THREE TEST PROBLEMS IN OPERATOR THEORY

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1. Introduction. In his tract [3] on infinite abelian groups, I. Kaplansky proposes three problems with which to test the adequacy of a purported structure theory for the subject. The problems are general with a certain intrinsic interest, and he comments there that they provide a worthy test in other subjects. In particular, Kaplansky has suggested these problems, suitably rephrased, in conversation as a test of a unitary equivalence theory for operators on a Hilbert space. In the order we treat them they are:

1. If A and B are operators acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and the operators $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, acting in the obvious way on $\mathscr{H} \oplus \mathscr{H}$ and $\mathscr{K} \oplus \mathscr{K}$, are unitarily equivalent, is it true that A and B are unitarily equivalent?

2. If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ are unitarily equivalent is it true that B and C are unitarily equivalent?

3. If A and B are unitarily equivalent to direct summands of each other (that is, A equivalent to BF and B equivalent to AE, where E and F commute with A and B, respectively), are A and B unitarily equivalent?

A superficial examination provides examples which show that Problem 2 must, in general, be answered negatively. In fact infinite projections for B and C, one with an infinite and the other with a finite-dimensional orthogonal complement, and A an infinite-dimensional projection with an infinite-dimensional complement illustrates this. On the other hand, all three problems have an affirmative answer in the finite-dimensional case—Problem 3, trivially so, since E and F must be the identity operator on simple numerical-dimension grounds, and the other problems not at all trivially so (especially when approached from an elementary viewpoint).

Problem 3 has an affirmative answer, and a simple adaptation of the usual Cantor-Bernstein argument proves this. We shall give this problem no further attention except to note that it can be settled by

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use of ring of operators techniques as well as by the direct argument mentioned. We shall show that Problem 1 can always be answered affirmatively, and Problem 2 has an affirmative answer provided the rings generated by the operators in question are, together with their commutants, of finite type—a most satisfactory result in view of the negative example presented and the finite-dimensional situation. The proofs make use of some of the sophisticated techniques of the theory of rings of operators (and in some sense these techniques must be used). It seems to us a pleasant circumstance that this theory is capable now of solving some of the primitive problems of the subject. Our primary interest in the questions discussed is in their role of test problems, for which reason, we have refrained from dealing with such obvious generalizations as the one obtained from Problem 1 by replacing the two-fold copies of A and B by n-fold copies (even though the proof would suffice).

2. The test questions. The first of the test questions we shall discuss is that of the unitary equivalence of the operators A and B given that $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$ are unitarily equivalent. A large share of the solution to this question is contained in the process of phrasing it properly in the terminology of rings of operators and taking full advantage of the hypotheses in these terms. Let $\mathscr{M}$ be the ring of operators generated by $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and φ the $*$-isomorphism of $\mathscr{M}$ onto $\mathscr{N}$, the ring generated by $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, determined by $\varphi\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\right)=\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$. The projections $E'=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $F'=\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ commute with $\mathscr{M}$ and are equivalent in $\mathscr{M}'$ via the partial isometry $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ (in $\mathscr{M}'$); moreover $E'+F'=I$. These same properties hold for the projections M', N' given by the same matrix description relative to $\mathscr{N}'$. In these terms, our result becomes:

THEOREM 1. *The mapping ψ defined on $\mathscr{M}E'$ by $\psi(TE')=\varphi(T)M'$ is implemented by a unitary transformation when φ is implemented by a unitary transformation.*

Proof. Let U be a unitary transformation which implements φ, and let us denote by φ again the unitary equivalence induced on all bounded operators by U. Clearly then, φ so extended carries E' and F' into projections $\varphi(E')$ and $\varphi(F')$ in $\mathscr{N}'$ such that $\varphi(E')$ and $\varphi(F')$ are equivalent and have sum I. We shall note that $\varphi(E')$ and M' are equivalent under these conditions; but let us assume this for the moment, and let W' be a partial isometry in $\mathscr{N}'$ effecting this equivalence. We assert that the unitary transformation $W'UE'$ of the range of E' onto the range of M' implements ψ. Indeed,

$$(*)\quad W'UE'(TE')E'U^{-1}W'^* = W'UTU^{-1}UE'U^{-1}W'^* = W'\varphi(T)\varphi(E')W'^*$$
$$= \varphi(T)W'\varphi(E')W'^* = \varphi(T)W'W'^* = \varphi(T)M' = \psi(TE').$$

That $\varphi(E')$ and M' are equivalent may be accepted as a consequence of the elementary comparison theory of projections in a ring of operators (all projections equivalent to their orthogonal complements are equivalent to each other), or may be reduced to more apparent facts of this theory. In fact if $\varphi(E')$ is not equivalent to M' then for some nonzero central projection P in $\mathscr{N}$ we have, say, $P\varphi(E') \prec PM'$. Restricting consideration to the range of P, we may assume that $\varphi(E') \prec M'$ whence $\varphi(F') \prec N'$ and $I = \varphi(E') + \varphi(F') \prec M' + N' = I$, a contradiction. Establishing this last relation in all detail, however, would require in effect an easy but lengthy development of the cardinal-valued dimension function for projections in a ring of operators. We shall let these remarks suffice as an indication of the proof that $\varphi(E')$ and M' are equivalent.

