LIE ALGEBRAS OF LOCALLY COMPACT GROUPS

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1. Introduction. We call an LP-group, a group which is the projective limit of Lie groups. Yamabe [8] has proved that every connected locally compact group is an LP-group. This permits the extension to locally compact groups of the notion of a Lie algebra. In §§ 2 and 3 we prove the existence and uniqueness of the Lie algebra of an LP-group and show the connection of the Lie algebra with the group by means of the exponential mapping.

In § 4, we extend the notion of a universal covering group for connected groups with the same Lie algebra. A covering group of a connected group g, in the extended sense used here, means a pair $(\bar{g}, w)$, where $\bar{g}$ is a connected LP-group and w is a continuous representation of $\bar{g}$ into g which induces an isomorphism of the Lie algebra of $\bar{g}$ onto the Lie algebra of g (see Definition 4.5). The universal covering group of a connected locally compact group is not necessarily locally compact and may not map onto the group. It turns out that the arc component of the identity in $\bar{g}$ is a covering space in the sense of Novosad [5] of the arc component of the identity of g (these components are dense subgroups, Lemma 3.7).

Finally, in § 5, we establish a one-to-one correspondence between "canonical LP-subgroups" of a group and subalgebras of its Lie algebra.

2. Projective limit of Lie algebras.

DEFINITION 2.1. By a *topological Lie algebra* (over the real numbers) we shall mean a (not necessarily finite dimensional) Lie algebra with an underlying topology such that the operations of addition, multiplication and scalar multiplication are continuous.

DEFINITION 2.2. Let J be an inductive set. Suppose given for each $a \in J$, a topological Lie algebra G_a such that if $a < b$ there exists a continuous representation $f_{ab}: G_b \to G_a$. Let $G = [\{X_a\} \in \prod_{a \in J} G_a$ such that $f_{ab} X_b = X_a$, all $a, b \in J$ with $a < b]$. Then G is a closed topological subalgebra of the direct product.

In analogy to A. Weil [7, p. 23], G will be called the *projective limit* of the G_a ($G = \lim G_a$) if the following hold:

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LP I: If $a < b < c$, $f_{ac}X_c = f_{ab}(f_{bc}X_c)$;
LP II: f_{ab} is a continuous open homomorphism of G_b onto G_a;
LP III: f_a, the natural projection of G into G_a, is continuous and onto.
(Remark: f_a open is implied by f_{ab} open, (see [7, p. 24])

In particular, if the G_a are finite dimensional Lie algebras with the usual topology as a Euclidean space, we get the following.

THEOREM 2.3. *Let G_a, $a \in J$, be a system of finite dimensional Lie algebras satisfying* LP I *and*

LP II′: *f_{ab} is a representation onto.*

Then G as defined in Definition 2.2 *will necessarily satisfy* LP II *and* LP III *and hence $G = \lim G_a$.*

Proof. LP II′ implies LP II, since for finite dimensional vector spaces a representation onto is both continuous and open in the usual topology. That LP III is satisfied follows directly from the theory of linearly compact vector spaces [3 Ch. III, § 27]. In fact, this result holds for an inverse system of finite dimensional vector spaces.

DEFINITION 2.4. If $G = \lim G_a$, G_a *finite* dimensional Lie algebras, then G will be called an *LP algebra*.

LEMMA 2.5. *Let $G = \lim G_a$, where the G_a's are complete topological Lie algebras, with homomorphisms f_a and f_{ab} satisfying* LP I, II, *and* III. *Let N_a be the kernel of f_a, then*

A. *Every neighborhood of zero in G contains an N_a*
B. *For each N_a, N_b; there exists an $N_c \subset N_a \cap N_b$.*
C. *G is complete.*

Proof. It is easy to show (see [7]) that a fundamental system of neighborhoods of zero in G is given by $f_a^{-1}(V_a)$, $a \in J$, and V_a running through a fundamental system of neighborhoods of zero in G_a. Condition A then follows directly.

If $c > a$, b then N_c obviously satisfies B. Condition C is immediate from the definition of G.

LEMMA 2.6. *Given a topological Lie algebra G containing closed ideals N_a satisfying* A, B, C; *then* $\lim G/N_a$ *exists and is isomorphic to G (where we define $a < b$ if $N_a \supset N_b$ and let $f_{ab}: G/N_b \to G/N_a$ be the natural homomorphism).*

Proof. Since the Conditions A, B, C are identical to those for topological groups [7, p. 25] it follows that G is isomorphic to $\lim G/N_a$ as an additive topological group. The lemma now follows since the N_a are ideals.

THEOREM 2.7. *Suppose $G=\lim G_a$, G_a finite dimensional. If K is a closed subalgebra of G, then $K=\lim K_a$, where K_a is the image of K in G_a. If K is a closed ideal in G, then $G/K=\lim G_a/K_a$. In particular, G/K is complete.*

Proof. If $f_{ab}:G_b \to G_a$ $(a<b)$, then $f_{ab}:K_b \to K_a$ satisfies LP I, II′. Hence $\lim K_a$ exists. Since K maps onto K_a, K is dense in $\lim K_a$, [7]. Since K is closed, $K=\lim K_a$.

Likewise G_a/K_a, $a \in J$, satisfy LP I, II where $\bar{f}_{ab}:G_b/K_b \to G_a/K_a$ is induced by f_{ab}. Hence $\lim G_a/K_a$ exists. The natural maps $p_a:G_a \to G_a/K_a$ evidently induce a map $p:G \to \lim G_a/K_a$ defined by: $p\{X_a\}=\{p_aX_a\}$, $\{X_a\} \in G$. This definition is legitimate since:

$$\bar{f}_{ab}(p_bX_b)=p_a(f_{ab}X_b)=p_a(X_a)\,, \qquad a<b.$$

This in turn induces a map $i:G/K \to \lim G_a/K_a$. We have to show that i is an isomophism.

