii remains true for $U_2(x)$ replaced by $-U_2(x)M^{-1}$, it follows that without loss of generality the matrices

U_1, U_2 of ii and iii may be so chosen that $\{U_1, U_2\}=-I$. With such a choice the matrix $U(x)=U_1(x)+U_2(x)$ satisfies $L[U]\simeq 0$ and $\{U, U\}=0$. Moreover, as $U(a)=U_2(a)$ and $U(b)=U_1(b)$, if $U(x)$ is singular at a point x_3 of ab then $a<x_3<b$. Now if $a<x_3<b$ and $U(x_3)\xi=0$, set $\eta(x)=U_1(x)\xi$ on ax_3 and $\eta(x)=-U_2(x)\xi$ on x_3b. Then $\eta(x)\in\Gamma_0(\omega)$ and the application of Lemma 2.2 to the separate intervals ax_3 and x_3b yields $I[\eta]=\xi^*U_1^*(x_3)V_1(x_3)\xi-\xi^*U_2^*(x_3)V_2(x_3)\xi$; moreover, as $U_1^*V_1=V_1^*U_1$ and $U_1\xi=-U_2\xi$ at $x=x_3$, it follows that $I[\eta]=\xi^*\{U_1, U_2\}\xi=-\xi^*\xi\leqq 0$. On the other hand, if $\xi(x)\equiv\xi$ on ax_3, and $\xi(x)=-U_1^{-1}(x)U_2(x)\xi$ on x_3b, then $\xi(x)$ is a.c. and $\eta(x)=U_1(x)\xi(x)$ on ab. Since $\eta(x)\in\Gamma_0(\omega)$ and $\{U_1, U_1\}\equiv 0$, $L[U_1]\simeq 0$, it follows from Lemma 2.2 that $I[\eta]=\int_{x_3}^{b}\xi^{*\prime}U_1^*RU_1\xi' dx$, and hence $I[\eta]\geqq 0$, in view of H_R. Consequently $0\leqq I[\eta]=-\xi^*\xi\leqq 0$, so that $\xi=0$ and $U(x_3)$ is non-singular for $a<x_3<b$, thus completing the proof of (f). The above method of proof of statement (f) is the same as that introduced initially by Hestenes in establishing a corresponding result for the second variation of a Bolza type variational problem with separated end-conditions, (see, for example, Bliss [2; Secs. 86, 87]). In this connection it is to be commented that the same method of proof establishes for hermitian functionals results of the same order of generality as the original result of Hestenes (Lemma 87.2 of Bliss [2]) for real symmetric functionals; apropos of this remark the reader is referred to the treatment of Part II of Reid [6].

It is to be noted that in view of criteria i and iv of Theorem 2.1 we have immediately the following results in the nature of separation and comparison criteria.

COROLLARY 1. *If $U(x)$ is an $n\times n$ matrix whose columns form a conjoined system of solutions of* (2.4) *of dimension n, and x_1, x_2, $(x_1<x_2)$, are points of ab which are mutually conjugate, then $U(x)$ is singular for at least one value on x_1x_2 in case hypothesis* H_R *is satisfied.*

COROLLARY 2. *Suppose that $L[u]\simeq 0$ and $L_1[u]\simeq 0$ are the Euler equations for corresponding functionals $I[\eta]$ and $I_1[\eta]$ of the type* (2.2), *with respective integrand forms ω and ω_1 whose coefficient matrices satisfy hypotheses* (H_1), (H_2), (H_3) *and* H_R *on ab. If $\Gamma_0(\omega_1)\subset\Gamma_0(\omega)$ and $I_1[\eta]-I[\eta]\geqq 0$ for arbitrary $\eta\in\Gamma_0(\omega_1)$ then whenever $L[u]\simeq 0$ is non-oscillatory on ab the equation $L_1[u]\simeq 0$ is also non-oscillatory on this interval.*

If $U(x)$ is a non-singular $n\times n$ matrix of $\mathfrak{L}(\omega)$ then $W(x)=V(x)U^{-1}(x)$ is a.c. on ab, and it is readily verified that, (see, for example, Reid [5]),

$$(2.9) \qquad \{U, U\}=U^*(W-W^*)U\ , \qquad U^*L[U]\simeq U^*K[W]U\ ,$$

where $K[W]$ is the corresponding Riccati matrix differential operator

$$K[W]\equiv W'+WA+A^*W+WBW-C\,. \tag{2.10}$$

Conversely, if $W(x)$ is an $n\times n$ matrix that is a.c. on ab, and $U(x)$ is a non-singular a.c. matrix satisfying $U'\simeq(A+BW)U$, then $U\in\mathfrak{L}(\omega)$ and $V=WU$. Consequently, conditions iv and v of Theorem 2.1 are equivalent to the following conditions, respectively:

iv$_R$. *There exists an $n\times n$ hermitian a.c. matrix $W(x)$ satisfying $K[W]\simeq 0$ on ab.*

v$_R$. *There exists an $n\times n$ hermitian a.c. matrix $W(x)$ such that a.e. on ab the hermitian matrix $K[W]$ is non-positive.*

The following results will be of use in the following sections.

