THE COEFFICIENT REGIONS OF STARLIKE FUNCTIONS

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1. The coefficient regions of schlicht functions have been studied at some length by Schaeffer, Schiffer, and Spencer [2, 3]. Properties of these coefficient regions are obtained only with difficulty, and in particular the actual coefficient regions can be computed only with a great deal of labor [2]. In fact, the computations necessary to determine the coefficient region of (a_2, a_3, a_4) probably would be prohibitive.

The class of starlike functions is of course much simpler in behavior. Since $f(z)=z+a_2z^2+a_3z^3+\cdots$ is starlike if and only if $zf'(z)/f(z)$ has a positive real part in $|z|<1$, one might say that everything is known about such functions. However, in practice, our rather complete knowledge about functions with positive real part proves difficult to apply back to the class of starlike functions. This is easily seen to be true by noting the number of papers on starlike functions which appear every year.

In an earlier paper, the writer presented a new variational method in the class of starlike functions. It is the purpose of this paper to apply this variational method to find the coefficient regions for starlike functions.

Let S^* be the class of all normalized functions $f(z)=z+a_2z^2+a_3z^3+\cdots$, schlicht and starlike in the unit circle. Let $\boldsymbol{V}_n^*$ be the $(2n-2)$ dimensional region composed of all points $(a_2, a_3, \cdots, a_n)$ belonging to the functions of S^*. Since the class of functions $p(z)$ with $p(0)=1$, regular and having a positive real part in $|z|<1$, is a compact family, so is S^*. Thus $\boldsymbol{V}_n^*$ is a closed domain (i.e., the closure of a domain).

We will study $\boldsymbol{V}_n^*$ by determining its cross sections with $a_2, a_3, \cdots, a_{n-1}$ held fixed. In § 2, a simple proof of the fact that each such cross section is convex is given. It is then shown that any point on the boundary of this cross section must lie on a particular circle, and thus that the cross section itself is a circle. The actual equations for the region $\boldsymbol{V}_n^*$ can be determined for each n by means of a simple recursion, but the calculation becomes tedious after the first few n.

2. For fixed $a_2, a_3, \cdots, a_{n-1}$, let $\boldsymbol{C}_n^*=\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ be the two dimensional cross section of $\boldsymbol{V}_n^*$ in which a_n varies.

LEMMA 1. *$\boldsymbol{C}_n^*$ is a closed, convex set.*

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Proof. $\boldsymbol{C}_n^*$ is certainly closed, since it is a cross section of the closed set $\boldsymbol{V}_n^*$. To show that it is convex, we introduce a new variation.

If $f(z)$ and $g(z)$ belong to S^*, define for any ε, $0 \leqq \varepsilon \leqq 1$,

$$h_\varepsilon(z) = f(z)^{1-\varepsilon} g(z)^\varepsilon \,. \tag{1}$$

Here, appropriate branches of the powers are chosen so that $h_\varepsilon(z)$ is regular at the origin and has a series expansion $z + \cdots$ there. Taking the logrithmatic derivative of (1), we have,

$$\frac{zh_\varepsilon'(z)}{h_\varepsilon(z)} = (1-\varepsilon)\frac{zf'(z)}{f(z)} + \varepsilon\frac{zg'(z)}{g(z)} \,.$$

Therefore, if f and g are in S^*, so is $h_\varepsilon(z)$, for all ε between 0 and 1.

If $f(z)$ and $g(z)$ are any two functions of S^* belonging to $\boldsymbol{C}_n^*$, say, $f(z) = f_0(z) + a_n z^n + \cdots$, $g(z) = f_0(z) + b_n z^n + \cdots$, where $f_0(z) = z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$, then by direct computation from (1),

$$h_\varepsilon(z) = f^{1-\varepsilon} g^\varepsilon = f_0\left(1 + \frac{a_n z^n}{f_0} + \cdots\right)^{1-\varepsilon}\left(1 + \frac{b_n z^n}{f_0}\right)^\varepsilon$$

$$= f_0 + [a_n - \varepsilon(a_n - b_n)] z^n + \cdots \,,$$

and so, as ε goes from 0 to 1, the n-th coefficient of $h_\varepsilon(z)$ moves along the line between a_n and b_n. Therefore this entire line segment is contained in $\boldsymbol{C}_n^*$, and the lemma is proved.