By its definition i is evidently continuous and one-to-one into. We show that it is an isomorphism into. Since the natural map of G onto G/K is open, it is sufficient to show that if W is a neighborhood in G then $p(W)=p(W+K)$ is open. Now if W is a neighborhood in G, take $V+V \subset W$. Then V contains an N_a, kernel of $f_a:G \to G_a$ (Lemma 2.5). Then $p(W)$ contains $p(V)+p(N_a)$. Now $(p(V))_a=p_af_a(V)$, is an open neighborhood in G_a/K_a; and the preimage in $\lim G_a/K_a$ of a neighborhood in G_a/K_a is a neighborhood in lim G_a/K_a. But if $(p(X))_a \in p_af_a(V)$, $X \in G$ then $X_a \in f_a(V)+K_a$ and hence $X \in V+N_a+K \subset W+K$. But this implies that i is open in $i(G/K)$ and hence is an isomorphism into.

It remains to show that i is onto. But this follows since as an abstract vector space G is linearly compact [3]. Hence

$$\bigcap_{a\in J} f_a^{-1}p_a^{-1}(Y_a) \neq \phi, \quad \{Y_a\} \in \lim G_a/K_a$$

since the intersection is nonempty for any finite subset of J.

LEMMA 2.8. *Let θ be a continuous representation of an* LP-*algebra G into an* LP-*algebra H. Then the image of G in H is a closed subalgebra of H.*

Proof. Suppose $G=\lim G_a$, $a \in J$; $H=\lim H_b$, $b \in K$; G_a and H_b finite dimensional. Consider the map $p_b:G \to H_b$, composed of θ and the projection of H onto H_b. This takes G onto a subalgebra H'_b of H_b. Obviously, the H'_b, $b \in K$, define a closed subalgebra $H' = \lim H'_b$ of H under the induced system of representations. Also $\theta(G) \subset H'$.

Let $Y = \{Y_b\}$ be in H'. The preimage $p_b^{-1}(Y_b)$ is a closed linear variety in G. Since G is linearly compact, $\bigcap p_b^{-1}(Y_b)$ is nonempty, as the intersection of a finite number is clearly nonempty. Hence there exists $X \in G$ such that $p_b(X) = Y_b$, all $b \in K$. Hence G maps onto H'. As H' is closed, this proves the Lemma.

LEMMA 2.9. *Let θ be a continuous representation of an* LP-*algebra G onto an* LP-*algebra H. Then θ is open.*

Proof. Let K be the kernel of θ. By Theorem 2.7, G/K is an LP-algebra. Hence it is clearly sufficient to prove the Lemma in the case that the map is also one-to-one.

Hence let θ be a continuous one-to-one representation of G onto H. Suppose $G = \lim G/K_a$; K_a, $a \in J$, closed ideals in G and G/K_a finite dimensional. By Lemma 2.8, $\theta(K_a)$ is a closed ideal in H. Further $H/\theta(K_a)$ is finite dimensional and hence is (topologically) isomorphic to G/K_a. Since θ is continuous every neighborhood of H contains some $\theta(K_a)$. It follows from Lemma 2.6 that

$$H = \lim H/\theta(K_a) = \lim G/K_a = G\,.$$

Clearly, the isomorphism so induced is the same map as θ.

THEOREM 2.10. *If G_1 and G_2 are two* LP-*algebras with the same underlying abstract algebra G, then G_1 and G_2 have the same topology.*

Proof. By Lemma 2.9 it is sufficient to construct an LP-algebra G_0 whose underlying abstract algebra is G and such that the identity maps of G_0 into G_1 and G_2 are both continuous. Let K_a, $a \in J$, be the set of all abstract ideals in G which have finite codimension. As is well known, if K_a and K_b have finite codimension, so does $K_a \cap K_b$. It follows that G/K_a, $a \in J$, satisfy the conditions of Theorem 2.3, where we define $b > a$ if $K_b \subset K_a$ and $f_{ab}:G/K_b \to G/K_a$ is the natural projection. Let $G_0 = \lim G/K_a$. We claim the underlying abstract algebra of G_0 is G.

Now G is linearly compact since it is the underlying algebra of G_1 and G_2. Let $p_a: G \to G/K_a$ be the natural map. If $Y = \{Y_a\} \in G_0$, then $\bigcap p_a^{-1}(Y_a)$ is nonempty, since the intersection of a finite number is nonempty. Hence there exists $X \in G$ such that $p_a(X) = Y_a$, all $a \in J$. Hence the map $p: X \to p_a(X)$ take G onto G_0. p is one-to-one, since for every $X \in G$ there is a K_a such that $p_a(X)$ is not zero, because the identity map of say G onto G_1 is one-to-one.

Now $G_1=\lim G_1/K_b$, $b \in J'$, J' a subset of J. $p^{-1}:G_0 \to G_1$ is an isomorphism of this underlying abstract algebras and is continuous since J' is a subset of J. Hence the theorem follows by Lemma 2.9.

COROLLARY 2.11. *Let G be an abelian* LP-*algebra. Then G is the direct product of* 1-*dimensional algebras. In particular, the underlying topological vector space of an* LP-*algebra is algebraically and topologically the direct product of* 1-*dimensional vector spaces.*

3. Lie algebra of an LP-group.

DEFINITION 3.1. Let g_a, $a \in J$ be Lie groups. Suppose $g=\lim g_a$, the limit satisfying LP I, II, III of A. Weil [7]. Then we call g an *LP-group*. (Note that g is complete since the g_a are complete.)

DEFINITION 3.2. Suppose $g=\lim g_a$, g_a connected Lie groups. Let G_a be the Lie algebra of g_a. Then the homomorphisms $f_{ab}:g_b \to g_a$ $(a<b)$ induce homomorphisms $df_{ab}:G_b \to G_a$ satisfying LP I, II of Definition 2.1. Hence the G_a, $a \in J$, have a limit G. G is called the *Lie algebra* of g.

We show in Lemma 3.4 below that G is independent (in a natural sense) of the representation of g as a limit of Lie groups.