THEOREM 2.2. *If condition* H$_+$ *is satisfied, and $u(x)$ is a solution of* (2.4), *then $I[\eta]\geqq I[u]$ for arbitrary $\eta(x)\in\Gamma(\omega)$ satisfying $\eta(a)=u(a)$, $\eta(b)=u(b)$, and $I[\eta]=I[u]$ only if $\eta(x)\equiv u(x)$ on ab.*

As $\eta_1=\eta-u$ is a.c. and $\eta_1(a)=0=\eta_1(b)$, the fact that $u(x)$ is a solution of (2.4) implies $I[\eta_1,u]=0$, and upon suitable expansion of the integrand of (2.2) we have $I[\eta]=I[u]+I[u,\eta_1]+I[\eta_1,u]+I[\eta_1]=I[u]+I[\eta_1]$. In particular, $\eta_1(x)\in\Gamma_0(\omega)$ and the stated result is a consequence of H_+.

For convenience we set $\omega_0(x,\eta,\pi)=\pi^*R(x)\pi$, and denote by $\Gamma(\omega_0)$ the class of a.c. vectors $\eta(x)$ such that

$$I_0[\eta]\equiv\int_a^b\eta^{*\prime}R(x)\eta' dx \tag{2.11}$$

exists and is finite; correspondingly, $\Gamma_0(\omega_0)$ is the subclass of $\Gamma(\omega_0)$ satisfying $\eta(a)=0=\eta(b)$.

THEOREM 2.3. *If $R(x)$ satisfies* H$_R$ *then for arbitrary $\eta(x)\in\Gamma_0(\omega_0)$,*

$$I_0[\eta]\geqq 4\eta^*(c)\left[\int_a^b R^{-1}(t)dt\right]^{-1}\eta(c)\,,\qquad a<c<b; \tag{2.12}$$

moreover, the inequality holds in (2.12) *if $\eta(x)\not\equiv 0$ and for each x_0 on $a<x_0<b$ with $|\eta(x_0)|\neq 0$ there is a corresponding neighborhood $(x_0)_\delta$: $x_0-\delta<x<x_0+\delta$ on which there is defined a continuous vector $\zeta(x)$ such that $\zeta(x)\simeq R(x)\eta'(x)$ on $(x_0)_\delta$.*

Clearly H$_R$ implies that $I_0[\eta]$ is positive definite of $\Gamma_0(\omega_0)$, and that

$$U_1(x)=\int_a^x R^{-1}(t)dt\,,\qquad U_2(x)=\int_x^b R^{-1}(t)dt=U_1(b)-U_1(x) \tag{2.13}$$

are $n\times n$ matrices which satisfy conditions ii and iii of Theorem 2.1 for the Euler equation $L_0[u]\equiv(R(x)u')'\simeq 0$ of the function $I_0[\eta]$. For a given $\eta(x) \in \Gamma_0(\omega_0)$ and $a < c < b$, let $u_1(x) = U_1(x)U_1^{-1}(c)\eta(c)$, and $u_2(x) = U_2(x)U_2^{-1}(c)\eta(c)$. Then $L_0[u_\gamma] \simeq 0$, $(\gamma = 1, 2)$, and $u_1(a) = 0 = \eta(a)$, $u_1(c) = \eta(c) = u_2(c)$, $u_2(b) = 0 = \eta(b)$, so that the application of Theorem 2.2 to $I_0[\eta]$ on the individual intervals ac and cb yields

$$(2.13') \qquad I_0[\eta; a, c] \geqq I_0[u_1; a, c] = \eta^*(c)U_1^{-1}(c)\eta(c) ,$$

$$(2.13'') \qquad I_0[\eta; c, b] \geqq I_0[u_2; c, b] = \eta^*(c)[U_1(b) - U_1(c)]^{-1}\eta(c) ,$$

and the inequality in (2.13′) or (2.13″) holds only if $\eta \equiv u_1$ or $\eta \equiv u_2$ on the respective interval. Relations (2.13′), (2.13″) imply

$$(2.14) \qquad I_0[\eta] \geqq \eta^*(c)(U_1^{-1}(c) + [U_1(b) - U_1(c)]^{-1})\eta(c) ,$$

and (2.12) follows from the fact that if A, B are hermitian matrices such that A and $B - A$ are positive then the hermitian matrix $A^{-1} + [B - A]^{-1} - 4B^{-1}$ is nonnegative. Clearly this stated result for hermitian matrices A, B is equivalent to this result for the special case when $B = I$, and if A and $I - A$ are positive hermitian matrices the desired result follows from the identity $A^{-1} + [I - A]^{-1} - 4I = (I - 2A)^2 A^{-1}(I - A)^{-1}$ and the fact that $(I - 2A)^2$, A^{-1} and $(I - A)^{-1}$ are individually nonnegative hermitian matrices that are mutually commutative under multiplication.

In order to establish the final statement of the lemma, it is to be noted first that if $|\eta(c)| = 0$ then the inequality holds in (2.12) for non-identically vanishing $\eta(x) \in \Gamma_0(\omega_0)$. On the other hand, if $|\eta(c)| > 0$ and there is a neighborhood $(c)_\delta$ on which there is a continuous vector $\zeta(x)$ such that $\zeta(x) \simeq R(x)\eta'(x)$ on $(c)_\delta$, then the relations $R(x)u_1'(x) \simeq U_1^{-1}(c)\eta(c)$ and $R(x)u_2'(x) \simeq -U_2^{-1}(c)\eta(c)$, together with the positiveness of $U_1^{-1}(c) + U_2^{-1}(c)$, implies that not both $\eta(x) \equiv u_1(x)$ on ac and $\eta(x) \equiv u_2(x)$ on cb are valid. Consequently, in either (2.13′) or (2.13″) the inequality sign holds, and hence the inequality sign holds in (2.12).

