REGULAR REGIONS FOR THE HEAT EQUATION

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1. Introduction. Let R be a region (open connected set) in the plane or in space ($x=[x_1, x_2]$ or $x=[x_1, x_2, x_3]$). We will say that R is a regular region for Laplace's equation

$$\Delta u=0 \tag{1}$$

if the Dirichlet problem for R always has a solution for continuous data. By this we mean: given a function $\psi(\xi) \in C$ (that is, continuous) for $\xi \in B$, the boundary of R, there is a unique function $u(x) \in C$ for $x \in \overline{R}=R \cup B$, for which

$$\Delta u=0 \qquad x \in R,$$

$$u(\xi)=\psi(\xi) \qquad \xi \in B.$$

We will further say that R is regular for the heat equation

$$\Delta u=u_t \tag{2}$$

if the "Dirichlet problem" for the heat equation has a solution for continuous data, that is, if for each

$$\phi(x) \in C \qquad x \in \overline{R}$$

and

$$\psi(\xi, t) \in C \qquad \xi \in B, \ t \geqq 0$$

where

$$\phi(\xi)=\psi(\xi, 0)$$

there is a unique function $u(x, t) \in C$, for $x \in \overline{R}$, $t \geqq 0$ for which

$$\Delta u=u_t \qquad x \in R, \ t>0$$

$$u(x, 0)=\phi(x) \qquad x \in \overline{R}$$

$$u(\xi, t)=\psi(\xi, t) \qquad \xi \in B, \ t \geqq 0 .$$

Tychonoff [4] has shown that if R is bounded and regular for

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Laplace's equation, then it is regular for the heat equation and conversely. We give here a new proof that regularity for Laplace's equation implies regularity for the heat equation.

2. The work of Tychonoff. In the first half of the memoir cited above, Tychonoff proves the following three theorems.

A. *Each bounded region which is a regular region for the heat equation is also regular for Laplace's equation.*

B. *Each bounded regular region for the equation* $\Delta u=\bar{\lambda}u$ *for a certain* $\bar{\lambda}\geqq 0$ *is also regular for the equation* $\Delta u=\lambda u$ *for arbitrary* $\lambda\geqq 0$.

C. *Each bounded region which is regular for all the equations* $\Delta u=\lambda u$ *for* $\lambda\geqq\lambda_0$ *is also regular for the heat equation.*

This cycle of theorems shows the equivalence of regular regions for the equations $\Delta u=0$, $\Delta u=\lambda u$ $(\lambda\geqq 0)$, and $\Delta u=u_t$.

In the proof of B Tychonoff observes that the solution of the boundary value problems

$$\begin{aligned} &\Delta u-\lambda u=0 \qquad && x\in R \\ &u(\xi)=\psi(\xi) \qquad && \xi\in B \end{aligned}$$

is equivalent to the solution of the integral equations

$$u=(\lambda-\bar{\lambda})\int_R G(x,\, y)u(y)dy+w(x)$$

where G is Green's function for the region R for the equation $\Delta u=\bar{\lambda}u$, and $w(x)$ is the solution to the problem

$$\begin{aligned} &\Delta w-\bar{\lambda}w=0 \qquad && x\in R \\ &w(\xi)=\psi(\xi) \qquad && \xi\in B\,. \end{aligned}$$

The existence of both w and G are guaranteed by the assumption that R is regular for $\Delta u=\bar{\lambda}u$. He then deduces, via the Hilbert-Schmidt theory, that the desired solutions of the integral equations exist and hence these solve the boundary value problems.

However, in establishing C, he forsakes his integral equation methods and bases his argument on a refinement of a differential-difference method due to Rothe [2].

We may note that to complete the cycle of theorems it is sufficient to prove that if R is regular for $\Delta u=0$ it is regular for $\Delta u=u_t$, and we give here a proof of this result using a modification of the integral equation argument mentioned above.

In our argument we will use the following theorem which was indicated in a footnote in the paper by Tychonoff. For the sake of completeness we present the proof.