3. In an earlier paper [**1**], the writer showed by use of a variational method in the class of starlike functions, that any function $f(z)$ in S^* which maximizes $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$ must be of the form

$$f(z) = \frac{z}{\prod_{\nu=1}^{m}(1-\kappa_\nu z)^{\mu_\nu}} \,, \quad \mu_\nu \geqq 0, \ \sum_{\nu=1}^{m} \mu_\nu = 2, \ m \leqq n-1 \tag{2}$$

and that $f(z)$ must satisfy the differential equation

$$\frac{zf'(z)}{f(z)} R(z) = Q(z) \,, \tag{3}$$

where

$$\left\{\begin{aligned} R(z) &= \sum_{\nu=2}^{n}\left[\lambda_\nu \sum_{\mu=1}^{\nu-1} \frac{a_\mu}{z^{\nu-\mu}} - \lambda_\nu^* \sum_{\mu=1}^{\nu-1} a_\mu^* z^{\nu-\mu}\right] , \\ Q(z) &= \sum_{\nu=2}^{n}\left[\lambda_\nu \sum_{\mu=1}^{\nu-1} \frac{\mu a_\mu}{z^{\nu-\mu}} + (\nu-1)\lambda_\nu a_\nu + \lambda_n^* \sum_{\mu=1}^{\nu-1} \mu a_\mu^* a^{\nu-\mu}\right] . \end{aligned}\right. \tag{4}$$

(Here, and throughout the paper, an asterisk attached to a value indicates the complex conjugate of that value.) The function $R(z)$ has m zeros on $|z|=1$ corresponding to the m poles of $f'(z)/f(z)$. The function $Q(z)$ has m zeros on $|z|=1$ corresponding to the tips of the m slits (where $f'(z)=0$). The functions $R(z)$ and $Q(z)$ have $2n-m-2$ additional zeros in common.

In order to study the coefficient regions, we will determine the nature of $\boldsymbol{C}_n^*$ $(a_2, \cdots, a_{n-1})$. Since $\boldsymbol{C}_n^*$ is convex, as shown above, the boundary points of $\boldsymbol{C}_n^*$ can be determined by finding a function which maximizes $\Re\{\lambda_n a_n\}$ for fixed $a_2, a_3, \cdots, a_{n-1}$ and for each $\lambda_n=e^{i\theta}$. If $f(z)$ maximizes $\Re\{\lambda_n a_n\}$, then it also maximizes $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$ where $\lambda_2, \lambda_3, \cdots, \lambda_{n-1}$ are a set of Lagrange multipliers which are determined by the fact that $a_2, \cdots, a_{n-1}$ must take on the prescribed values.

The desired results are obtained by use of $2n-m-2$ zeros which $R(z)$ and $Q(z)$ in (4) have in common. To this end, we obtain the GCD of $R(z)$ and $Q(z)$. The Euclidean algorithm is used in a simple form. That is, having two polynomials of the same degree.

$$p_1(z)=\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n ,$$

$$p_2(z)=\beta_0+\beta_1 z+\cdots+\beta_n z^n ,$$

two new polynomials of lesser degree are obtained by the process

$$\begin{cases} q_1(z)=\dfrac{1}{z}[\beta_0 p_1(z)-\alpha_0 p_2(z)] , \\ q_2(z)=\beta_n p_1(z)-\alpha_n p_2(z) . \end{cases} \tag{5}$$

This scheme is started by taking $Q(z)-R(z)$ and multiplying through by an appropriate power of z (the functions $Q(z)$ and $R(z)$ have no zeros at $z=0$ or $z=\infty$). From (4) this gives a polynomial

$$R_1(z)=\alpha_{0,1}+\alpha_{1,1}z+\cdots+\alpha_{n-2,1}z^{n-2}+\beta^*_{n-2,1}z^{n-1}+\cdots+\beta^*_{0,1}z^{2n-3}$$

where

$$\begin{cases} \alpha_{\nu,1}=(\nu+1)\lambda_n a_{\nu+2}+\nu\lambda_{n-1}a_{\nu+1}+\cdots+\lambda_{n-\nu}a_2 , \\ \beta_{\nu,1}=(\nu+2)\lambda_n a_{\nu+1}+(\nu+1)\lambda_{n-1}a_\nu+\cdots+2\lambda_{n-\nu} . \end{cases} \tag{6}$$

In a similar fashion, taking $Q(z)+R(z)$ we obtain

$$Q_1(z)=\beta_{0,1}+\beta_{1,1}z+\cdots+\beta_{n-2,1}z^{n-2}+\alpha^*_{n-2,1}z^{n-1}+\cdots+\alpha^*_{0,1}z^{2n-3} .$$