3. An equivalent real differential system. If $M \equiv M^1 + iM^2 \equiv \|M^1_{\alpha j} + iM^2_{\alpha j}\|$, $(\alpha = 1, \cdots, n; j = 1, \cdots, r)$, is an $n \times r$ matrix with complex elements the corresponding bold-face letter $\boldsymbol{M}$ will be used to denote the $2n \times r$ matrix $\|\boldsymbol{M}_{\beta j}\|$, $(\beta = 1, \cdots, 2n; j = 1, \cdots, r)$, of real elements $\boldsymbol{M}_{\alpha j} = M_{\alpha j}$, $\boldsymbol{M}_{n+\alpha, j} = M^2_{\alpha j}$. If $M = M^1 + iM^2$ is an $n \times n$ matrix the corresponding script letter $\mathscr{M}$ will designate the $2n \times 2n$ matrix of real elements.

$$(3.1) \qquad \mathscr{M} \equiv \left\| \begin{matrix} M^1 & -M^2 \\ M^2 & M^1 \end{matrix} \right\| \equiv \|\boldsymbol{M}\boldsymbol{N}\|, \quad \text{where} \quad N = iM .$$

In particular, for I the $n \times n$ identity matrix the matrix $\mathscr{I}$ is the $2n \times 2n$ identity matrix, while for $J = iI$ the corresponding $\mathscr{J}$ is the matrix already introduced in § 2. If $\eta \equiv (\eta^1_\alpha + i\eta^2_\alpha)$, $(\alpha = 1, \cdots, n)$, is an n-dimensional vector of $\Gamma(\omega)$ then $\boldsymbol{\eta} \equiv (\boldsymbol{\eta}_\beta)$, $(\beta = 1, 2, \cdots, 2n)$, with $\boldsymbol{\eta}_\alpha = \eta^1_\alpha$,

$\boldsymbol{\eta}_{n+\alpha}=\eta^2_\alpha$, is a $2n$-dimensional vector with real components, and the integral $I[\eta]$ of (2.2) becomes

$$\boldsymbol{I}[\boldsymbol{\eta}]\equiv\int_a^b \boldsymbol{\omega}(x,\ \boldsymbol{\eta},\ \boldsymbol{\eta}')dx\ , \tag{3.2}$$

where $\boldsymbol{\omega}(x, \boldsymbol{\eta}, \boldsymbol{\pi})\equiv\boldsymbol{\pi}^\blacktriangle\mathscr{R}(x)\boldsymbol{\pi}+\boldsymbol{\pi}^\blacktriangle\mathscr{Q}(x)\boldsymbol{\eta}+\boldsymbol{\eta}^\blacktriangle\mathscr{Q}^\blacktriangle(x)\boldsymbol{\pi}+\boldsymbol{\eta}^\blacktriangle\mathscr{P}(x)\boldsymbol{\eta}$ is a real quadratic form in the $4n$ real variables $(\boldsymbol{\eta}_\beta, \boldsymbol{\pi}_\beta)$. For (3.2) the class $\Gamma(\boldsymbol{\omega})$ consists of all a.c. $2n$-dimensional vectors $y(x)$ such that $\boldsymbol{I}[y]$ exists and is finite; if the components of $y(x)$ are real-valued then clearly $y(x)\in\Gamma(\boldsymbol{\omega})$ if and only if $y(x)=\boldsymbol{\eta}(x)$, where $\eta(x)\in\Gamma(\omega)$.

Corresponding to the notation of the preceding section, $\mathfrak{L}(\boldsymbol{\omega})$ denotes the set of a.c. $2n$-rowed matrices $Y(x)$ for which there is a corresponding a.c. $Z(x)$ of the same dimensions as $Y(x)$ such that $Z(x)\simeq\mathscr{R}(x)Y'(x)+\mathscr{Q}(x)Y(x)$. If $Y(x)\in\mathfrak{L}(\boldsymbol{\omega})$, then $Y'(x)\simeq\mathscr{A}(x)Y(x)+\mathscr{B}(x)Z(x)$, $\mathscr{Q}^\blacktriangle(x)Y'(x)+\mathscr{P}(x)Y(x)\simeq\mathscr{C}(x)Y(x)-\mathscr{A}^\blacktriangle(x)Z(x)$, and $\mathscr{L}[Y]\equiv Z'(x)-\mathscr{C}(x)Y(x)+\mathscr{A}^\blacktriangle(x)Z(x)\in\mathfrak{L}$. By a solution of the Euler equation of (3.2) will be understood a vector $y(x)\in\mathfrak{L}(\boldsymbol{\omega})$ such that

$$\mathscr{L}[y]\simeq 0\ . \tag{3.3}$$

In particular, a solution $y(x)$ of (3.3) with real components is of the form $y(x)=\boldsymbol{u}(x)$, where $u(x)$ is a solution of (2.4). Indeed, if $U(x)\in\mathfrak{L}(\omega)$ and $V(x)$ is an a.c. matrix such that $V\simeq RU'+QU$ then $\boldsymbol{U}(x)\in\mathfrak{L}(\boldsymbol{\omega})$, $\boldsymbol{V}(x)\simeq\mathscr{R}\boldsymbol{U}'+\mathscr{Q}\boldsymbol{U}$, and $\mathscr{L}[\boldsymbol{U}]\simeq\boldsymbol{M}$, where $M=L[U]$; moreover, in case $U(x)$ is an $n\times n$ matrix then $\mathscr{V}\simeq\mathscr{R}\mathscr{U}'+\mathscr{Q}\mathscr{U}$ and $\mathscr{L}[\mathscr{U}]\simeq\mathscr{M}$, where $M=L[U]$. It is to be noted also that if $U(x)\in\mathfrak{L}(\omega)$ then $\mathscr{J}\boldsymbol{U}=\boldsymbol{U}_0$, where $U_0=iU$, and $\mathscr{L}[\mathscr{J}\boldsymbol{U}]=\mathscr{J}\mathscr{L}[\boldsymbol{U}]$; in particular, if $y(x)$ is a solution of (3.3) then $\mathscr{J}y(x)$ is also a solution of this equation.