The argument contained in $(*)$ can be applied more generally to prove a fact which will be of later use. We state this fact in:

REMARK 2. If φ is a unitary equivalence carrying $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ onto $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ and $\varphi\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right)$ is equivalent to $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ in the commutant of the ring generated by $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$, then A and C are unitarily equivalent (via the natural restriction of φ). A curious consequence of this remark is the fact that if $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ generates a factor of type III (on a separable space) then the existence of the unitary equivalence φ implies the unitary equivalence of A, C, B, and D.

It might be thought that some simple construction with the unitary transformation which effects the original equivalence alone in Problem 1 might yield the appropriate unitary operator for demonstrating the equivalence of A and B. That this is not the case can be seen by taking A and B to be I, so that an arbitrary unitary transformation effects the original equivalence.

The next test question we take up is that of the unitary equivalence of B and C given the unitary equivalence of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$. We have noted that the unitary equivalence of B and C, under these conditions, does not follow, in general. Our example illustrating this possibility relies upon an "improper mixture of finiteness and infiniteness". The following theorem shows that, when such a mixture is not possible, B is unitarily equivalent to C. This mixture is not possible when the ring of operators $\mathscr{M}$ generated by $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is finite with finite commutant $\mathscr{M}'$. Our hypothesis tells us that the $*$-isomorphism ψ of $\mathscr{M}$ onto the ring $\mathscr{N}$ generated by $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$, determined by $\psi\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)=\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ is implemented by a unitary transformation, and, with E' the projection $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathscr{M}'$, F' the projection $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathscr{N}'$, the mapping η of $\mathscr{M}E'$ onto $\mathscr{N}F'$ defined by $\eta(TE')=\psi(T)F'$ is a $*$-isomorphism which is implemented by a unitary transformation. We shall denote the unitary equivalences induced on the rings of all bounded operators on $\mathscr{H}$ and $\mathscr{H}E'$ by unitary transformations which implement ψ and η respectively, by ψ and η again, so that it will be meaningful to speak, for example, of $\psi(E')$. In the notation just described our statement becomes:

THEOREM 3. *If $\mathscr{M}$ and $\mathscr{M}'$ are finite the mapping φ of $\mathscr{M}(I-E')$ onto $\mathscr{N}(I-F')$ defined by $\varphi(T(I-E'))=\psi(T)(I-F')$, for T in $\mathscr{M}$, is a $*$-isomorphism which is implemented by a unitary transformation.*

Proof. Note first that the definition of η and the fact that it is a $*$-isomorphism implies that $\psi(C_{E'})=C_{F'}$, in view of [**2**; Lemma 3.1.3], and by this same result, it will suffice to show that $\psi(C_{I-E'})=C_{I-F'}$ in order to establish that φ is a $*$-isomorphism. Now $I-C_{I-E'}$ is the union of all central projections contained in E', whence, from the symmetry of this situation, it will suffice to show that if P is a central projection in $\mathscr{M}$ contained in E' then $\psi(P)\leqq F'$. We make use of the dimension functions in the various rings, and we shall denote these functions by D for $\mathscr{M}$, $\mathscr{M}'$, $\mathscr{N}$ and $\mathscr{N}'$ and by D_0 for $\mathscr{M}E'$, $E'\mathscr{M}'E'$, $\mathscr{M}(I-E')$, $(I-E')\mathscr{M}'(I-E')$, $\mathscr{N}F'$, $F'\mathscr{N}'F'$, $\mathscr{N}(I-F')$, and $(I-F')\mathscr{N}'(I-F')$. By definition $\eta(P)=\eta(PE')=\psi(P)F'$, and η is a unitary equivalence so that

$$\eta[D_0(E'PE')]=\eta(P)=\psi(P)F'=D_0(F'\psi(P)F')$$

$$=\frac{D[\psi(P)]F'}{D(F')}=\frac{\psi(P)F'}{D(F')}$$

(recall that, with G' in $F'\mathscr{N}'F'$, $D_0(G')=F'D(G')/D(F')$). Thus $\psi(P)(D(F')-I)F'=0$, so that $\psi(P)C_{F'}(D(F')-I)=0$, by [**2**; Lemma 3.1.1], and $\psi(P)(D(F')-I)=0$, since $\psi(P)\leqq\psi(C_{E'})=C_{F'}$. It follows that $D(\psi(P)-\psi(P)F')=0$ and $\psi(P)-\psi(P)F'=0$; that is, $\psi(P)\leqq F'$, and φ is a $*$-isomorphism of $\mathscr{M}(I-E')$ onto $\mathscr{N}(I-F')$.