DEFINITION 3.3. Suppose $g=\lim g_a$, g_a connected Lie groups. Let G, $G=\lim G_a$, be the Lie algebra of g. Then we define a continuous map

$$\exp:G \to g, \quad \exp\{X_a\}=\{\exp X_a\}, \quad \{X_a\} \in G\,.$$

This mapping is legitimate, since if $f_{ab}:g_b \to g_a$ then

$$f_{ab}(\exp X_b)=\exp df_{ab}X_b=\exp X_a$$

and hence $\{\exp X_a\} \in g$.

LEMMA 3.4. *Suppose $g=\lim g_a$, g_a connected Lie groups, $a \in J$; $h=\lim h_b$, h_b connected Lie groups, $b \in K$. Let $G=\lim G_a$, $H=\lim H_b$ be the corresponding Lie algebras. If $\theta: g \to h$ is an isomorphism then we can define an isomorphism $d\theta:G \to H$ such that*

$$\theta(\exp X)=\exp d\theta(X), \quad X \in G$$

Proof. Let $f_a:g \to g_a$ and $\bar{f}_b:h \to h_b$ be the natural maps. Let n_a and $\bar{n}_b$ be the kernels of f_a and $\bar{f}_b$ respectively. Let $b \in K$. Since h_b is a Lie group there is a neighborhood V_b of h_b which contains no nontrivial subgroups (that is, h_b doesn't have arbitrarily small subgroups). Since $\bar{f}_b\theta: g \to h_b$ is continuous, there is a neighborhood W in g which maps into V_b. But W contains some n_a and this n_a must go into the unit element of h_b. This defines a homomorphism $\theta_{ba}: g_a \to h_b$ such that

$$\theta_{ba} f_a = \bar{f}_b \theta ,$$

this condition characterizing θ_{ba}.

If $a > a'$, a and a' in J, then $f_a = f_{aa'} f_{a'}$ and $\theta_{ba} f_{aa'} f_{a'} = \bar{f}_b \theta$. Hence $\theta_{ba} f_{aa'} = \theta_{ba'}$. Similarly, if $b' < b$, $\bar{f}_{b'b} \theta_{ba} = \theta_{b'a}$. The induced homomorphisms of the corresponding Lie algebras therefore satisfy

$$d\theta_{ba} df_{aa'} = d\theta_{ba'}, \quad d\bar{f}_{b'b} d\theta_{ba} = d\theta_{b'a} .$$

It follows that the maps $d\theta_{ba}$ define a continuous representation $d\theta$ of G into H, where $d\theta(X)$, $X = \{X_a\} \in G$, is defined by

$$(d\theta(X))_b = d\theta_{ba}(X_a) .$$

This map is well defined because of the conditions satisfied above, and is continuous because $d\theta_{ba}$ is continuous.

Similarly for each $a \in J$, we can find a $b \in K$ and a homomorphism $\psi_{ab}: h_b \to g_a$ such that $f_a \theta^{-1} = \psi_{ab} \bar{f}_b$. This defines a continuous representation $d\psi$ of H into G. Because of the conditions satisfied by the maps one sees easily that: $d\theta d\psi: H \to H$ and $d\psi d\theta: G \to G$ are the identities, and hence that $d\theta$ is an isomorphism.

Since $d\theta_{ba} df_a = d\bar{f}_b d\theta$ by definition, we have

$$\bar{f}_b \theta(\exp X) = \theta_{ba} f_a(\exp X) = \exp d\theta_{ba}\, df_a(X) = \exp d\bar{f}_a d\theta(X) .$$

By Definition 3.3, this implies that $\theta(\exp X) = \exp d\theta(X)$.

THEOREM 3.5. *Suppose $g = \lim g_a$, g_a connected Lie groups. Let $G = \lim G_a$ be the corresponding Lie algebra. Then g is the closed subgroup generated by the elements of the form $\exp X$, $X \in G$.*

Proof. Since G maps onto G_a, $\exp X_a$ for $X \in G$ generates g_a, and $\exp G$ generates a dense subgroup of g, proving the theorem.

LEMMA 3.6. *Suppose $G = \lim G_a$, G_a finite dimensional Lie algebras. Then the underlying space of G is arcwise connected.*

Proof. Since G is a topological vector space it is arcwise connected by straight lines.

LEMMA 3.7. *Suppose $g = \lim g_a$, g_a connected Lie groups. Then g is* *connected and the arcwise connected component of g is dense in g.*

Proof. The map $\exp: G \rightarrow g$ is continuous. Hence if A is the image of G, A is arcwise connected. Hence A^n is arcwise connected. Hence $\bigcup_1^\infty A^n$ is arcwise connected. By Theorem 3.5 this is dense in g. Hence g is connected.

THEOREM 3.8. *Let (g_a), $a \in J$ be a system of connected Lie groups satisfying* LP I, II *of A. Weil* [1]; *then* $g = (\{x_a\} \in \prod_{a \in J} G_a;\ f_{ab}x_b = x_a,$ *all* $a, b \in J$ *with* $a < b)$ *satisfies* LP III, *and hence*:

$$g = \lim g_a$$

Proof. Let G_a be the Lie algebra of g_a. Then the (G_a), $a \in J$ satisfy LP I, II of Definition 1.1. Hence they have a limit G. Let $X \in G$, then if $X = \{X_a\}$ we have $\exp X_a \in g_a$ and $\{\exp X_a\} \in g$, since:

$$f_{ab} \exp X_b = \exp df_{ab} X_b = \exp X_a, \quad a < b$$

But elements of the form $\exp X_a$ generate g_a since G maps onto G_a. Hence g maps onto g_a. Hence $g = \lim g_a$.