The coefficients of $Q_1(z)$ are exactly the conjugates of the coefficients of $R_1(z)$ in reverse order. This is easily seen from (4), except that it must be noted that for the extremal $f(z)$, the center term $\sum_{\nu=2}^{n}(\nu-1)\lambda_\nu a_\nu$ is a purely real number, (see [3])

The polynomials $R_1(z)$ and $Q_1(z)$ have in common the same $2n-m-2$ zeros that $R(z)$ and $Q(z)$ have in common, and each has in addition $m-1$ other zeros. The latter zeros are distinct in $R_1(z)$ and $Q_1(z)$ since any common zero of $R_1(z)$ and $Q_1(z)$ must be a common zero of $R(z)$ and $Q(z)$.

This process may then be continued, combining $R_1(z)$ and $Q_1(z)$ as in the scheme (5) to produce two new polynomials $R_2(z)$ and $Q_2(z)$, each one lower in degree. It is easily seen from (5) that the relationship between the coefficients of $R_1(z)$ and $Q_1(z)$ will be preserved in the reduced polynomial. Thus, as this scheme is continued, pairs of polynomials $R_k(z)$ and $Q_k(z)$ of degree $2n-k-2$ will be produced. The coefficients of $Q_k(z)$ will be the conjugates of the coefficients of $R_k(z)$, in reverse order. $R_k(z)$ and $Q_k(z)$ will have in common the $2n-m-2$ zeros that $R(z)$ and $Q(z)$ have in common, and $m-k$ others, not in common. The process will terminate with $R_m(z)$ and $Q_n(z)$, for these two will then be identical up to a constant factor.

Because of the relationship between the coefficients, we need to determine only $R_k(z)$ for each k. The corresponding $Q_k(z)$ can be computed as needed.

LEMMA 2. *For $1 \leq k \leq m$, the polynomial $R_k(z)$ is of the form*

$$R_k(z)=\alpha_{0,k}+\alpha_{1,k}z+\cdots+\alpha_{n-k-1,k}z^{n-k-1}+\cdots+\beta^*_{n-k-1,k}z^{n-1}$$
$$+\beta^*_{n-k-2,k}z^{n}+\cdots+\beta^*_{0,k}z^{2n-k-2},$$

with

$$\alpha_{\mu,k}=\lambda_n A_{\mu+1,k}+\lambda_{n-1}A_{\mu,k}+\cdots+\lambda_{n-\mu}A_{1,k},$$
$$\beta_{\mu,k}=\lambda_n B_{\mu+1,k}+\lambda_{n-1}B_{\mu,k}+\cdots+\lambda_{n-\mu}B_{1,k}.$$

Here, each $A_{j,k}$ and each $B_{j,k}$ is a polynomial in the a_ν and their conjugates (independent of the λ_ν), and the $A_{j,k}$ and $B_{j,k}$ satisfy the recursion relations

$$(7)\qquad \begin{cases} A_{j,\nu+1}=B_{1,\nu}A_{j+1,\nu}-A_{1,\nu}B_{j+1,\nu}, \\ B_{j,\nu+1}=B^*_{1,\nu}B_{j,\nu}-A^*_{1,\nu}A_{j,\nu}. \end{cases}$$

Proof. We first remark that the coefficients of $R_k(z)$ belonging to powers of z between z^{n-k-1} and z^{n-1} are of no interest to us here. From

(6) we see that the form of the coefficients is as asserted in the lemma for $k=1$. Suppose now that the form is correct for $k=\nu$. Then using the scheme (5) (removing a common factor of λ_n) we can compute

$$\alpha_{\mu,\nu+1}=\frac{1}{\lambda_n}\beta_{0,\nu}\alpha_{\mu+1,\nu}-\frac{1}{\lambda_n}\alpha_{0,\nu}\beta_{\mu+1,\nu}$$

$$=B_{1,\nu}[\lambda_n A_{\mu+2,\nu}+\cdots+\lambda_{n-\mu}A_{2,\nu}+\lambda_{n-\mu-1}A_{1,\nu}]$$

$$-A_{1,\nu}[\lambda_n B_{\mu+2,\nu}+\cdots+\lambda_{n-\mu}B_{2,\nu}+\lambda_{n-\mu-1}B_{1,\nu}]\,.$$

Thus, $\alpha_{\mu,\nu+1}$ has the form asserted in the lemma, with the $A_{j,\nu+1}$ determined by the recursion formula (7). The other recursion formula and the remainder of the lemma is proved in an exactly similar fashion.

