ADDITION THEOREMS FOR SOLUTIONS OF THE WAVE EQUATION IN PARABOLIC COORDINATES

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1. Introduction. The wave equation

$$\Delta U+k^2U=0$$

admits solutions of the form

$$U_{\kappa,\mu}=A_{\kappa,\mu}(\xi)B_{\kappa,\mu}(\eta)C_{\kappa,\mu}(\phi)$$

if the coordinate system is such that separation of variables is possible. ξ, η and ϕ are the three independent variables, and κ and μ represent arbitrary complex parameters. In general $U_{\kappa,\mu}$ will not be regular and one-valued over the whole space, but will be so for special values of κ and μ. Let ξ', η' and ϕ' be functions of ξ, η, and ϕ resulting from a translation or rotation of the coordinate system; then a relation which expresses $U_{\kappa,\mu}(\xi', \eta', \phi')$ as a summation of terms of the form $U_{\kappa,\mu}(\xi, \eta, \phi)$ is called an addition theorem.

Addition theorems for cylindrical and spherical coordinate systems are well known. These are the addition theorems for Bessel and Hankel functions, Legendre polynomials, spherical harmonics, Mathieu functions and spheroidal wave functions (see Meixner and Schäfke [5] and Erdélyi [2]).

It is proposed to derive such addition theorems for those functions of the paraboloid of revolution which are regular and one-valued in the whole space. As will be seen subsequently, these restrictions are not always necessary. That such theorems might exist can be inferred from the invariance of ΔU under rotations and translations of space, and from the fact that the family of solutions that are everywhere regular and one-valued will be mapped onto itself by motions of space.

It is possible to derive several of these theorems by using known addition theorems. For example, it is possible to derive linear relations between the functions of the paraboloid of revolution and spherical harmonics. Since an addition theorem under a rotation of coordinates is known for the latter functions, it is possible to derive one for the functions of the paraboloid of revolution.

2. The functions of the paraboloid of revolution. The introduction

Received November 14, 1956. The work was sponsored by the Office of Scientific Research under Contract No. Af 18(600)-367. The author wishes to express his thanks to Prof. W. Magnus for suggesting the problem considered here and for his help and interest during the course of the investigation.

of the parabolic coordinates

$$x=2\sqrt{\xi\eta}\cos\phi$$

$$y=2\sqrt{\xi\eta}\sin\phi$$

$$z=\xi-\eta$$

into the wave equation

$$\Delta U+k^2U=0$$

leads to the equation

$$\frac{1}{2(\xi+\eta)}\left\{\frac{\partial}{\partial\xi}2\xi\frac{\partial U}{\partial\xi}+\frac{\partial}{\partial\eta}2\eta\frac{\partial U}{\partial\eta}+\frac{\xi+\eta}{2\xi\eta}\frac{\partial^2 U}{\partial\phi^2}\right\}+k^2U=0\ .$$

The method of separation of variables then shows, that the solution U can be expressed in terms of functions of the type

$$U=f_1(\xi)f_2(\eta)e^{-i\mu\phi}\ .$$

In the notation of Buchholz [1], these can be represented by

$$f_1(\xi)=m_\chi^\mu(-2ik\xi)=(-2ik\xi)^{\mu/2}e^{ik\xi}\frac{{}_1F_1\left(\frac{1+\mu}{2}-\chi;\ 1+\mu;\ -2ik\xi\right)}{\Gamma(1+\mu)}$$

and

$$f_1(\xi)=w_\chi^\mu(-2ik\xi)=\frac{\pi}{\sin\pi\mu}\left[\frac{m_\chi^{-\mu}(-2ik\xi)}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)}-\frac{m_\chi^\mu(-2ik\xi)}{\Gamma\left(\frac{1-\mu}{2}-\chi\right)}\right].$$