If $y_\gamma(x)$, $(\gamma=1, 2)$, are solutions of (3.3), and $z_\gamma(x)$ are corresponding a.c. $2n$-dimensional vectors such that $z_\gamma(x)\simeq\mathscr{R}y'_\gamma+\mathscr{Q}y_\gamma$, then $\langle y_1, y_2\rangle\equiv y_1^\blacktriangle z_2-z_1^\blacktriangle y_2$ is constant on ab. According to the terminology due to von Escherich (see, for example, Bliss [2, p. 233], or Morse [**4**; p. 46]), y_1 and y_2 are termed (mutually) *conjugate* if $\langle y_1, y_2\rangle=0$. In the case of real-valued solutions of (3.3) clearly the concept of conjugate solutions is equivalent to that of conjoined solutions as defined in the preceding section. In general, if $u_\gamma(x)$, $(\gamma=1, 2)$, are solutions of (2.4), then $y_\gamma(x)=\boldsymbol{u}_\gamma(x)$ are real solutions of (3.3), and

$$\{u_1,\ u_2\}=\langle\boldsymbol{u}_1,\ \boldsymbol{u}_2\rangle+i\langle\mathscr{J}\boldsymbol{u}_1,\ \boldsymbol{u}_2\rangle\ .$$

From the above relation is seen that if u_1 and u_2 are conjoined solutions of (2.4), then $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are conjugate solutions of (3.3); on the other hand, if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are conjugate solutions of (3.3) it does not follow that u_1 and u_2 are conjoined solutions of (2.4) but merely that $\Re\mathrm{e}\{u_1, u_2\}=0$.

In view of the above remarks it is clear that the result of Theorem 2.1 is true with the conditions i–v of that theorem replaced by corresponding conditions i′–v′ for the real system (3.3); moreover, from the details of proof of Theorem 2.1 it is evident that in the statements of conditions i′–v′ attention may be restricted to matrices with real-valued elements. As the wording of conditions i′–v′ should be obvious to the reader, they will not be stated explicitly here. Clearly conditions i and i′ are equivalent. Moreover, if $a \leqq x_0 \leqq b$ and $U(x)$ is an $n \times n$ matrix whose column vectors are solutions of (2.4) with $U(x_0)=0$ and $V(x_0)$ non-singular, it follows readily that the most general $2n \times 2n$ matrix $Y(x)$ with column vectors solutions of (3.3) and $Y(x_0)=0$ is of the form $Y(x)=\mathscr{U}(x)D$, with D a $2n \times 2n$ constant matrix. Consequently, conditions ii′ and iii′ are equivalent to the respective conditions ii and iii. Finally, the conditions iv′ and v′ for (3.3) corresponding to iv and v for (2.4), and involving $2n \times 2n$ matrices $Y(x)$ of real elements corresponding to the $n \times n$ matrices $U(x)$ of iv and v, may be expressed as follows in terms of $n \times n$ matrices with complex coefficients:

iv′. *There exist $n \times n$ matrices $U_1(x)$, $U_2(x)$ of $\mathfrak{L}(\omega)$ such that on ab the $2n \times 2n$ real matrix $\| U_1 \;\, U_2 \|$ is non-singular, while $L[U_\gamma]=0$, $(\gamma=1, 2)$, and $\Re\{U_\gamma, U_\mu\}=0$, $(\gamma, \mu=1, 2)$, on this interval.*

v′. *There exist $n \times n$ matrices $U_1(x)$, $U_2(x)$ of $\mathfrak{L}(\omega)$ such that on ab the $2n \times 2n$ real matrix $\| U_1 \;\, U_2 \|$ is non-singular, and $\Re\{U_\gamma, U_\mu\} \equiv 0$, $(\gamma, \mu=1, 2)$, while the (necessarily symmetric) real part of the $2n \times 2n$ matrix $\| U_\gamma^* L[U_\mu] \|$, $(\gamma, \mu=1, 2)$, is non-positive on this interval.*

If a matrix $U(x)$ satisfies iv or v then clearly $U_1(x)=U(x)$, $U_2(x) = iU(x)$ satisfies the corresponding condition iv′ or v′, and $\| U_1 \;\, U_2 \| = \mathscr{U}$.