D. *Let R be a regular region for $\Delta u=0$, and let $\psi(\xi, t)$ be defined on B and be k times differentiable with respect to t, $0 \leqq t < T \leqq \infty$, and let ψ and each of its k derivatives respect to t be continuous for $\xi \in B$, $0 \leqq t < T$. Further, let $u(x, t)$ be the solution to the problem*

$$\Delta u(x, t)=0 \qquad x \in R$$

$$u(\xi, t)=\psi(\xi, t), \qquad \xi \in B,\ 0 \leqq t < T .$$

Then $u(x, t)$ has k continuous derivatives with respect to t and

$$v=\frac{\partial^j u}{\partial t^j}, \quad 0 \leqq j \leqq k ,$$

solves the problem

$$\Delta v(x, t)=0 \qquad x \in R$$

$$v(\xi, t)=\frac{\partial^j}{\partial t^j}\psi(\xi, t), \quad \xi \in B,\ 0 \leqq t < T .$$

Proof. Choose t_0, $0 \leqq t_0 < T$. By the maximum and minimum principles for harmonic functions

$$|u(x, t)-u(x, t_0)| \leqq \max_{\xi \in B} |\psi(\xi, t)-\psi(\xi, t_0)| .$$

But by the uniform continuity of $\psi(\xi, t)$ for $\xi \in B$, and t in a (sufficiently small) closed t interval about t_0, this maximum tends to zero as t tends toward t_0. So that $u(x, t)$ is continuous in t.

Since R is a regular region for $\Delta u=0$ there is a solution to the problem

$$\Delta v(x, t)=0 \qquad x \in R$$

$$v(\xi, t)=\frac{\partial}{\partial t}\psi(\xi, t), \quad \xi \in B,\ 0 \leqq t T .$$

Then

$$\left|\frac{u(x, t)-u(x, t_0)}{t-t_0}-v(x, t)\right| \leqq \max_{\xi \in R}\left|\frac{\psi(\xi, t)-\psi(\xi, t_0)}{t-t_0}-\frac{\partial}{\partial t}\psi(\xi, t_0)\right|$$

by the same argument used above. But

$$\frac{\psi(\xi, t)-\psi(\xi, t_0)}{t-t_0}=\frac{\partial \psi}{\partial t}(\xi, \bar{t}(\xi)) ,$$

where $\bar{t}(\xi)$ lies between t and t_0. Again by the uniform continuity of

$\frac{\partial\psi}{\partial t}(\xi,,t)$ this maximum vanishes as t tends toward t_0. Hence $u(x, t)$ is differentiable with respect to t and this derivative attains the continuous boundary data $\frac{\partial\psi}{\partial t}(\xi, t)$. Hence by the first part of the proof $\frac{\partial u}{\partial t}(x, t)$ is continuous in t. By iterating this argument k times the proof is completed.

We will need the following, also taken from Tychonoff.

E. *Let R be bounded and regular for $\Delta u=0$, and let $G(x, y)$ be the Green's function for this equation and this region:*

$$G(x, y)=\begin{cases}\frac{1}{2\pi}\log\frac{1}{r_{xy}}-g(x, y) & n=2\\[2ex] \frac{1}{4\pi}\cdot\frac{1}{r_{xy}}-g(x, y) & n=3\end{cases}$$

where $g(x, y)$ is the solution to the problem

$$\Delta_x g(x, y)=0 \qquad x\in R,\ y\in R$$

$$g(\xi, y)=\begin{cases}\frac{1}{2\pi}\log\frac{1}{r_{\xi y}} & \xi\in B,\ y\in R,\ n=2\\[2ex] \frac{1}{4\pi}\cdot\frac{1}{r_{\xi y}} & \xi\in B,\ y\in R,\ n=3\,.\end{cases}$$

Then $G(x, y)=G(y, x)$, $x\in R$, $y\in R$.

