COMPLETELY MONOTONIC FUNCTIONS ON CONES

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1. Introduction. A function $f(x)$, $0 \leq x < \infty$, is said to be completely monotonic on $0 \leq x < \infty$ if $(-1)^n f^{(n)}(x) \geq 0$ for $0 < x < \infty$ and $f(0) = f(0+)$. A similar and equivalent definition involving differences is available. A fundamental theorem regarding such functions, proved (independently) by Hausdorff, Bernstein and Widder, states that they are the class of Laplace-Stieltjes transforms of bounded monotone functions. Several of the many known proofs are given in Widder [3], which also gives references for other proofs. The corresponding theorem for two dimensions has been proved by Schoenberg [2]. It is not difficult to construct a proof for n-dimensions along the lines of the original proof of Hausdorff and in the process establish the equivalence of the corresponding derivative and difference criteria.

In this note we wish to introduce a class of functions, defined on n-dimensional polyhedral cones with vertex at the origin, which we call completely monotonic (A), and, in analogy with the theorem of Hausdorff-Bernstein-Widder, show that they are the Laplace-Stieltjes transforms of bounded monotone functions on the "conjugate space" $t = (t_1, \cdots, t_n)$ with $\sum_{i=1}^{n} x_i t_i \geq 0$. We then show that a function completely monotonic (A) on each of a set of overlapping cones may be represented by a single integral, which may then be used to extend the function to the convex closure of the set of cones. Lastly, we show by an example that a function may be completely monotonic along every line with nonnegative slope in the first quadrant without being completely monotonic as a function of two variables.

2. Functions completely monotonic on cones. We commence with some notations and definitions. We shall write x in place of $(x_1, \cdots, x_n)$, xt' in place of $(x_1 t_1 + \cdots + x_n t_n)$, and where these appear in integrands we shall use a single integral sign to denote a multiple integral.

For a given convex cone D, D^* will be the set of all t such that $\sum_{i=1}^{n} x_i t_i \geq 0$ for all x in D. By an n-cone we shall mean a convex cone in E_n spanned by n linearly independent vectors $x^i = (x_1^i, \cdots, x_n^i)$, and such that there is a hyperplane having only the origin in common with

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the cone. We shall say that $\{D_\sigma\}$, $\sigma \in S$, is a collection of overlapping n-cones if it is impossible to divide the index set S into subsets S' and S'', $S=S' \cup S''$, so that $\bigcup_{S'} D_\sigma$ and $\bigcup_{S''} D_\sigma$ as point sets in E_n have only the origin in common.

Let $f(x)$ be defined on an n-cone D and be continuous on the boundary. Then $f(x)$ will be said to be completely monotonic (A) on D if

$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \binom{m_1}{i_1} \cdots \binom{m_n}{i_n} (-1)^{i_1+\cdots+i_n} f(x+i_1\delta_1 x^1 + \cdots + i_n \delta_n x^n) \geqq 0$$

for any x in D and any $\delta_i \geqq 0$. If D is the positive orthant, a function completely monotonic (A) on D is completely monotonic in the ordinary sense.

For reference purposes we now state the ordinary form of the Hausdorff-Bernstein-Widder Theorem for several variables. A proof paralleling the proof given for one dimension in Widder [3], p. 162, is not difficult.

THEOREM 2.1. *A necessary and sufficient condition that $f(x)$ be completely monotonic on $0 \leqq x_i < \infty$, $i=1, 2, \cdots, n$, is that*

$$f(x) = \int_0^\infty e^{-xt'} d\varphi(t) \ ,$$

where $\varphi(t)$ is bounded and monotone (in the sense of [1]) and the integral is convergent for $0 \leqq x_i$, $i=1, 2, \cdots, n$. $\varphi(t)$ is essentially unique.

From this we proceed to the corresponding theorem for n-cones.