For $Y(x)$ a non-singular matrix of $\mathfrak{L}(\boldsymbol{\omega})$ and $Z(x)$ an a.c. matrix such that $Z(x) \simeq \mathscr{R}(x)Y'(x)+\mathscr{Q}(x)Y(x)$ the matrix $T(x)=Z(x)Y^{-1}(x)$ is a.c. on ab, and corresponding to (2.11) we have

$$(3.4) \qquad \langle Y, Y \rangle = Y^{\blacktriangle}(T-T^{\blacktriangle})Y\,, \qquad Y^{\blacktriangle}\mathscr{L}[Y]=Y^{\blacktriangle}\mathscr{K}[T]Y\,,$$

where $\langle Y, Y \rangle \equiv Y^{\blacktriangle}Z-Z^{\blacktriangle}Y$ and $\mathscr{K}[T]$ is the corresponding Riccati matrix differential operator

$$(3.5) \qquad \mathscr{K}[T] \equiv T'+T\mathscr{A}+\mathscr{A}^{\blacktriangle}T+T\mathscr{B}T-\mathscr{C}\,.$$

The following conditions iv'_R and v'_R are then equivalent to the above conditions iv′ and v′, respectively:

iv'_R. *There exists on ab an a.c. real symmetric $2n \times 2n$ matrix $T(x)$ such that $\mathscr{K}[T] \simeq 0$.*

v'_R. *There exists an a.c. real symmetric $2n \times 2n$ matrix $T[x]$ such that the real symmetric matrix $\mathscr{K}[T]$ is non-negative a.e. on ab.*

For an arbitrary a.c. $n \times n$ matrix $W(x)$ it is seen readily that

$\mathscr{K}[\mathscr{W}]=\mathscr{M}$, where $M=K[W]$, so that the validity of iv_R or v_R for a matrix $W(x)$ implies that the corresponding condition iv'_R or v'_R holds with $T=\mathscr{W}$. If $T(x)$ is an a.c. $2n\times 2n$ matrix we have $\mathscr{J}^{\blacktriangle}\mathscr{K}[T]\mathscr{J}=\mathscr{K}[\mathscr{J}^{\blacktriangle}T\mathscr{J}]$, and the corresponding matrices $T_1=\frac{1}{2}(T+\mathscr{J}^{\blacktriangle}T\mathscr{J})$ and $T_2=\frac{1}{2}(T-\mathscr{J}^{\blacktriangle}T\mathscr{J})$ satisfy the relation

$$\mathscr{K}[T_1]=\tfrac{1}{2}\mathscr{K}[T]+\tfrac{1}{2}\mathscr{J}^{\blacktriangle}\mathscr{K}[T]\mathscr{J}-T_2\mathscr{B}T_2\,. \tag{3.6}$$

Moreover, if $T(x)$ has real elements and $U_1(x)$, $U_2(x)$ are $n\times n$ matrices such that $T\equiv\|U_1\ U_2\|$, then $T_1\equiv\mathscr{W}$ where $W\equiv\frac{1}{2}(U_1-iU_2)$, and if T is symmetric then T_1 and T_2 are symmetric and W is hermitian. Consequently, from (3.6) it follows that if condition v'_R holds for a matrix $T(x)$, and hypothesis H_R is satisfied, then for W the hermitian matrix such that $T_1=\mathscr{W}$ the condition v'_R holds for $T=\mathscr{W}$, and condition v_R holds for this matrix W.

From the result of §§ 2 and 3 we have the following result on the character of solutions of (2.4).

THEOREM 3.1. *Under hypotheses* (H_1), (H_2), (H_3) *and* (H_R) *each of the conditions* (H_+), ii, iii, iv, v, iv_R, v_R, iv′, v′, iv'_R *and* v'_R *is a necessary and sufficient condition for* (2.4) *to be non-oscillatory on ab.*

In regard to these various criteria, all but v, v_R, v′ and v'_R are of the general type of condition that has been frequently used in the calculus of variations, and particularly in the extension of the Sturmian theory to self-adjoint systems with real coefficients. A systematic use of conditions of the remaining type is much more recent. For a single differential equation with real coefficients see Wintner [12] and Taam [8] for the use of a Riccati inequality condition. For self-adjoint systems with real coefficients, and of the generality of the accessory equations for a variational problem of Lagrange type, Sternberg [7] has presented criteria of the forms v′ and v'_R.

4. Specific criteria for oscillation and non-oscillation. For (2.4) one may derive various specific criteria for oscillation and non-oscillation that are extensions of known criteria for a scalar second order differential equation. Attention here will be confined to the presentation of a few such criteria that appear of particular interest, either because of their range of application or for the type of proof involved.

Let $G(x)$ be a non-singular $n\times n$ a.c. matrix such that $R(x)G'(x)+Q(x)G(x)\simeq 0$, that is, $G'(x)\simeq A(x)G(x)$. Under the substitution $\eta(x)=G(x)\eta_1(x)$ the integral (2.2) becomes

$$I[\eta]=I_1[\eta_1]=\int_a^b \omega_1(x,\ \eta_1,\ \eta_1')dx\ , \tag{4.1}$$

where $\omega_1(x, \eta, \pi) \equiv \pi^* R_1(x)\pi + \eta^* P_1(x)\eta$, and $P_1 = G^*[P - Q^* R^{-1} Q]G$, $R_1 = G^* R G$ are matrices satisfying the conditions specified for P and R in (H_1) and (H_2). Clearly $\eta(x) \in \mathfrak{L}(\omega)$ if and only if $\eta_1(x) \in \mathfrak{L}(\omega_1)$, and the Euler expression $L_1[\eta]$ for (4.1) satisfies $G^* L[\eta] \simeq L_1[\eta_1]$. In view of these remarks, for the consideration of specific criteria attention will be limited to an equation (2.4) with $Q \equiv 0$; moreover, we shall choose to replace $P(x)$ by $-F(x)$, so that for the specific problem under consideration (2.2) becomes

$$I[\eta] \equiv \int_a^b [\eta^{*\prime} R(x)\eta' - \eta^* F(x)\eta] dx \;, \tag{4.2}$$