Proof. Let R_j be a sequence of regions, $\overline{R}_j\subset R_{j+1}\subset R$, which tend to R with the property that the corresponding boundaries B_j are surfaces having continuous curvature and such that the distance from each point on B_j to B is not greater than δ_j where the sequence $\delta_j\to 0$ as $j\to\infty$. For such a construction see Kellog [1].

Let $G_j(x, y)$ be the Green's function for R_j. Under the hypotheses on R_j it is well known that $G_j(x, y)$ is symmetric (see Tamarkin and Feller [3]). It is therefore sufficient to prove that

$$\lim_{j\to\infty} G_j(x, y)=G(x, y)\,.$$

To this end we note that $G\geqq 0$: since it vanishes on B and is large and positive near the pole y it must be nonnegative by the minimum principle.

Let $\varepsilon>0$ be given, then if j is sufficiently large we have $0\leqq G(x, y)\leqq\varepsilon$ for each point $x\in\overline{R}-R_j$, and in particular on B_j. Hence

$$0 \leqq G(x, y) - G_j(x, y) = g(x, y) - g_j(x, y) < \varepsilon$$

everywhere in R_j since that inequality is true on B_j. This completes the argument.

3. Reduction of the data. We return now to the problem

$$\Delta u = u_t \qquad x \in R,\ t > 0$$

$$u(x, 0) = \phi(x) \qquad x \in \overline{R}$$

$$u(\xi, t) = \psi(\xi, t) \qquad \xi \in B,\ t > 0$$

under the assumption that R is regular for $\Delta u = 0$. We show that $\phi(x)$ may be assumed to be zero. Let R' be a sphere (or circle) containing $\overline{R}$ in its interior, and let $\phi'(x)$ be a continuous bounded extension of $\phi(x)$ into R'. Define

$$u_1(x, t) = \int_{R'} k(x-y, t)\phi'(y)dy\ ,$$

dy being the element of area or volume, and $k(x, t)$ being the fundamental solution

$$k(x, t) = (4\pi t)^{-n/2} \exp [\|x\|^2/4t]$$

where

$$\|x\|^2 = \sum_{j=1}^{n} x_j^2 \qquad n = 2, 3.$$

If $u(x, t)$ be the solution to our problem, the function

$$v(x, t) = u(x, t) - u_1(x, t)$$

solves the problem

$$\Delta v = v_t \qquad x \in R,\ t > 0$$

$$v(x, 0) = 0 \qquad x \in \overline{R}$$

$$v(\xi, t) = \psi(\xi, t) - u_1(\xi, t), \qquad \xi \in B,\ t \geqq 0$$

and

$$v(\xi, t)|_{t=0} = \psi(\xi, 0) - u_1(\xi, 0) = \psi(\xi, 0) - \phi(\xi) = 0.$$

4. The integral equations. We study now the problem

$$\Delta u = u_t \qquad x \in R,\ t > 0$$

$$u(x, 0) = 0 \qquad x \in \overline{R}$$

$$u(\xi,\ t)=\psi(\xi,\ t) \qquad \xi\in B,\ t\geqq 0$$

with

$$\psi(\xi,\ 0)=0, \qquad \xi\in B\ .$$

Since R is assumed regular for $\Delta u=0$, let $\bar{u}(x,\ t)$ be the solution to the problem

$$\Delta\bar{u}(x,\ t)=0 \qquad x\in R$$

$$\bar{u}(\xi,\ t)=\psi(\xi,\ t), \qquad x\in R$$

Also since R is regular for $\Delta u=0$, the Green's function $G(x,\ y)$ exists and is symmetric function by E, and if $f(x)$ is differentiable the function

$$g(x)=-\int_R G(x,\ y)f(y)dy$$

solves the problem

$$\Delta g=f(x) \qquad x\in R$$

$$g(\xi)=0 \qquad \xi\in B\ .$$

(See Tamarkin and Feller [3]). Hence if $u(x,\ t)$ be the solution to our problem it must also satisfy the integral equation

$$u(x,\ t)=\bar{u}(x,\ t)-\int_R G(x,\ y)\frac{\partial}{\partial t}u(y,\ t)dy\ . \tag{3}$$

Conversely any solution of our integral equation which is differentiable in x (and which attains the proper initial values) must also solve our problem.