LEMMA 3. *For each j, k, $0\leqq j\leqq n-k-1$, $1\leqq k\leqq m$, the $A_{j,k}$ and $B_{j,k}$ of Lemma 2 satisfy the following:*

(i) *$A_{j,k}$ is a polynomial in $a_2, a_3, \cdots, a_{j+k}$ and $a_2^*, a_3^*, \cdots, a_{k-1}^*$.*

(ii) *$B_{j,k}$ is a polynomial in $a_2, a_3, \cdots, a_{j+k-1}$ and $a_2^*, a_3^*, \cdots, a_k^*$.*

(iii) *$B_{1,k}$ is real for any choice of $a_2, a_3, \cdots, a_k$.*

(iv) $A_{j,k}=(j+k-1)B_{1,1}B_{1,2}\cdots B_{1,k-1}a_{j+k}-A_{1,1}B_{1,2}B_{1,3}\cdots B_{1,k-1}B_{j+k-1,1}$
$-A_{1,2}B_{1,3}B_{1,4}\cdots B_{1,k-1}B_{j+k-2,2}-\cdots-A_{1,k-2}B_{1,k-1}B_{j+2,k-2}$
$-A_{1,k-1}B_{j+1,k-1}$.

(v) *For any ν, $1\leqq\nu<k$*

$$B_{1,k}=B_{1,\nu}^2B_{1,\nu+1}B_{1,\nu+2}\cdots B_{1,k-1}-|A_{1,k-1}|^2-B_{1,k-1}|A_{1,k-2}|^2$$
$$-B_{1,k-1}B_{1,k-2}|A_{1,k-3}|^2-\cdots$$
$$-B_{1,k-1}B_{1,k-2}\cdots B_{1,\nu+1}|A_{1,\nu}|^2\,.$$

Proof. From (6) we see that

$$\begin{cases}A_{j,1}=ja_{j+1}\,,\\ B_{j,1}=(j+1)a_j,\ \ (B_{1,1}=2)\,,\end{cases}\tag{8}$$

hence properties (i), (ii), and (iii) of the lemma hold true for $k=1$. Using the recursion formulas (7), properties (i) and (ii) can be verified inductively for all $k\leqq m$. Property (iii) is obvious from (7) since $B_{1,k}=|B_{1,k-1}|^2-|A_{1,k-1}|^2$.

Property (iv) is clearly true for $k=1$ from (8). It also can be verified simply by induction on k.

Finally, property (v) is clearly true by (7) and (iii) for $\nu=k-1$ and any k, $1 < k \leqq m$. It can then be proved in general by backward induction on ν. Thus, from (7)

$$B_{1,\nu} = B_{1,\nu-1}^2 - |A_{1,\nu-1}|^2$$

and substituting this for one of the $B_{1,\nu}$ factors in the first term of (v), the corresponding formula for $\nu-1$ is obtained.

4. The reduction process given above must lead to $R_{m+1}(z) \equiv 0$ since $R_m(z)$ and $Q_m(z)$ have all of their roots in common. Therefore the extremal function $f(z)$, maximizing $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$, must have $|A_{1,m}| = |B_{1,m}|$ because of (7). We may now prove.

THEOREM 1. *Let* $(a_2, a_3, \cdots, a_{n-1}) \in \boldsymbol{V}_{n-1}^*$. *If* $(a_2, a_3, \cdots, a_{n-1})$ *is an interior point of* $\boldsymbol{V}_{n-1}^*$ *then* $\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ *is a circular disc determined by* $|A_{1,n-1}| = B_{1,n-1}$; *furthermore* $|A_{1,k}| < B_{1,k}$ *for* $k < n-1$. *If* $(a_2, \cdots, a_{n-1})$ *is a boundary point of* $\boldsymbol{V}_{n-1}^*$ *then* $\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ *consists of a single point.*[1]

Proof. Note that the statement of this theorem makes the tacit assumption that $B_{1,k}$ (which is real by Lemma 3) is always non-negative. This of course will be true by (7) if we merely prove $|A_{1,k}| \leqq |B_{1,k}|$ for all k.