In case μ is an integer, $w_\chi^\mu(-2ik\xi)$ must be derived by a limit process from the above definition. Similarly

$$f_2(\eta)=m_\chi^\mu(2ik\eta)=(2ik\eta)^{\mu/2}e^{-ik\eta}\frac{{}_1F_1\left(\frac{1+\mu}{2}-\chi;\ 1+\mu;\ 2ik\eta\right)}{\Gamma(1+\mu)}$$

and

$$f_2(\eta)=w_\chi^\mu(2ik\eta)=\frac{\pi}{\sin\pi\mu}\left[\frac{m_\chi^{-\mu}(2ik\eta)}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)}-\frac{m_\chi^\mu(2ik\eta)}{\Gamma\left(\frac{1-\mu}{2}-\chi\right)}\right].$$

When μ is an integer the function $m_\chi^\mu(z)$ is regular and single-valued over the entire space; $w_\chi^\mu(z)$ in general is neither single-valued nor regular.

For the case $\chi=n+\frac{1+\mu}{2}$ the function $m_{\chi}^{\mu}(z)$ can be expressed in terms of the more familiar Laguerre polynomials

$$m^{\mu}_{n+\frac{1+\mu}{2}}(z)=\frac{n!}{\Gamma(n+\mu+1)}z^{\mu/2}e^{-z/2}L_n^{\mu}(z) .$$

However, the more general notation introduced by Buchholz in his book on confluent hypergeometric functions will be used throughout this article.

The generating function for the functions

$$\Omega_n^{\mu}(P)=\frac{\Gamma(1+n+\mu)}{n!}m^{\mu}_{n+\frac{1+\mu}{2}}(-2ik\xi)m^{\mu}_{n+\frac{1+\mu}{2}}(2ik\eta)e^{-i\mu\phi} \qquad n=0,\ 1,\ 2,\ \cdots$$

is known as the Hardy-Hille expansion (for proof and additional reference see [1].) For the sake of completeness, it will be stated as a theorem.

THEOREM 1. *For* $|t|<1,\ \mu\neq-1,\ -2,\ \cdots$

$$(1) \qquad G_{\mu}(P,\ t)=\sum_{n=0}^{\infty}\Omega_n^{\mu}(P)(-t)^n=\frac{\exp\left[ik(\xi-\eta)\frac{1-t}{1+t}\right]J_{\mu}\left(\frac{4k\sqrt{\xi\eta t}}{1+t}\right)e^{-i\mu\phi}}{t^{\mu/2}(1+t)} .$$

The case in which μ is a negative integer must be treated with some care. From the limit relationship [1]

$$\lim_{\mu\to-m} m^{\mu}_{n+\frac{1+\mu}{2}}(-2ik\xi)m^{\mu}_{n+\frac{1+\mu}{2}}(2ik\eta)$$

$$=\begin{cases}\left[\frac{n!}{(n-m)!}\right]^2 m^{m}_{n+\frac{1-m}{2}}(-2ik\xi)m^{m}_{n+\frac{1-m}{2}}(2ik\eta) , & n\geqq m\\ 0 , & n<m\end{cases}$$

it follows that

$$\lim_{\mu\to-m} G_{\mu}(P,\ t)=(-t)^m G_m(P,\ t)\,e^{2im\phi} .$$

A relationship between the spherical wave functions and the parabolic functions can now be established. The Fourier expansions of a plane wave in cylindrical and spherical coordinates respectively are [4]

$$\exp\left(ik[z\cos\Psi+\rho\cos\phi\sin\Psi]\right)=\sum_{m=0}^{\infty}i^m\varepsilon_m J_m(k\rho\sin\Psi)\,e^{ikz\cos\Psi}\cos m\phi ,$$

$$e^{ikr\cos\gamma}=\sqrt{\frac{\pi}{2kr}}\sum_{0}^{\infty}(2n+1)i^n J_{n+1/2}(kr)P_n(\cos\gamma) ,$$

$$\cos \gamma = \cos \theta \cos \Psi + \sin \theta \sin \Psi \cos \phi ,$$

$$P_n(\cos \gamma) = \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \Psi) \cos m\phi .$$