LEMMA 3.9. *Let $g = \lim g_a$, g_a arbitrary Lie groups. Let g_a^0 be the connected component of the identity of g_a. Then the (g_a^0), $a \in J$ form a system of groups satisfying* LP I, II *of A. Weil. Let $g^0 = \lim g_a^0$, then g^0 is the connected component of g.*

Proof. Since $f_{ab}: g_b \rightarrow g_a$ is continuous, open and g_b^0 is open in g_b, it takes g_b^0 onto g_a^0. Hence the (g_a^0), $a \in J$ satisfy LP I, II. By Theorem 3.8 they have a limit g^0. g^0 may obviously be considered as a subgroup of g, closed since complete.

By Lemma 3.7, g^0 is connected and hence contained in the connected component of g. On the other hand, if g_1 is the connected component of g, $f_a(g_1)$ is connected and hence contained in g_a^0. Hence g_1 is contained in the limit of the g_a^0. Hence $g_1 = g^0$.

DEFINITION 3.10. Let g be a topological group. If the connected component of the identity of g is an LP-group, we define the Lie algebra of g as the Lie algebra of its connected component.

REMARK. According to the result of Yamabe [8], every locally compact group is a generalized Lie group. This implies in particular that its connected component is an LP-group. Hence every locally compact group has a Lie algebra.

THEOREM 3.11. *Let g and h be topological groups for which Lie Algebras G and H are defined (Definition 3.10). Let f be a continuous representation of g into h, then f induces a unique continuous representation of G into H such that $f(\exp X)=\exp df(X)$, $X \in G$.*

Proof. Obviously f defines a continuous representation of the connected component of g into the connected component of h. Assume therefore that g and h are connected.

Suppose $g=\lim g_a$, $h=\lim h_b$ (g_a, h_b connected Lie groups). The map $g \to h \to h_b$ induces a map of $g_a \to h_b$ for some a, since h_b doesn't have arbitrarily small subgroups. This in turn induces a map of $G_a \to H_b$ and hence of G into H_b. As in the proof of Lemma 3.4, it is easy to see that the maps $G \to H_b$ induce a continuous representation of G into H, $df: G \to H$; such that $f(\exp X)=\exp df(X)$.

Suppose there are two such representations, say df and $\overline{df}$. Then $(dfX)_b \neq (\overline{df}X)_b$ some b and X. Since any neighborhood of zero generates G, and since df, $\overline{df}$ are linear; we can chose X such that dfX and $\overline{df}X$ are in any desired neighborhood of 0 in H_b. For a sufficiently small neighborhood exp is one-to-one on H_b. Hence $(\exp dfX)_b \neq (\exp \overline{df}X)_b$, a contradiction.

COROLLARY. *If g is connected and f, f_1 are two representations of g into h such that $df=df_1$, then $f=f_1$.*

Proof. Since $f(\exp X)=\exp dfX=\exp df_1X=f_1(\exp X)$ and since $\exp G$ generates a dense subgroup of g, we have $f=f_1$.

LEMMA 3.12. *Let g and h be locally compact topological group and g connected, G and H their Lie algebras. Let f be a continuous open homomorphism of g onto h, then df is a continuous open homomorphism of G onto H.*

Proof. According to A. Weil [7], if k is the kernel of f, we may take $g=\lim g_a$, $k=\lim k_a$, k_a the image of k in g_a, and $h=\lim g_a/k_a$, g_a and k_a Lie groups.

Then $G=\lim G_a$, G_a the Lie algebra of g_a. Let K be the Lie algebra of k; then $K=\lim K_a$, K_a the Lie algera of k_a. Then the Lie algebra of g_a/k_a is G_a/K_a [1]. Hence $H=\lim G_a/K_a$. By Theorem 2.5, $H=G/K$. It is easy to check that for $df: G \to G/K$ that $f(\exp X)=\exp dfX$. So that the natural map of G onto G/K is df. (Note: By a generalization of Pontrjagin's theorem on groups satisfying the second axiom of countability; if f is continuous and onto it is automatically open.)

4. Universal covering group.

DEFINITION 4.1. Suppose $g=\lim g_a$, (g_a), $a\in J$ connected Lie groups. Let $\bar{g}_a$ be the simply connected covering group of g_a. The map f_{ab} taking g_b onto g_a $(a<b)$ induces an open homomorphism $\bar{f}_{ab}$ of $\bar{g}_b$ onto $\bar{g}_a$. Hence the $(\bar{g}_a)$, $a\in J$ satisfy LP I, II and therefore have a limit $\bar{g}$ (Theorem 3.8). $\bar{g}$ is a complete, connected group. $\bar{g}$ is called the *universal covering group* of g.

PROPOSITION 4.2. *Let G be the Lie algebra of g. Then $\bar{g}$ has the Lie algebra G, and there exists a continuous representation w taking $\bar{g}$ onto a dense subgroup of g such that $dw:G\to G$ is the identity.*

Proof. The covering homomorphisms $w_a:\bar{g}\to g_a$ induce a continuous representation of $\bar{g}$ into g. Since $dw_a:G_a\to G_a$ is the identity, it follows that dw is the identity. Since w_a maps $\bar{g}_a$ onto g_a it follows that $w(\bar{g})$ is dense in g.

PROPOSITION 4.3. *The kernel of w is totally disconnected and is in the center.*

Proof. If k is the kernel of w, it follows from the definition of w that the image $f_a(k)$ of k in g_a belongs to the kernel of w_a. But this kernel is discrete and hence $f_a(k)$ is discrete, and therefore closed in g_a. It follows that the $f_a(k)$ satisfy LP I, II and since k is closed, $k=\lim f_a(k)$. Hence k is the projective limit of discret groups. It follows from Lemma 3.9 that k is totally disconnected. Further, a totally disconnected normal subgroup of a connected group belongs to the center.