where the coefficient matrices satisfy the following hypothesis: (H) $R(x)$ *and* $F(x)$ *are hermitian, of class* $\mathfrak{L}$, *and there exists an hermitian matrix* $R^{-1}(x) \in \mathfrak{L}$ *such that* $R(x)R^{-1}(x) \simeq I$ *on* ab. With the conventions prescribed in § 2, for (4.2) the Euler equation and corresponding Riccati differential operator are, respectively,

$$L[u] \equiv (R(x)u')' + F(x)u \simeq 0 \;, \tag{4.3}$$

$$K[W] \equiv W' + WR^{-1}(x)W + F(x) \;. \tag{4.4}$$

If Δ is a given interval on the x-axis then $\mathrm{H}\{\Delta\}$ will denote the condition that $R(x)$, $F(x)$ satisfy (H) on arbitrary compact subintervals ab of Δ; correspondingly, $\mathfrak{L}\{\Delta\}$ signifies the class of matrices in $\mathfrak{L}$ for each such subinterval, and a.c. $\{\Delta\}$ is the property of absolute continuity on each such subinterval. As a first instance of specific criteria for non-oscillation the following result is presented.

1°. *If* $R(x)$, $F(x)$ *satisfy* $\mathrm{H}\{\Delta\}$ *and* $R(x) > 0$ *a.e. on* Δ, *then each of the following conditions is sufficient for* (4.3) *to be non-oscillatory on* Δ:

(a) *there exists an hermitian matrix* $F_1(x) \in \mathfrak{L}\{\Delta\}$, *and an hermitian matrix* $M(x)$ *that is a.c.* $\{\Delta\}$ *and such that* $M'(x) \simeq F(x) + F_1(x)$, $F_1(x) \geqq M(x)R^{-1}(x)M(x)$ *a.e. on* Δ;

(b) Δ *is a subinterval of* $0 < x < \infty$, *and there exists a constant* $k > 0$ *and an hermitian matrix* $M(x)$ *which is a.c.* $\{\Delta\}$ *and such that* $M'(x) \simeq -xF(x)$, $R(x) \geqq kI$, *and* $k^2 I - 4M^2(x) \geqq 0$ *a.e. on* Δ.

The sufficiency of (a) results from the fact that $W(x) = -M(x)$ satisfies condition v_R of § 2 for the equation (4.3). As special cases of (a) one has the following criteria which for $n=1$ reduce to results given by Wintner [12]:

(a′) Δ: $0 \leqq x \leqq 1$, $R(x) \equiv I$, $M_0(x) \equiv \int_0^x F(s)ds$ *satisfies* $F(x) \geqq 4M_0^2(x)$ *a.e. on* Δ;

(a'') Δ: $0<x<\infty$, $R(x)\equiv I$, $\int_1^\infty F(s)ds=\lim_{x\to\infty}\int_1^x F(s)ds$ *exists and is finite, and the matrix* $M_1(x)\equiv\int_x^\infty F(s)ds$ *satisfies either* (i) $F(x)\geq 4M_1^2(x)$ *a.e. on* Δ, *or* (ii) $-3I\leq 4xM_1(x)\leq I$ *a.e. on* Δ.

Indeed (a') implies (a) with $F_1(x)=F(x)$, $M(x)=2M_0(x)$; similarly, (i) of (a'') implies (a) with $F_1(x)=F(x)$, $M(x)=-2M_1(x)$, and (ii) of (a'') implies (a) with $F_1(x)=(4x^2)^{-1}I$, $M(x)=-M_1(x)-(4x)^{-1}I$.

The sufficiency of (b) is established by noting that $kR^{-1}(x)\leq I$, and consequently $W(x)=(2x)^{-1}[2M(x)+kI]$ satisfies $K[W]\leq(4kx^2)^{-1}[4M^2(x)-k^2I]\leq 0$ a.e. on Δ. The result of (b) corresponds to that of Theorem 5.3 of Sternberg [7] for systems that have real coefficients, but which may be more general than those considered here in the involvement of auxiliary differential equations as restraints.

If Δ is of the form $a\leq x<\infty$ then (4.3) will be said to be *non-oscillatory for large* x on Δ if there is a subinterval Δ_1: $a_1\leq x<\infty$ on which (4.3) is non-oscillatory. For (4.3) one has the following result, which may be established by the same type of argument as used by Sternberg [7] to prove a similar theorem for systems of equations with real coefficients; these results for systems are extensions of a result of Hille [3] for a single linear differential equation of the second order.

2°. *Suppose that* $R(x)$, $F(x)$ *satisfy* $H\{\Delta\}$ *and* $R(x)>0$ a.e. *on* Δ: $a\leq x<\infty$, *and* $\varphi(x)$ *is a positive function of* $\mathfrak{L}\{\Delta\}$ *such that* $\int^\infty\varphi(s)ds$ *is divergent, while* $I\geq\varphi(x)R(x)$ *and* $F(x)\geq 0$ *a.e. on* Δ. *Then* (4.3) *is non-oscillatory for large* x *if and only if* $\int_a^\infty F(s)ds=\lim_{x\to\infty}\int_a^x F(s)ds$ *exists, and there is a subinterval* Δ_1: $a_1\leq x<\infty$ *on which there is defined an hermitian matrix* $W(x)$ *such that* $W(x)\geq 0$, $\int_x^\infty W(s)R^{-1}(s)W(s)ds$ *exists, and*

$$W(x)=\int_x^\infty W(s)R^{-1}(s)W(s)ds+\int_x^\infty F(s)ds\ .$$

Moreover, if (4.3) *is non-oscillatory for large* x *then for each such* $W(x)$ *and arbitrary constant vectors* η *satisfying* $|\eta|=1$,

$$\limsup_{x\to\infty}\left(\int_a^x\varphi(s)ds\right)(\eta^* W(x)\eta)\leq 1\ .$$