Comparison of coefficients of $\cos m\phi$ leads to

$$\exp(ikz \cos \Psi) J_m(k\rho \sin \Psi)$$
$$= \sum_{n=|m|}^{\infty} i^{n-m}(2n+1)\frac{(n-m)!}{(n+m)!} j_n(kr) P_n^m(\cos \theta) P_n^m(\cos \Psi) ,$$
$$m=0, \pm 1, \pm 2, \cdots ,$$

where

$$j_n(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+1/2}(kr) .$$

If we substitute $\frac{1-t}{1+t}$ for $\cos \Psi$ here, introduce parabolic coordinates, and then use Theorem 1, we obtain an expression for $G_n(P, t)$ in terms of spherical harmonics:

$$(2) \quad G_m(P, t) = \sum_{n=m}^{\infty} i^{n-m}(2n+1)\frac{(n-m)!}{(n+m)!} j_n(kr) P_n^m(\cos \theta) \frac{P_n^m\left(\frac{1-t}{1+t}\right) e^{-im\phi}}{t^{m/2}(1+t)} ,$$

$$r = \xi + \eta , \qquad \cos \theta = \frac{\xi - \eta}{\xi + \eta} .$$

The right-hand side of (2) can be expanded in a power series in t by using

$$\frac{P_n^m\left(\frac{1-t}{1+t}\right)}{t^{m/2}(1+t)} = \frac{(-)^m \frac{(n+m)!}{(n-m)!} {}_2F_1\left(m-n, m+n+1; 1+m; \frac{t}{1+t}\right)}{m!(1+t)^{m+1}} ,$$

The left-hand side of (2) has been defined as a power series in t by equation (1). Comparing coefficients of equal powers of t in this series leads to

$$\Omega_s^m(P) = \sum_{n=m}^{\infty} a(n; m, s) j_n(kr) P_n^m(\cos \theta) e^{-im\phi} ,$$

(3)

$$a(n; m, s) = \frac{i^{n+m}(2n+1)}{m!} \sum_{r=0}^{s} \frac{(-)^r (m-n)_{(r)} (m+n+1)_{(r)} (r+m+1)_{(s-r)}}{(m+1)_{(r)} (s-r)!\, r!} ,$$
$$m=0, 1, 2, \cdots .$$

That the above series converges everywhere follows from the fact that $a(n;\, m,\, s)P_n^m(\cos\theta)$ behaves like a power of n for large n, but $j_n(kr)$ is $O\left(\frac{1}{n!}\right)$.

In order to find the inverse to the above relationship, the variable t is replaced by $\frac{w}{1-w}$ in (2). From the resulting power series expansion it now follows that

$$(4)\quad \sum_{s=0}^{l}(-)^s\frac{l![(m+l)!]^2}{(l-s)!(m+s)!}\Omega_s^m(P)$$
$$=\sum_{n=l+m}^{\infty} i^{n+m+2l}(2n+1)\frac{(n-m)!}{(n+m)!}b(n;\, m,\, l)j_n(kr)P_n^m(\cos\theta)e^{-im\phi}\ ,$$

where

$$b(n;\, m,\, l)=\frac{(n+m+l)!}{(n-m-l)!}\ ,\qquad m=0,\, 1,\, 2,\, \cdots .$$

The following vectors and matrices can now be defined:

$$a_l(m)=\sum_{s=0}^{l}(-)^{s+l}\frac{l![(m+l)!]^2}{(l-s)!(m+s)!}\Omega_s^m(P)\ ,$$

$$A(m)=\begin{pmatrix} a_0(m)\\ a_1(m)\\ a_2(m)\\ \vdots \end{pmatrix},$$

$$\beta_n^{(m)}=i^{n+m}(2n+1)\frac{(n-m)!}{(n+m)!}j_n(kr)P_n^m(\cos\theta)e^{-im\phi}\ ,$$

$$B(m)=\begin{pmatrix} \beta_m(m)\\ \beta_{m+1}(m)\\ \beta_{m+2}(m)\\ \vdots \end{pmatrix},$$

$$C(m)=\begin{pmatrix} b(m;\, m,\, 0) & b(m+1;\, m,\, 0) & b(m+2;\, m,\, 0) & \cdots\\ 0 & b(m+1;\, m,\, 1) & b(m+2;\, m,\, 1) & \cdots\\ 0 & 0 & b(m+2;\, m,\, 2) & \cdots\\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