Given $(a_2, \cdots, a_{n-1})$ in $\boldsymbol{V}_{n-1}^*$, Lemma 1 shows that the cross section $\boldsymbol{C}_n^*$ is convex. Hence, given any point a_n on the boundary of $\boldsymbol{C}_n^*$, there is a line of support for $\boldsymbol{C}_n^*$ passing through this point, and therefore a λ_n such that the function (or functions) belonging to this point satisfy (2) and (3). The reduction process described above then leads to $|A_{1,m}| = |B_{1,m}|$ for some m, $1 \leqq m \leqq n-1$.

We now procede to prove the first half of the theorem by induction. If $n=2$, then m must be $n-1=1$, and hence the function corresponding to each boundary point of $\boldsymbol{C}_2^*$ must satisfy $|A_{1,1}| = B_{1,1}$, or, using the values from (8), there is some θ such that $a_2 = 2e^{i\theta}$. Therefore each boundary point of $\boldsymbol{C}_2^*$ is a point of this circle and hence $\boldsymbol{C}_2^*$ consists of the disc $|a_2| \leqq 2$. However, a_2 is an interior point of $\boldsymbol{C}_2^*$ if and only if $|A_{1,1}| < B_{1,1}$.

Now suppose $(a_2, \cdots, a_{n-1})$ is an interior point of $\boldsymbol{V}_{n-1}^*$. Then a_ν is an interior point of $\boldsymbol{C}_\nu^*(a_2, \cdots, a_{\nu-1})$ for $\nu = 2, \cdots, n-1$, and hence by the inductive hypothesis $|A_{1,\nu}| < B_{1,\nu}$ for $\nu = 1, 2, \cdots, n-2$. Therefore $m = n-1$ and each boundary point of $\boldsymbol{C}_n^*$ must, from (iv) of Lemma 3, satisfy

[1] Professor G. Pólya has shown the writer that the fact that the cross sections are circular discs can easily be proved with the help of the Carathéodory theory for functions with positive real part. The exact expressions for these cross sections found from (9), (8), and (7) do not seem to be obtainable from the Carathéodory theory in any simple way however.

$$a_n = \frac{A_{1,1}B_{n-1,1}}{(n-1)B_{1,1}} + \frac{A_{1,2}B_{n-2,2}}{(n-1)B_{1,1}B_{1,2}} + \cdots + \frac{A_{1,n-2}B_{2,n-2}}{(n-1)B_{1,1}B_{1,2}\cdots B_{1,n-2}}$$
$$+ e^{i\theta}\frac{B_{1,n-1}}{(n-1)B_{1,1}B_{1,2}\cdots B_{1,n-2}}$$
$$= C_n + e^{i\theta}R_n \,, \tag{9}$$

for some θ, $0 \leqq \theta \leqq 2\pi$. Then expressions C_n and R_n are rational functions of the a_ν and their conjugates and are defined by (9). In particular R_n is real and positive since $B_{1,n-1} = |B_{1,n-2}|^2 - |A_{1,n-2}|^2 > 0$.

From (9), each a_n on the boundary of $\boldsymbol{C}_n^*$ must lie on the circle with center C_n and radius R_n. This means that $\boldsymbol{C}_n^*$ is itself this circle. Thus if a_n is interior point of $\boldsymbol{C}_n^*$, we must have $|A_{1,n-1}| < B_{1,n-1}$. By induction, the first half of the theorem is proved.

Now suppose that $(a_2, \cdots, a_{n-1})$ is a boundary point of $\boldsymbol{V}_{n-1}^*$. Then there is a unique smallest $\nu \leqq n-1$ such that a_μ is an interior point of $\boldsymbol{C}_\mu^*(a_2, \cdots, a_{\mu-1})$ for $\mu = 2, \cdots, \nu-1$ and a_ν is a boundary point of $\boldsymbol{C}_\nu^*(a_2, \cdots, a_{\nu-1})$. But then $|A_{1,\nu-1}| = B_{1,\nu-1} > 0$, $|A_{1,\mu}| < B_{1,\mu}$ for $\mu < \nu-1$ (and in particular $B_{1,\mu} > 0$ for $\mu = 1, 2, \cdots, \nu-1$), and $B_{1,\nu} = 0$. Choose a sequence of interior points $\{(a_2^{(j)}, \cdots, a_{n-1}^{(j)})\}$ of $\boldsymbol{V}_{n-1}^*$ which approach $(a_2, \cdots, a_{n-1})$. For each such point, $a_n^{(j)}$ is contained in a circle (9) of center $C_n^{(j)}$ and radius $R_n^{(j)}$. Now C_n is a rational function of the coefficients and their conjugates. Hence as $j \to \infty$, $C_n^{(j)}$ must approach some limit, finite or infinite. However this limit cannot be infinite since C_n is always bounded (indeed $|C_n| \leqq n$ because $|a_n| \leqq n$ for starlike functions). Thus the limiting value C_n must exist and be finite. On the other hand, the radius $R_n^{(j)} \to 0$, since by (v) of Lemma 3