THEOREM 2.2. *Let D be an n-cone. Then a necessary and sufficient condition that a function $f(x)$ be completely monotonic (A) on D is that*

$$f(x) = \int_{D^*} e^{-xt'} d\varphi(t) \ , \tag{I}$$

where $\varphi(t)$ is bounded and monotone in D^*, and the integral is convergent for x in D. $\varphi(t)$ is essentially unique.

Proof. Suppose that $f(x)$ is completely monotonic (A) on an n-cone D. Let T be a linear transformation which carries D into the positive orthant P. Let $g(x)=f[T^{-1}(x)]$. Then $g(x)$ is completely monotonic, since the differences taken along lines parallel to the edges of the cone are transformed into differences taken parallel to the axes. Then

$$g(x) = \int_0^\infty e^{-xt'} d\psi(t) \ ,$$

where ψ is bounded and monotone, by Theorem 2.1. Let U be the linear transformation on t such that $\sum_{i=1}^{n} x_i t_i = \sum_{i=1}^{n} x_i^\circ t_i^\circ$, where $x^\circ = T(x)$ and $t^\circ = U(t)$. For any set S in the t domain let $\varphi(S) = \psi[U^{-1}(S)]$. $\varphi(t)$ is clearly monotone and so (I) holds.

Suppose, on the other hand, that we have (I) with $\varphi(t)$ bounded and monotone in D^*. We use a linear transformation T to carry D onto the positive orthant and a U, defined as above, to carry D^* onto P^*. We then have a function $g(x)$ defined on P and equal there to

$$\int_0^\infty e^{-xt'} d\psi(t)$$

for a bounded monotone $\psi(t)$. The function $g(x)$ is thus completely monotonic, from Theorem 2.1, and this property will carry over into complete monotonicity (A) for $f(x)$ when we apply T^{-1}. This completes the proof of the theorem.

We now consider functions which are completely monotonic (A) on each of a collection of overlapping n-cones.

THEOREM 2.3. *Suppose that $f(x)$ is completely monotonic (A) on each of a collection $\{D_\sigma\}$, $\sigma \in S$, of overlapping n-cones. Suppose also that if the intersection of any two cones in $\{D_\sigma\}$ contains a point other than the origin it contains an open set. Then all of the $\varphi_\sigma(t)$ as defined in Theorem 2.2 are equal, and are zero outside $(\bigcup_S D_\sigma)^*$.*

Proof. We note that always $(\bigcup D_\sigma)^* = \bigcap D_\sigma^*$. To begin with, suppose that D consists of two cones, D_1 and D_2. Consider a point x in their intersection. From Theorem 2.2

$$f(x) = \int_{D_1^*} e^{-xt'} d\varphi_1(t) \, .$$

This may be extended to an integral over $D_1^* \cup D_2^*$ by defining $\varphi_1(t) = 0$ for t in $D_2^* - D_1^*$. At the same time

$$f(x) = \int_{D_2^*} e^{-xt'} d\varphi_2(t) \, ,$$

and $\varphi_2(t)$ can likewise be defined to be zero outside D_1^*. Since both of these representations are valid in an n-cone contained in $D_1 \cap D_2$, $\varphi_1(t) = \varphi_2(t)$ by the uniqueness condition.

Consider the general case, and suppose the theorem to be false. Then there are two cones, D_1 and D_2, say, such that $\varphi_2(t)$ is not zero somewhere outside D_1^*. Let the collection of cones $\{D_\sigma\}$, $\sigma \in S'$, be those for which φ_σ is zero outside D_1^*; let the other cones form $\{D_\sigma\}$, $\sigma \in S''$.

Neither S' nor S'' is empty, $S'\cup S''=S$, and $S'\cap S''=\phi$. Thus $\bigcup_{S'} D_\sigma$ and $\bigcup_{S''} D_\sigma$ have a point other than the origin in common, and there is a $D_{\sigma'}$, $\sigma'\in S'$, and a $D_{\sigma''}$, $\sigma''\in S''$, whose intersection contains an open set. By the first part of the theorem $\varphi_{\sigma'}(t)=\varphi_{\sigma''}(t)$, and this is a contradiction.