LEMMA 4.4. *Let g be a connected* LP-*group with Lie algebra G. Let h be any other connected* LP-*group with Lie algebra H isomorphic to G. Then there exists an isomorphism of their universal covering groups $f:\bar{g}\to\bar{h}$ such that df is the given isomorphim of G onto H.*

Proof. Suppose $g=\lim g_a$, $h=\lim h_b$; then $\bar{h}=\lim\bar{h}_b$, $\bar{g}=\lim\bar{g}_a$ and $G=\lim G_a$, $H=\lim H_b$. The homomorphism $G\to H\to H_b$ induces a homomorphism $G_a\to H_b$ for some a, since H_b has no small subgroups when considered as an additive group. This in turn induces a homomorphism of $\bar{g}_a$ onto $\bar{h}_b$. Similary there exist homomorphisms of $\bar{h}_b$ onto $\bar{g}_a$, $a\in J$, some b. As in the proof of Lemma 3.4, it is easy to see that this implies that there exists an isomorphism $f:\bar{g} \to \bar{h}$. The induced homomorphism $df:G \to H$ such that $f(\exp X) = \exp df(X)$, is obviously the original isomorphism of G onto H.

DEFINITION 4.5. Let g and h be connected LP-groups, G and H their Lie algebras. A continuous representation w of g into h such that dw is an isomorphism of G onto H is called a *covering map* and g is called a *covering group* of h, (g, w) is called the *covering*.

PROPOSITION 3.6. *$w(g)$ is dense in h.*

Proof. In fact $w(\exp X) = \exp dw(X)$, $X \in G$. But $\exp dw(X)$, $X \in G$ generates a dense subgroup of h since dw is onto.

We now give a purely topological definition of covering space for arcwise connected spaces due to Novosad [5] and show that the arc component of the identity g^c of g in Definition 4.5 is actually a covering space in this sense of the arc component of the identity h^c of h. (Note that g^c is dense in g, Lemma 3.7) Similarly, we show the arc component of the identity of the universal covering group is a universal covering space.

DEFINITION 4.7. (Novosad) Let A be an arcwise connected space, $a \in A$. Let $f:(B, b) \to (A, a)$ be a continuous map of an arcwise connected space B into A taking b into a. Then (f, B, b) is called a *covering space* of (A, a), if given any contractible space C, and point $c \in C$ which is a deformation retract of C, and a map $\alpha:(C, c) \to (A, c)$, then there exists a map $\bar{\alpha}:(C, c) \to (B, b)$ which is unique with respect to the property $f\bar{\alpha} = \alpha$.

Let $(P_A, a\sharp)$ be the pair consisting of the space of paths starting from $a \in A$, with the compact open topology, of an arbitrary topological space A, and the constant path $a\sharp$ at $a \in A$. A continuous map $f:(B, b) \to (A, a)$ induces a continuous map $f\sharp:(P_B, b\sharp) \to (P_A, a\sharp)$ defined by:

$$f\sharp(p)(t) = f(p(t)), \quad p \in P_B, \quad t \in I \text{ (the unit interval)}$$

It is then easy to see that Definition 4.7 is equivalent to the following.

DEFINITION 4.7′. Let $f:(B, b) \to (A, a)$ be a continuous map of an arcwise connected space B into an arcwise connected space A taking $b \in B$ into $a \in A$. Then (f, B, b) is called a covering space of (A, a) if $f\sharp:(P_B, b\sharp) \to (P_A, a\sharp)$ is a homeomorphism onto.

PROPOSITION 4.8. *Definitions* 4.7 *and* 4.7′ *are equivalent.*

Proof.

(4.7) implies (4.7′) by Lemma 2.3 of [5].

(4.7′) implies (4.7); since, let $\lambda_A : P_A \to A$, $\lambda_A(p)=p(1)$, $p \in P_A$. Then λ_A defines a fiber space (in the sense of Serre) which obviously satisfies the covering homotopy theorem for arbitrary spaces. Hence if $\alpha:(C, c) \to (A, a)$ is homotopic to tne constant map the homotopy may be lifted to $(P_A, a\sharp)$ and hence to $(P_B, b\sharp)$ by the homeomorphism of (4.7′). λ_B maps the image into (B, b). The endpoint of the homotopy gives the desired covering of (4.7). The uniquenes follows since any point of C describes a path under the retraction and the image of this path in B is unique since covering paths are unique by (4.7′).

DEFINITION 4.9. An arcwise connected space A is called *simply connected* if every covering space (4.7) of (A, a) is trivial. This property is independent of the base point $a \in A$ (see [5]).

Let Ω_A be the (closed) subspace of P_A consisting of closed paths (that is, the loop space). Let Ω_A^c be the arc component of $a\sharp$ in Ω_A.

THEOREM 4.10. *Let A be an arcwise and locally arcwise connected space, $a \in A$; then if Ω_A is connected* (*not necessarily arcwise connected*) *and Ω_A^c is dense in Ω_A, A is simply connected* (*Definition* 4.9).

Proof. Let (f, B, b) be a covering space of (A, a). Then $f\sharp : P_B \to P_A$ is a homeomorphism, and hence $f\sharp$ maps Ω_B homeomorphically into Ω_A. But every loop in A contractible to a may be lifted to a unique loop in B (see proof of 4.8), hence $f\sharp(\Omega_B) \supset \Omega_A^c$. But $f\sharp(\Omega_B)$ is closed in P_A. Hence $f\sharp$ maps Ω_B onto Ω_A.

Now this means that $f: B \to A$ is one-to-one. For if $f\sharp(p)(1)=f\sharp(p')\cdot(1)$, p, $p' \in P_B$, then $p(1)=p'(1)$; since $f\sharp p$ and $f\sharp p'$ having the some endpoint form a loop in A, and hence must come from a loop in B by the above. Also $p(1)=p'(1)$ implies $(f\sharp p)(1)=(f\sharp p')(1)$. Hence since $\lambda_B : P_B \to B$, and $\lambda_A : P_A \to A$ (the endpoint maps) are onto, f is one-to-one.

Further λ_A is both continuous and open [4, Lemma 4] since A is locally arcwise connected, hence the continuous map $\lambda_B f^{-1} : P_A \to B$ induces a continuous map $f^{-1} : A \to B$. This proves the theorem.