If Δ: $0\leq x<\infty$ and $\psi(x)$ is a function belonging to $\mathfrak{L}\{\Delta\}$, then the integral $\int_0^\infty\psi(t)dt$ is said to be summable (C, k), $(k\geq 0)$, to the value λ if

$$\mu[x; k|\psi]\equiv\int_0^x(1-t/x)^k\psi(t)dt\to\lambda \text{ as } x\to\infty\ .$$

As $\Psi(x)=\int_0^x \psi(t)dt$ satisfies the relation $\mu[x; k|\psi]=kx^{-k}\int_0^x (x-t)^{k-1}\Psi(t)dt$ on $0<x<\infty$ for $k>0$, one has immediately that if $\Psi_0(x)=\Psi(x)$, $\Psi_{j+1}(x)=\int_0^x \Psi_j(t)dt$, $(j=1, 2, \cdots)$, then $\mu[x; k|\psi]=(k!)x^{-k}\Psi_k(x)$ on $x>0$ for $k=1, 2, \cdots$. The result of the following theorem for $k=1$ applied to a second order differential equation $u''+f(x)u=0$ provides a criterion for oscillation that extends a result of Wintner [11].

THEOREM 4.1. *If Δ: $0\leqq x<\infty$, the matrices $R(x)$, $F(x)$ satisfy* H$\{\Delta\}$, *and $R(x)>0$ a.e. on Δ, then* (4.3) *is oscillatory on every subinterval $a\leqq x<\infty$ of Δ if there is a nonzero constant vector π and a constant $k\geqq 1$ such that*

$$\limsup_{b\to\infty}\,(\mu[b; 2k|\pi^*F\pi]-b^{-2}k^2\mu[b; 2k-2|\pi^*R\pi])=\infty\ . \tag{4.5}$$

In particular, (4.5) *holds if $\int_0^\infty \pi^*F(t)\pi dt$ is summable $(C, 2k)$ to ∞, and $\pi^*R(x)\pi=0(x)$ as $x\to\infty$.*

For arbitrary $0<a<c<b$ set $\eta(x)=\pi(x-a)/(c-a)$ on ac and $\eta(x)=\pi(b-x)^k/(b-c)^k$ on cb. Since $k\geqq 1$ we have that $\eta(x)$ belongs to the class $\Gamma_0\{\omega_1\}$ for (4.2) on the interval ab, and a direct computation yields

$$I[\eta; a, b]=I'+\left(\frac{b}{b-c}\right)^{2k}(b^{-2}k^2\mu[b; 2k-2|\pi^*R\pi]-\mu[b; 2k|\pi^*F\pi])\ ,$$

where I' is a function of a, c, b, k and π such that for fixed values of a, c, k and π we have $I'\to(c-a)^{-2}I[(x-a)\pi; a, c]+\int_a^c \pi^*F(s)\pi ds$ as $b\to\infty$. Consequently, condition (4.5) implies that as a function of b the integral $I[\eta; a, b]$ has limit inferior equal to $-\infty$ as $b\to\infty$, and hence for b chosen so that $I[\eta; a, b]<0$ it follows from Theorem 2.1 that the interval $a<x\leqq b$ contains a point conjugate to $x=a$. The last statement of the theorem is an immediate consequence of the fact that if $\pi^*R(x)\pi=0(x)$ as $x\to\infty$ then $\mu[b; 2k-2|\pi^*R\pi]=0(b^2)$ as $b\to\infty$.

It is to be commented that if one has the additional condition that $R(x)\in\mathfrak{L}_\infty$ on arbitrary compact subintervals ab of Δ, then for $k>1/2$ it is assured that the above defined vector $\eta(x)$ belongs to $\Gamma_0\{\omega_1\}$ on ab, and the above proof establishes the validity of the result obtained upon replacing "$k\geqq 1$" by "$k>1/2$."

The following criterion for non-oscillation on a compact interval ab is a generalization of a result of Liapounoff for a second order differential equation of the form $u''+f(x)u=0$, (see, for example, Wintner [12]). The proof here presented is based on the variational result of

Theorem 2.3, and for the case of a single equation of the second order differs from earlier proofs of the criterion.

THEOREM 4.2. *Suppose that $R(x)$, $F(x)$ satisfy* (H) *and $R(x)>0$ a.e. on a compact interval ab, and that $\theta(x)$ is a non-negative function of class $\mathfrak{L}$ such that $\theta(x)I\geqq F(x)$ a.e. on ab. If the constant hermitian matrix*

$$D=4\left[\int_a^b R^{-1}(x)dx\right]^{-1}-\left(\int_a^b \theta(x)\,dx\right)I$$

is nonnegative, then (4.3) *is non-oscillatory on ab.*

Suppose that ab contains points x_1, x_2, $(x_1<x_2)$, which are mutually conjugate with respect to (4.3), and let $u(x)$ be a non-identically vanishing solution of this equation satisfying $u(x_1)=0=u(x_2)$. For $\eta(x)=u(x)$ on x_1x_2, $\eta(x)\equiv 0$ elsewhere on ab, let $x=c$ be a value at which $|\eta(x)|$ assumes its maximum value on ab. Then $x_1<c<x_2$, $\eta(x)$ is a solution of (4.3) in a neighborhood of $x=c$, and by Theorem 2.3 relation (2.12) holds as a strict inequality. As