We apply the Laplace transform: let

$$\mathscr{L}\{u(x,\ t)\}=w(x,\ s),\quad \mathscr{L}\{\bar{u}(x,\ t)\}=v(x,\ s),$$

so that (3) becomes

$$w(x,\ s)=v(x,\ s)-s\int_R G(x,\ y)w(y,\ s)ds \tag{4}$$

which is a Fredholm integral equation with a symmetrical kernel $-G(x,\ y)$.

5. Restricted solution of the problem. To facilitate the solution of our integral equations (3) and (4) we make additional restrictions which will be removed later. We assume

(i) there exists $T>0$ such that $\psi(\xi,\ t)=0$ for $t>T$.

(ii) $\psi(\xi, t)$ in addition to being continuous with respect to (ξ, t), has four derivatives with respect to t which are also continuous with respect to (ξ, t) and

$$\psi_t(\xi, 0)=\psi_{tt}(\xi, 0)=\psi_{ttt}(\xi, 0)=0 , \quad \xi \in B .$$

From D it follows that $\bar{u}(x, t)$ has four continuous derivatives with respect to t; and

$$\bar{u}_t(x, 0)=\bar{u}_{tt}(x, 0)=\bar{u}_{ttt}(x, 0)=0$$

for $x \in R$, by the maximum principle. From (i) it follows that

$$\bar{u}(x, t)=0, \quad \text{for} \quad t>T, \quad x \in \bar{R}.$$

Since $-G(x, y)$ is symmetric in (x, y) it follows that the eigenvalues of our problem are all real and in fact it is well known that they are all negative. (See for example, Tamarkin and Feller [3]).

The solution of (4) is

$$w(x, s)=v(x, s)+\sum_n \frac{sv_n(s)}{\lambda_n - s}\phi_n(x) , \tag{5}$$

where $\phi_n(x)$ are the eigenfunctions for the kernel $-G(x, y)$ and where

$$v_n(s)=\int_R \phi_n(x)v(x, s)dx .$$

We must now invert the Laplace transform and show that $\mathscr{L}^{-1}\{w(x, s)\}$ is the solution to our restricted problem. To this end we examine some of the properties of $w(x, s)$. We begin with an examination of $v(x, s)$.

By its definition we have

$$v(x, s)=\int_0^\infty e^{-st}\bar{u}(x, t)dt ,$$

the integral being uniformly and absolutely convergent for $x \in \bar{R}$, and $\mathscr{R}\, s \geqq 0$. In fact any of the x derivatives of v can be computed under the integral sign, since the resulting integral is uniformly and absolutely convergent for $\mathscr{R}\, s \geqq 0$ and x in any closed sub-domain of R. So that, in particular,

$$\Delta v(x, s)=\int_0^\infty e^{-st}\Delta \bar{u}(x, t)dt=0 .$$

Furthermore $v(x, s)$ is analytic for $\mathscr{R}\, s>0$, and bounded for $\mathscr{R}\, s \geqq 0$, and by integrating by parts, under of course the restrictions (i) and (ii) we get

$$v(x, s)=\frac{1}{s^4}\int_0^\infty \bar{u}_{tttt}(x, t)e^{-st}dt .$$

From this we see that

$$|v(x, s)|\leqq K_1/|s^4| , \qquad \mathscr{R}\, s\geqq 0,\ x\in \bar{R}$$

which is of interest only for large $|s|$ since $v(x, s)$ is bounded.