To show that φ is implemented by a unitary transformation, it will suffice, of course, to establish this for each projection of an orthogonal family of central projections in $\mathscr{M}(I-E')$ with sum $I-E'$; whence it suffices to consider the case in which the center of $\mathscr{M}$, and hence $\mathscr{M}$ itself as well as $\mathscr{M}'$, $\mathscr{M}(I-E')$, $(I-E')\mathscr{M}'(I-E')$, $\mathscr{N}$, $\mathscr{N}'$, $\mathscr{N}(I-F')$, $(I-F')\mathscr{N}'(I-F')$, is countably-decomposable. Choose unit vectors x and y such that $M=[\mathscr{M}'x]$, $M'=[\mathscr{M}x]$, $N=[\mathscr{N}'y]$, and $N'=[\mathscr{N}y]$ are maximal cyclic projections in $\mathscr{M}$, $\mathscr{M}'$, $\mathscr{N}$, and $\mathscr{N}'$, respectively. (The existence of such projections follows from [2; Lemma 3.3.7].

Suppose that we can show

$$(1)\qquad \psi[D(E')]=D(F')\,,$$

In this case $\psi(E')$ and F' are equivalent, whence, by the finiteness of $\mathscr{N}'$ $\psi(I-E')$ and $I-F'$ are equivalent and our theorem follows from Remark 2. Our task then is to prove (1).

Let G and G' be paired projections (that is, ones having a joint generating vector) in $\mathscr{M}E'$ and $E'\mathscr{M}'E'$, respectively. Then, for each vector z,

$$D_0(G')D_0([E'\mathscr{M}'E'z])=D_0(G)D_0([\mathscr{M}E'z])\,,$$

by The Coupling Theorem (see [1], for example, or [2; Theorem 3.3.8]). From this, we have

$$(2)\qquad D_0(G')D([\mathscr{M}'E'z])D(E')=D_0(G)D([\mathscr{M}E'z])\,.$$

Now

$$(3)\qquad D([\mathscr{M}'E'z])D(M')=D([\mathscr{M}E'z])D(M)\,,$$

whence, multiplying (2) by $D(M)$ and combining with (3) we have

$$D_0(G')D([\mathscr{M}'E'z])D(E')D(M)=D_0(G)D([\mathscr{M}'E'z])D(M')$$

so that

$$(4)\qquad (D_0(G')D(E')D(M)-D_0(G)D(M'))C_{[\mathscr{M}'E'z]}=0\,.$$

Since E' contains a cyclic projection with central carrier $C_{E'}$, z can be so chosen that $C_{[\mathscr{M}'E'z]}=C_{E'}$, and since $C_{E'}D_0(G')=D_0(G')$, $C_{E'}D_0(G)=D_0(G)$, (4) becomes

$$(5)\qquad D_0(G')D(E')D(M)=D_0(G)D(M')\,.$$

Similarly,

$$D_0(\eta(G'))D(F'')D(N) = D_0(\eta(G))D(N') , \tag{6}$$

since η is a unitary equivalence and $\eta(G')$, $\eta(G)$ are paired projections in $F'\mathcal{N}'F'$, $F'\mathcal{N}$. Writing (5) as $D_0(G')D(E')E'D(M)E' = D_0(G)D(M')E'$ and applying η to it we have

$$D_0(\eta(G'))\psi(D(E'))\psi(D(M)) = D_0(\eta(G))\psi(D(M')) . \tag{7}$$

Since ψ is a unitary equivalence and N, N', M, M' are maximal cyclic, we have $\psi(D(M)) = D(N)$, $\psi(D(M')) = D(N')$, so that, comparing (6) and (7),

$$D_0(\eta(G'))D(N)(D(F'') - \psi(D(E'))) = 0 .$$

This being true for each cyclic projection $\eta(G')$ in $F'\mathcal{N}'F'$, $D(N)(D(F'') - \psi(D(E'))) = 0$, whence $C_N(D(F'') - \psi(D(E'))) = 0$. But $C_N = I$, so that $D(F'') = \psi(D(E'))$, and the proof is complete.

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