We now apply the above to LP-groups. We remark that if g is a topological group; then P_g, the space of paths of g beginning at the identity, may be made into a topological group by pointwise multiplication of paths. Then Ω_g is a closed normal subgroup.

LEMMA 4.11. *Let $g=\lim g_\alpha$, g_α Lie groups; then $P_g=\lim P_{g_\alpha}$.*

Proof. First $f_a: g \to g_a$ has a local cross-section. In fact, it is obvious that $df_a: G \to G_a$ has a cross-section since these spaces are linear, further g_a has a neighborhood which is homeomorphic to a neighborhood of G_a and $\exp: G \to g$ is continuous.

Now P_{g_a} is arcwise connected, hence $f_a^{\#}: P_g \to P_{g_a}$ will be onto if the image covers a neighborhood of the identity. But this follows from the local cross-section of g_a in g. In fact, a fundamental system of neighborhoods of the identity in P_a is obtained from a fundamental system of neighborhoods of the identity in g by taking all those paths that are contained in a given neighborhood of the identity in g. Also it follows that $f_a^{\#}$ is open.

If V is a neighborhood of the identity in g that contains the kernel k_a of f_a, then the corresponding neighborhood in P_g contains P_{k_a}, and this last is clearly the kernel of $f_a^{\#}$. Finally, P_g is complete since g is complete. Hence all the conditions for a projective limit are satisfied and the lemma follows.

LEMMA 4.12. *If $g = \lim g_a$, g_a simply connected Lie groups, then $\Omega_g = \lim \Omega_{g_a}$.*

Pooof. The proof is the same as above, using the fact that since g_a is simply connected, Ω_{g_a} is arcwise connected.

LEMMA 4.13. *If $g = \lim g_a$, g_a simply connected, then Ω_g^c is dense in Ω_g and Ω_g is connected.*

Proof. Since $f_a: g \to g_a$ has local cross-sections (see 4.11), it defines a principal fiber bundle and hence satisfies the covering homotopy theorem of [6]. Hence since every loop in g_a is contractible each loop may be lifted to a contractible loop in g. Hence Ω_g^c maps onto Ω_{g_a}, all a. Hence Ω_g^c is dense in Ω_g. The last statement then follows.

LEMMA 4.14. *If $g = \lim g_a$, g_a simply connected, then $P_g/\Omega \approx g^c$.*

Proof. For Lie groups, $P_{g_a}/\Omega_{g_a} \approx g_a$, since g_a is locally arcwise connected and hence the map of P_{g_a} onto g_a is open (see [7]). Since P_g maps continuously onto P_{g_a}, P_g maps continuously onto $P_{g_a}/\Omega_{g_a} \approx g_a$. The induced map of P_g into $g = \lim g_a$ is obviously λ_g, and this induces a continuous one-to-one map of P_g/Ω_g onto $g^c \subset g$. Hence it is sufficient to show that P_g/Ω_g has the proper topology as a subgroup of g.

The neighborhoods of the identity in P_g/Ω_g are of the form $V\Omega_g$, where V is the preimage in P_g of a neighborhood V_a in P_{ga}. But this is the preimage in P_g/Ω_g of the neighborhood $V_a\Omega_{g_a}$ in $P_{g_a}/\Omega_{g_a} \approx g_a$. Hence the Lemma follows.

THEOREM 4.15. *If* $g = \lim g_a$, g_a *simply connected Lie groups, then* g^c *(the arc-component of the identity) is arcwise connected, locally arcwise connected, and simply connected (Definition* 4.9).

Proof. g^c is locally arcwise connected since P_g is obviously so. Hence by Lemmas 4.12 and 4.13 and Theorem 4.10, g^c is simply connected.

COROLLARY 4.16. *If* g *is the universal covering group* (4.1) *of a metrisable* LP-*group or a connected locally compact group, then* g *is arcwise connected, locally arcwise connected and simply connected.*

Proof. For metrisable groups, P_g is metrisable and complete. Hence $P_g/\Omega_g \approx g^c$ is complete and thus $g^c = g$. The result for locally compact groups will follow from Theorem 4.25.

We write again w for the map of Definition 4.5 cut down to g^c.

LEMMA 4.17. (w, g^c, e) *is a covering space (Definition* 4.7) *of* (h^c, e).

Proof. First assume $g = \bar{h}$ the universal covering group of h. Then if $h = \lim h_a$, $\bar{h} = \lim \bar{h}_a$ and since $P_{\bar{h}_a} \approx P_{h_a}$ (this is obvious for Lie groups), $P_{\bar{h}} \approx \lim P_{\bar{h}_a} \approx \lim P_{h_a} \approx P_h$. Further, this isomorphism is clearly induced by the covering map w.

Now let (g, w) be any covering group of h. Then $\bar{h} \approx \bar{g}$ by (4.4) and hence $P_g \simeq P_{\bar{g}} \approx P_{\bar{h}} \approx P_h$. Since $\bar{g} \to g \to h$ is the same as $\bar{g} \to \bar{h} \to h$. The isomorphism $P_g \approx P_h$ is induced by w. Hence (w, g^c, e) satisfies (4.7)′.

THEOREM 4.18. *If* g *and* h *are* LP-*groups,* G *and* H *their Lie algebras,* $\bar{g}$ *and* $\bar{h}$ *their universal covering groups,* P_g *and* P_h *their group of paths,* Ω_g^c *and* Ω_h^c *the arc component of the identity in their group of loops, respectively; then the following are equivalent:*

(a) G *isomorphic to* H (b) $\bar{g}$ *isomorphic to* $\bar{h}$

(c) P_g *isomorphic to* P_h *such that the isomorphism takes* Ω_g^c *onto* Ω_h^c.

Proof.

(a) implies (b) follows from (4.4).

(b) implies (a) follows from (4.2).