$$\int_a^b \eta^*(x)F(x)\eta(x)dx\leqq\int_a^b \theta(x)|\eta(x)|^2dx\leqq|\eta(c)|^2\int_a^b\theta(x)dx\ ,$$

it then follows that $I[\eta]>\eta^*(c)D\eta(c)\geqq 0$. On the other hand, since $\eta\equiv 0$ outside x_1x_2 and $\eta(x_1)=0=\eta(x_2)$, $L[\eta]\simeq 0$ on x_1x_2, it follows that $I[\eta]=0$, thus presenting a contradiction under the assumption that the hypotheses of the theorem hold and (4.3) is oscillatory on ab.

5. Conditions of non-oscillation for more general equations. In this section we shall consider a vector equation of the form

$$E[u]\equiv A_2(x)u''+A_1(x)u'+A_0(x)u\simeq 0\ , \tag{5.1}$$

under the assumption that on a certain interval $\varDelta$ of the x-axis the matrices $A_0(x)$, $A_1(x)$ belong to $\mathfrak{L}\{\varDelta\}$, while $A_2(x)$ is a.c. $\{\varDelta\}$ and nonsingular. By a solution of (5.1) is understood a vector $u(x)$ which is of class C' on $\varDelta$, with $u'(x)$ a.c. $\{\varDelta\}$, and such that (5.1) is valid on $\varDelta$.

Now if $u(x)$ is a solution of (5.1) for which $u(x_1)=0=u(x_2)$, then

$$0=-\int_{x_1}^{x_2}u^*E[u]dx=\int_{x_1}^{x_2}(u^{*\prime}A_2u'+u^*[A_2'-A_1]u'-u^*A_0u)dx\ . \tag{5.2}$$

For the right-hand integral in (5.2) we shall introduce the notation $I[u;x_1,x_2]$, and set $R_0(x)=\frac{1}{2}[A_2(x)+A_2^*(x)]$, $R_1(x)=\frac{1}{2}i[A_2^*(x)-A_2(x)]$, $P_0(x)=-\frac{1}{2}[A_0(x)+A_0^*(x)]$, $P_1(x)=-\frac{1}{2}i[A_0^*(x)-A_0(x)]$, $Q_0(x)=\frac{1}{2}[A_2^{*\prime}-A_1^*(x)]$, $Q_1(x)$

$=iQ_0(x)$. It follows readily that $I[u; x_1, x_2]=I_0[u; x_1, x_2]+iI_1[u; x_1, x_2]$, where

$$I_\beta[u; x_1, x_2]=\int_{x_1}^{x_2}\omega_\beta(x, u, u')dx\ , \qquad (\beta=0, 1)$$

and $\omega_\beta(x,\eta,\pi)$ is the hermitian form $\pi^*R_\beta(x)\pi+\pi^*Q_\beta(x)\eta+\eta^*Q_\beta^*(x)\pi+\eta^*P_\beta(x)\eta$. Moreover, for λ_0, λ_1 real and $R(x;\lambda)=\lambda_0R_0(x)+\lambda_1R_1(x)$, $Q(x;\lambda)=\lambda_0Q_0(x)+\lambda_1Q_1(x)$, $P(x;\lambda)=\lambda_0P_0(x)+\lambda_1P_1(x)$, and $\omega(x,\eta,\pi;\lambda)=\lambda_0\omega_0(x,\eta,\pi)+\lambda_1\omega_1(x,\eta,\pi)$ we have

$$0=I[u; x_1, x_2; \lambda]\equiv\int_{x_1}^{x_2}\omega(x, u, u'; \lambda)dx\ . \tag{5.3}$$

In particular, from Theorem 2.1 we have that (5.1) *is non-oscillatory on Δ if for each compact subinterval ab of Δ there are real constants λ_0, λ_1 such that $R(x;\lambda)>0$ on ab, and the self-adjoint equation*

$$L[u;\lambda]\equiv(R(x,\lambda)u'+Q(x,\lambda)u)'-(Q^*(x,\lambda)u'+P(x,\lambda)u)\simeq 0 \tag{5.4}$$

is non-oscillatory on this subinterval. Of special interest is the case in which the above conditions are satisfied by a choice of λ_0, λ_1 that is independent of the subinterval ab.

It is to be commented that for the consideration of (5.1) one may assume without loss of generality that $A_2(x)$ is positive hermitian on Δ, as this property holds for $B(x)E[u]=0$, where $B(x)$ is a.c. $\{\Delta\}$ and such that $B(x)A_2(x)$ is positive hermitian; in particular, these conditions hold for $B(x)=A_2^{-1}(x)$ and $B(x)=A_2^*(x)$. If $A_2(x)$ is positive hermitian then $R(x,\lambda)\equiv\lambda_0A_2(x)$, and in the application of the above criterion for non-oscillation one may assume without loss of generality that $\lambda_0=1$.

It is to be remarked that all the general criteria for non-oscillation of a single second-order differential equation considered by Taam ([9] and [10]) are of the type discussed above, and for most cases in which specific criteria of non-oscillation occur in his treatment these criteria are of the sort for which 1° of § 4 provides a generalization to the case of systems herein considered.

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