$$R_n^{(j)} = \frac{B_{1,n-1}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,n-2}^{(j)}} \leqq \frac{B_{1,\nu}^{(j)2} B_{1,\nu+1}^{(j)} \cdots B_{1,n-2}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,n-2}^{(j)}}$$
$$= \frac{B_{1,\nu}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,\nu-1}^{(j)}} \to 0 \,.$$

Therefore, the cross section $\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ consists of the single point $C_n = \lim_{j\to\infty} C_n^{(j)}$. This completes the proof of the theorem.

5. With the help of the above theorem, we may now describe something of the nature of the coefficient region $\boldsymbol{V}_n^*$. The region $\boldsymbol{V}_n^*$ is $(2n-2)$-dimensional and its boundary is a $(2n-3)$-dimensional manifold. This manifold, however may be decomposed into $n-1$ parts. That is, the boundary of $\boldsymbol{V}_n^*$ is composed of $\Pi_n^{(1)}, \Pi_n^{(2)}, \cdots, \Pi_n^{(n-1)}$, where $\Pi_n^{(\nu)}$ is a $(2\nu-1)$-dimensional manifold lying on the surface of $\boldsymbol{V}_n^*$ and such

that $(a_2, a_3, \cdots, a_n)$ is in $\prod_n^{(\nu)}$ if and only if $(a_2, \cdots, a_\nu)$ is an interior point of V_ν^* and $(a_2, \cdots, a_{\nu+1})$ is a boundary point of $V_{\nu+1}^*$.

For example, from (9) we can explicitly calculate the first few cross sections C_2^*, C_3^*, C_4^*. The boundaries of these cross sections are given by

$$a_2 = 2e^{i\theta}, \tag{10}$$

$$a_3 = \frac{3a_2^2}{4} + e^{i\theta}\frac{4-|a_2|^2}{4}, \tag{11}$$

$$a_4 = \frac{4a_2a_3}{6} + \frac{(4a_3-3a_2^2)(6a_2-2a_2^*a_3)}{6(4-|a_2|^2)} + e^{i\theta}\frac{(4-|a_2|^2)^2-|4a_3-3a_2^2|^2}{6(4-|a_2|^2)}. \tag{12}$$

Taking for example V_4^*, the 5-dimensional manifold $\prod_4^{(3)}$ is defined by (10), (11), and (12) as a_2 varies in the interior of the disc (10), a_3 varies in the interior of the disc (11), and θ varies from 0 to 2π. The 3-dimensional manifold $\prod_4^{(2)}$ is determined by (10), (11), and

$$a_4 = \frac{4a_2a_3}{6} + e^{i\theta}\frac{6a_2-2a_2^*a_3}{6}$$

as a_2 varies in the interior of the disc (10) and θ varies from 0 to 2π. Finally, the 1-dimensional manifold $\prod_4^{(1)}$ is determined by $a_2=2e^{i\theta}$, $a_3=3e^{2i\theta}$, $a_4=4e^{3i\theta}$ and θ varies from 0 to 2π.

As a final remark, we may note that the coefficient regions V_n^* become quite "thin" as n becomes large. In fact, using (v) of Lemma 3

$$R_n = \frac{B_{1,n-1}}{(n-1)B_{1,1}\cdots B_{1,n-2}} \leqq \frac{B_{1,1}^2B_{1,2}\cdots B_{1,n-2}}{(n-1)B_{1,1}\cdots B_{1,n-2}} = \frac{2}{n-1},$$

and hence the radius of any cross section C_n^* is less than or equal to $2/(n-1)$. This estimate in sharp since it is attained for $a_2=a_3=\cdots=a_{n-1}=0$, the functions being

$$f(z)=z(1-e^{i\theta}z^{n-1})^{-2/(n-1)}.$$

Since a function $f(z)$ is convex if and only if the function $zf'(z)$ is starlike, the structure of the coefficient regions for convex functions can be determined directly from the structure of the coefficient regions of starlike functions.

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