(b) implies (c): $P_g \simeq P_{\bar{g}}$ by (4.17). Also $\Omega_{\bar{g}}^c \simeq \Omega_g^c$ under the same map since every contractible loop in g may be lifted to $\bar{g}$ (see proof of 4.8).

(c) implies (b): writing $\bar{\Omega}_g^c$ for the closure of Ω_g^c in P_g, we have $\bar{g}^c \simeq P_{\bar{g}}/\Omega_{\bar{g}} \simeq P_g/\bar{\Omega}_g^c \simeq P_h/\bar{\Omega}_h^c \simeq P_{\bar{h}}/\Omega_{\bar{h}} \simeq \bar{h}^c$, since Ω_g^c is dense in $\Omega_{\bar{g}}$. Hence $\bar{g} \simeq \bar{h}$.

LEMMA 4.19. *Every* LP-*algebra is the Lie algebra of an* LP-*group.*

Proof. By assumption, if G is an LP-algebra then $G = \lim G_a$, G_a finite dimensional. Let g_a be the simply connected groups corresponding to G_a. The homomorphisms of G_a onto G_b $(a < b)$, induce homomorphisms of g_a onto g_b which satisfy LP I, II. Hence they have a limit g (Theorem 3.8). But g obviously has Lie algebra G.

DEFINITION 4.20. The group g defined in the proof of (4.19) is called the *universal group corresponding to G.*

LEMMA 4.21. *Let g be a universal* LP-*group then every covering group* (h, w) (*Definition* 4.5) *is trivial, that is, w is an isomorphism of h onto g.*

Proof. h^c is a trivial covering of g^c. Since h and g are complete and h^c and g^c are dense, the lemma follows.

THEOREM 4.22. *Let h be an* LP-*group and let H be its Lie algebra. Let G be an* LP-*algebra and g the universal group corresponding to G. Let* θ *be a continuous representation of G into H. Then* θ *induces a continuous representation f of g into h such that* $df = \theta$.

Proof. If $h = \lim h_b$, h_b Lie groups; then $H = \lim H_b$, H_b finite dimensional Lie algebras. Suppose $G = \lim G_a$, G_a finite dimensional. Let g_a be the simply connected group corresponding to G_a, $g = \lim g_a$. The map $G \to H \to H_b$ induces a map $G_a \to H_b$, some a. But this in turn induces a map $g_a \to h_b$ and hence a map of $g \to h_b$ for every b. It is easy to see that this defines a map $f: g \to h$ such that $df = \theta$.

THEOREM 4.23. *The universal covering group* $\bar{g}$ *of a connected locally compact group g is the direct product of simply connected Lie groups. More explicitly* $\bar{g} \simeq h \times a \times s$, *where h is a simply connected Lie group, a is the* (*possibly infinite*) *direct product of the reals and s is the* (*possibly infinite*) *direct product of simple simply connected compact Lie groups.*

Proof. According to Yamabe [8] and Iwasawa (Theorem 11 of [2]), g is locally the direct product of a local Lie group h' and a compact normal subgroup k. Now $k=\lim k_a$, where $k_a=k/n_a$, n_a normal in k and hence in g (Theorem 4 of [2]). Hence $g=\lim g_a$, where $g_a=g/n_a$. Evidently g_a is locally isomorphic to $h'\times k_a$; and hence to $h'\times k_a^0$, k_a^0 connected component of k_a (since k_a a Lie group).

Since $f_{ab}:g_b\to g_a$ takes k_b onto k_a, it takes k_b^0 onto k_a^0. Hence f_{ab} induces a homomorphism of $h\times\bar{k}_b^0$ onto $h\times\bar{k}_a^0$, where h is the simply connected group associated with h' and $\bar{k}_a^0$ is the simply connected covering group of k_a^0. If $\bar{k}^0=\lim\bar{k}_a^0$, then $k\times\bar{k}^0=\lim h\times\bar{k}_a^0$; $h\times\bar{k}_a^0$ is the simply connected covering group of g_a and hence $h\times\bar{k}^0$ is the universal covering group of g.

Since $\bar{k}^0$ is the universal covering group of k^0, k^0 connected component of k; the problem is reduced to considering the universal covering group of a compact connected group.

According to A. Weil (p. 91 of [7]), k^0 is isomorphic to $(a'\times s)/d$, where a' is a compact abelian connected group, s is the (possibly infinite) direct product of simple simply connected compact Lie groups, and d is totally disconnected. It is evident that $\bar{k}^0=a\times s$, where a is the universal covering group of a'. Since a' in the projective limit of toroidal groups, a is the projective limit of vector groups, and hence is the direct product of the reals (2.11). This proves the theorem.

COROLLARY 4.24. *If g is a locally compact group, then its Lie algebra G has the form $G=H\times A\times S$, where H is a finite dimensional Lie algebra, A is the product (possibly infinite) of 1-dimensional Lie algebras, and S is the (possibly infinite) direct product of simple compact Lie algebras.*

EXAMPLE 1. Let $g=\Pi T_a$, $a\in J$, T_a isomorphic to the torus group, all a. Then g is compact. But $\bar{g}=\Pi R_a$, R_a isomorphic to the additive group of reals, $a\in J$. Hence for J infinite, $\bar{g}$ is not locally compact.

EXAMPLE 2. Let P be the p-adic solenoid (See for example: Eilenberg and Steenrod, *Foundations of algebraic topology*, p. 230). P is a compact connected group and is the projective limit of torus groups. If T is the multiplicative group of all complex numbers z with $|z|=1$, the projections $\varphi:T\to T$ are given by $\varphi(z)=z^p$, p an integer. φ induces the map $\bar{\varphi}:R\to R$, $\bar{\varphi}(x)=px$, which is an isomorphism of the additive group of the reals onto itself. Hence the universal covering group of P, which is the projective limit of the reals under these isomorphisms, is itself the additive group of reals. Hence the Lie algebra of P is 1-

dimensional. As is well known (see above reference) R maps continuously, one-to-one onto a dense subgroup of P, not the whole group. The map is not open and not onto P. More generally we have the following.