Since

$$w(x, s)=v(x, s)-\sum_n \frac{sv_n(s)}{(s/\lambda_n)-1}\cdot\frac{\phi_n(x)}{\lambda_n}$$

we get

$$|w(x, s)|\leqq|v(x, s)|+\left[\sum_n \frac{|s|^2|v_n(s)|^2}{|(s/\lambda_n)-1|^2}\cdot\sum_n\frac{\phi_n^2(x)}{\lambda_n^2}\right]^{1/2}$$

Now $\lambda_n\leqq 0$ so that $|(s/\lambda_n)-1|\geqq 1$, and hence

$$|w(x, s)|\leqq v(x, s)+|s|\left[\int_R |v(x,\ s)|^2\,dx\cdot\int_R G^2(x, y)dy\right]^{1/2} .$$

But $\int_R G^2(x, y)dy$ is bounded since G is continuous except for a singularity at x like $\log\|x-y\|$ or $1/\|x-y\|$, as the case may be. Hence

$$|w(x, s)|\leqq \frac{K_1}{|s|^4}+\frac{K_2}{|s|^3}\leqq\frac{K_3}{|s|^3} \quad \text{for } |s|\geqq 1$$

uniformly for $x\in R$, $\mathscr{R}\, s\geqq 0$, and

$$|w(x, s)|\leqq K_4, \qquad |s|\leqq 1,\ \mathscr{R}\, s\geqq 0$$

since $v(x, s)$ is bounded there.

Hence $w(x, s)$ is also bounded for all $x\in \bar{R}$, $\mathscr{R}\, s\geqq 0$, and for large $|s|$,

$$w(x, s)=O(1/|s|^3)$$

uniformly for $x\in \bar{R}$.

The inverse transform

$$(6) \qquad u(x, t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} w(x, s)e^{ts}ds \qquad \sigma>0$$

exists, and since e^{st} is bounded and $w(x, s)=O(1/|s|^3)$ converges uniformly. Also

$$\frac{\partial u}{\partial t}(x,\ t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} sw(x,\ s)e^{ts}ds\ ,$$

since the integral converges uniformly.

Since $w(x, s)$ satisfies (4), by applying the inverse transform to each side we are led back to (3), the integration under the integral sign being permissible by the uniform convergence of the integrals involved. Hence $u(x, t)$ as given by (6) where $w(x, s)$ is given by (5) is the solution to the integral equation (3), and as such is a solution to the heat equation in R and attains the proper boundary conditions. Let us examine the initial values of $u(x, t)$:

$$u(x,\ 0)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} w(x,\ s)ds\ , \qquad \sigma>0,\ x\in \overline{R}$$

$$|u(x,\ 0)|\leqq\frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}|w(x,\ s)|\cdot|ds|\ ,$$

$$\leqq\frac{K_3}{2\pi}\int_{-\infty}^{\infty}\frac{d\tau}{(\sigma+i\tau)^3}=\frac{K_1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\frac{dv}{|1+iv|^3}$$

which tends to zero as σ becomes infinite. Hence $u(x, 0)=0$, $x\in\overline{R}$. This completes the solution in the restricted case.

6. Removal of the restrictions. We first remove the restrictions (ii).

Let $\psi(\xi, t)$ be continuous, $\xi\in B$, $t\geqq 0$, with $\psi(\xi, 0)=0$, $\xi\in B$, and $\psi(\xi, t)=0$, $t>T$. By the Weierstrass approximation theorem there is a polynomial $p_n(\xi, t)$ such that

$$|\psi(\xi,\ t)-p_n(\xi,\ t)|<1/4n\ , \qquad \xi\in B,\ 0\leqq t\leqq T.$$

By the uniformity of the continuity of $\psi(\xi, t)$ there exists t_n, t_n' such that

$$|\psi(\xi,\ t)|<1/4n,\ \text{for}\begin{cases}0\leqq t\leqq t_n, & \xi\in B\\ t_n'\leqq t\leqq T, & \xi\in B\end{cases}$$

and without loss of generality we may, assume $t_n<1/2n$ and $T-t_n'<1/2n$.

Let $q_n(t)\in C^5$ $0\leqq t$, increase from 0 to 1 as t increases from 0 to t_n and be identically 1 for $t_n\leqq t\leqq t_n'$ and decrease to zero again at $t=T$, and have four vanishing derivatives at $t=0$ and at $t=T$.