It is known that D^{**} is the convex closure of D, where D is any (possibly non-convex, non-polyhedral) cone. Also,

$$\int_{D^*} e^{-xt'}d\varphi(t)\leqq\int_{D^*}d\varphi(t)$$

for any x in D^{**}. Thus we may use the integral representation to extend a function of the sort described in Theorem 2.3 to the convex closure of $\bigcup_S D_\sigma$. We state this as a corollary.

COROLLARY. *Suppose $f(x)$ is completely monotonic (A) on each of a set $\{D_\sigma\}$ of n-cones as in Theorem 2.3. Then $f(x)$ may be continued to the convex closure K of $\bigcup_S D_\sigma$ in such a fashion that it will be completely monotonic (A) on any n-cone in K.*

3. **Functions completely monotonic on lines.** Using Theorem 2.3 we can deduce complete monotonicity (A) on large cones from complete monotonicity (A) on small cones. The conclusion is false, however, if the small cones are replaced by lines. In fact we can exhibit a function completely monotonic along any line with suitable slope, through the origin or not, which fails to be completely monotonic in several variables. For the sake of simplicity we will discuss an example in two dimensions. Consider the function

$$\varphi(t_1, t_2)=\begin{cases}1 \text{ for } (0\leqq t_1\leqq 3,\ 0\leqq t_2\leqq 3) \text{ except for } (1\leqq t_1\leqq 2,\ 1\leqq t_2\leqq 2)\\ -1 \text{ for } (1\leqq t_1\leqq 2,\ 1\leqq t_2\leqq 2)\\ 0 \text{ otherwise.}\end{cases}$$

Let

$$f(x_1, x_2)=\int_0^\infty\int_0^\infty e^{-t_1x_1-t_2x_2}\varphi(t_1, t_2)dt_1dt_2 .$$

By Theorem 2.1 $f(x_1, x_2)$ cannot be completely monotonic. Let

$$g_\theta(x_1', x_2')=f(x_1'\cos\theta-x_2'\sin\theta,\ x_2'\cos\theta+x_1'\sin\theta);$$

that is, rotate the axes through an angle θ. We will now show that $g_\theta(x_1', x_2')$ is completely monotonic on $0\leqq x_1'<\infty$ for any fixed value of x_2' and any $0\leqq\theta\leqq\pi/2$. To this end let

$$u_1=t_1\cos\theta+t_2\sin\theta \text{ and } u_2=-t_1\sin\theta+t_2\cos\theta ,$$

so that

$$g_\theta(x_1', x_2') = \iint e^{-x_1'u_1 - x_2'u_2}\psi(u_1, u_2)du_1du_2 ,$$

where $\psi(u_1, u_2)$ is zero outside a rotated square. We can replace the multiple integral by a repeated integral:

$$g_\theta(x_1', x_2') = \int_0^{3\cos\theta + 3\sin\theta} e^{-x_1'u_1}du_1 \int e^{-x_2'u_2}\psi(u_1, u_2)du_2 .$$

Now the inner integral will always be positive, because any line which intersects the square on which ψ is not zero will have a greater length in the positive region than in the negative, and can thus make only a positive contribution to the integral. Since the inner integral is positive, $g_\theta(x_1', x_2')$ must be completely monotonic from the one dimensional version of Theorem 2.1.

If θ is allowed outside the interval $0 \leqq \theta \leqq \pi/2$, the inner integral will remain positive but the range of the outer integral will extend outside $0 \leqq u_1 < \infty$. The function $g_\theta(a+b, c+d)$ will then be a "kernel of positive type" in a and b for fixed c and d (and vice versa), as discussed in Chapter VI, §§ 20–21 of Widder [3]. If the square upon which $\varphi(t_1, t_2)$ is nonzero is moved sufficiently far from the origin along the line $t_1 = t_2$ the corresponding $g_\theta(x_1', x_2')$ may be made completely monotonic for $0 \leqq x_1' < \infty$ for $-\pi/4 + \delta \leqq \theta \leqq 3\pi/4 - \delta$ for any small positive δ and any fixed x_2'.

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