EXAMPLE 3. Let g be a connected, but not locally connected, locally compact group. Let $(\bar{g}, w)$ be its universal covering group. Then $w:\bar{g} \to g$ is not both open onto. Consequently, if $\bar{g}$ is locally compact, w is neither open (on the image) nor onto.

In fact $\bar{g}$ is locally connected. Hence if w is open and onto, g would be locally connected. If $\bar{g}$ is locally compact then if w is open on $w(\bar{g})$, $w(\bar{g})$ would be locally compact, hence closed, hence $w(\bar{g})=g$. On the other hand, if w is onto and $\bar{g}$ connected locally compact, w is open.

Hence, in particular we have the following. Let g be a connected, but not locally connected finite dimensional locally compact group. Then $\bar{g}$ is locally compact (a Lie group) and hence w is neither open nor onto.

EXAMPLE 4. Not every complete topological Lie algebra is an LP-algebra. In fact an infinite dimensional Banach space cannot contain arbitrarily small subspaces and cannot be an abelian LP-algebra.

5. Subgroups and subalgebras.

DEFINITION 5.1. Let g be an LP-group. An LP-group h is called an LP-*subgroup* of g if h is an abstract subgroup and the inclusion map $f:h \to g$ is a continuous representation such that $df:H \to G$ is an isomorphism into.

THEOREM 5.2. *Let $g=\lim g_a$ be a* LP-*group; $G=\lim G_a$ its Lie algebra. Let H be a closed subalgebra of G, then $H=\lim H_a$ where H_a is the image of H in G_a. Let h_a be the analytic subgroup of g_a corresponding to H_a. Then $h=\lim h_a$ exists and is a connected* LP-*subgroup of g with Lie algebra H.*

Proof. $H=\lim H_a$ follows from Theorem 2.5. Now $f_{ab}:g_b \to g_a$ induces $\bar{f}_{ab}:h_b \to h_a$, onto since the image of h_b in g_a is the analytic subgroup of g_a whose Lie algebra is H_a [1]. Therefore the h_a satisfy LP I, II and hence have a limit h (Theorem 3.8).

Obviously h is an abstract subgroup of g and the maps $h_a \to g_a$ induce a continuous one-to-one representation of h into g, namely the inclusion map. Obviously df is the inclusion map of H into G and hence an isomorphism into.

LEMMA 5.3. *Let h be the* LP-*subgroup of g defined in Theorem* 5.2. *Let h' be any other connected* LP-*subgroup with Lie algebra H' and such that the inclusion map $f': h' \to g$ induces an isomorphism df' of H' onto H. Then h' is a covering group of h and the covering map is abstractly an inclusion.*

Proof. Suppose $h' = \lim h'_b$, h'_b Lie groups. The map $h' \to g \to g_a$ induces a map $h'_b \to g_a$, for some b. But the image of h'_b in g_a is the analytic subgroup whose Lie algebra is H_a, the image of H'_b in G_a. Hence, $h'_b \to h_a$ is continuous and open, and induces $h' \to h_a$ continuous, open. This induces a continuous representation $\theta: h' \to h$ such that $d\theta$ is the isomorphism dg'. Since h' is contained abstractly in g, its elements are determined by their coordinates in g_a. Hence θ is an abstract inclusion of h' in h.

COROLLARY. *The subgroup h defined in Theorem* 5.2 *is uniquely characterized by Lemma* 5.3.

Proof. Suppose h and h' are two connected LP-subgroups such that h is a covering group of h' and h' is a covering group of h, and such that the covering maps are abstract inclusions. Then the maps $h \to h' \to h$ and $h' \to h \to h'$ are both the identity. Hence $h = h'$.

DEFINITION 5.4. The LP-subgroup h of g defined in Theorem 5.2 is called the *canonical* LP-*subgroup* corresponding to the subalgebra H of G.

We have proved the following.

THEOREM 5.5. *Let g be an* LP-*group, G its Lie algebra. There exists a one to one correspondence between canonical* PL-*subgroups of g and closed subalgebras of G.*

THEOREM 5.6. *Let g be the universal group corresponding to an* LP-*algebra G. Let k be a closed normal connected subgroup of g. Then k is the canonical* LP-*subgroup corresponding to an ideal in G. Conversely, the canonical* LP-*subgroup corresponding to an ideal in G is a closed normal connected topological subgroup of g.*

Proof. Let k be a closed, normal, connected subgroup of g. Suppose $g = \lim g_a$, g_a simply connected Lie groups. Let the image of k in g_a be k_a. The closure $\bar{k}_a$ of k_a in g_a is a closed connected normal subgroup of g_a. Let K_a be the ideal of G_a corresponding to $\bar{k}_a$.

The image by f_{ab} of $\bar{k}_b$ in g_a $(a < b)$, is the analytic subgroup of g_a corresponding to the image of K_b in G_a. But the image is an ideal in

G_a, hence the corresponding analytic subgroup is a closed normal connected subgroup of g_a. Hence $\bar{k}_a \subset f_{ab}(\bar{k}_b)$. On the other hand since f_{ab} is continuous, $f_{ab}(\bar{k}_b) \subset \bar{k}_a$. Hence $f_{ab}(\bar{k}_b) = \bar{k}_a$.

Hence $k' = \lim \bar{k}_a$ exists and is a closed normal subgroup of g, and is the canonical LP-subgroup corresponding to $K = \lim K_a$. On the other hand, $k < K'$; but k maps onto k_a, a dense subgroup of $\bar{k}_a$. Hence k is dense in k' and $k = k'$ since k is closed.

The converse is obvious since the analytic subgroups of g_a corresponding to ideals are closed topological subgroups.

EXAMPLE. Consider the p-adic solenoid of Example 2, § 4. The additive group of reals R may be considered as an LP-subgroup of the p-adic solenoid P. Then P itself is the canonical LP-subgroup corresponding to the Lie algebra of R.

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