Now let $\psi_n(\xi, t)=q_n(t)p_n(\xi, t)$. This function is an admissible boundary function under the restricted proof, which we have already completed. Hence for each n there is a solution $u_n(x, t)$ of the heat equation assuming these boundary values and of course zero initial values. To show that this sequence converges to the solution to our present problem we consider first

$$|\psi(\xi, t)-\psi_n(\xi, t)|=|\psi(\xi, t)-p_n(\xi, t)|<\frac{1}{4n}$$

for $t_n<t<t'_n$. For $0\leqq t\leqq t_n$ and $t'_n\leqq t\leqq T$,

$$|\psi(\xi, t)-\psi_n(\xi, t)|\leqq|\psi(\xi, t)|+|\psi_n(\xi, t)|$$

$$\frac{1}{4n}+|q_n(t)|\cdot|p_n(\xi, t)|\leqq\frac{1}{4n}+|p_n(\xi, t)|,$$

but

$$|p_n(\xi, t)|\leqq|p_n(\xi, t)-\psi_n(\xi, t)|+|\psi_n(\xi, t)|\leqq\frac{1}{4n}+\frac{1}{4n}=\frac{1}{2n},$$

so that

$$|\psi(\xi, t)-\psi_n(\xi, t)|<\frac{1}{2n}, \qquad 0\leqq t\leqq T,$$

and consequently

$$|\psi_n(\xi, t)-\psi_m(\xi, t)|\leqq\frac{1}{\min(m, n)}, \qquad 0\leqq t\leqq T.$$

For $x\in\overline{R}$, $0\leqq t\leqq T$

$$u_n(x, t)-u_m(x, t)$$

is a solution of $\Delta u=u_t$ in R and continuous for $x\in\overline{R}$, $0\leqq t\leqq T$. Hence by the maximum and minimum principles for the heat equation this function attains its maximum and its minimum on the bottom or lateral parts of the space time cylinder defined by $x\in\overline{R}$, $0\leqq t\leqq T$

It follows that

$$|u_n(x, t)-u_m(x, t)|\leqq\max_{\xi\in B,\ 0\leqq t\leqq T}|\psi_n(\xi, t)-\psi_m(\xi, t)|\leqq\frac{1}{\min(m, n)}$$

from which the uniform convergence of the sequence $u_n(x, t)$ in the cylinder is clear. The limit function, $u(x, t)$, clearly attains the proper initial values, since each of the approximating functions does. And for $\xi\in B$,

$$u(\xi, t)=\lim_{n\to\infty}u_n(\xi, t)=\lim_{n\to\infty}\psi_n(\xi, t)=\psi(\xi, t),$$

so that $u(x, t)$ is the solution to our problem under the restriction (i).

Consider now any $\psi(\xi, t)$, continuous for $\xi \in B$, $t \geqq 0$, which vanishes for $t=0$. Then let

$$r_n(t)=\begin{cases} 1 & 0 \leqq t \leqq n \\ 1+(n-t) & n \leqq t \leqq n+1 \\ 0 & n+1 \leqq t . \end{cases}$$

and this time let

$$\psi_n(\xi, t)=\psi(\xi, t) r_n(t) .$$

If $u_n(x, t)$ be the solution to the problem with data ψ_n we will again show convergence. For let (x, t) be any point, $x \in \bar{R}$, $t \geqq 0$, and let n and m each be greater than, say $2t$. Then

$$|u_n(x, t)-u_m(x, t)| \leqq \max_{0 \leqq \tau \leqq 2t} |\psi_n(\xi, \tau)-\psi_m(\xi, \tau)|$$

where the maximum is computed over all $\xi \in B$, $0 \leqq \tau \leqq 2t$. But this maximum vanishes, hence $u_n(x, t)=u_m(x, t)$ for n, m sufficiently large. So that $\lim_{n \to \infty} u_n(x, t)$ exists and is a solution of the heat equation and takes on the prescribed initial and boundary values.

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