With this notation the system of equations represented by (4) can be written as

$$A(m)=C(m)B(m)\,, \qquad m=0,1,\cdots. \tag{5}$$

In order to express the spherical functions in terms of parabolic functions it is necessary to invert the system (5). The inverse of the matrix $C(m)$ is given by

$$C^{-1}(m)=\begin{pmatrix} \gamma(m;\,m,\,0) & \gamma(m;\,m,\,1) & \gamma(m;\,m,\,2) & \cdots \\ 0 & \gamma(m+1;\,m,\,1) & \gamma(m+1;\,m,\,2) & \cdots \\ 0 & 0 & \gamma(m+2;\,m,\,2) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where

$$\gamma(n;\,m,\,l)=\frac{(-)^{n+m+l}(2n+1)}{(m-n+l)!(m+l+n+1)!}\,.$$

To prove the assertion that this matrix is really the inverse of $C(m)$, it must be shown that

$$\sum_{i=j}^{k}\gamma(m+j;\,m,\,i)b(m+k;\,m,\,i)=\delta_{jk}\,.$$

We have

$$\begin{aligned}&\sum_{i=j}^{k}\gamma(m+j;\,m,\,i)b(m+k;\,m,\,i)\\ &\quad=\sum_{i=j}^{k}\frac{(-)^{i+j}(2m+2j+1)(2m+k+i)!}{(i-j)!(2m+j+i+1)!(k-i)!}\\ &\quad=\frac{(2m+k+j)!}{(k-j)!(2m+2j)!}\,{}_2F_1(j-k,\,2m+k+j+1;\,2m+2j+2;\,1)\\ &\quad=\frac{(2m+k+j)!}{(k-j)!(2m+2j)!}\,\frac{\Gamma(2m+2j+2)\Gamma(1)}{\Gamma(2m+k+j+2)\Gamma(1+j-k)}=\begin{cases}0, & k\geqq 1+j\\ 1, & k=j\,.\end{cases}\end{aligned}$$

Use of the inverse matrix allows one to write

$$\begin{aligned}&j_n(kr)P_n^m(\cos\theta)e^{-im\phi}\\ &\quad=\frac{(n+m)!}{(n-m)!}\sum_{j=n-m}^{\infty}\frac{i^{n+m}[(m+j)!]^2}{(j-n+m)!(m+n+j+1)!}\sum_{s=0}^{j}(-)^s\frac{j!}{(j-s)!(m+s)!}\Omega_s^m(P)\,.\end{aligned} \tag{6}$$

One can now state

Theorem 2. *For* $m=0,1,2,\cdots$

$$\Omega_s^m(P)=\sum_{n=m}^{\infty} a(n;\, m,\, s) j_n(kr) P_n^m(\cos\theta) e^{-im\phi}\,,$$

$$a(n;\, m,\, s)=\frac{i^{n+m}(2n+1)}{m!}\sum_{r=0}^{s}\frac{(-)^r(m-n)_{(r)}(n+m+1)_{(r)}(r+m+1)_{(s-r)}}{(m+1)_{(r)}(s-r)!\, r!}\,,$$

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi}$$
$$=\frac{(n+m)!}{(n-m)!}\sum_{j=n-m}^{\infty}\frac{i^{n+m}[(m+j)!]^2}{(j-n+m)!(m+n+j+1)!}\sum_{s=0}^{j}(-)^s\frac{j!}{(j-s)!\,(m+s)!}\Omega_s^m(P)\,.$$

It is not permissible to interchange the two summations in (6) because the coefficient of the inner summation is $O(1/j)$. Although the series does not converge absolutely it can be shown to converge conditionally. The inverse Laplace transform of the Kummer function is given by [2]

$${}_2F_1(-\sigma;\, 1+m;\, -2ik\xi)=\frac{m!(-2ik\xi)^{-m}}{2\pi i}\int_C\frac{\exp\left[-2ik\xi z\left(1-\frac{1}{z}\right)^{\sigma}\right]}{z^{m+1}}dz$$

where C is a circle enclosing the origin and $z=1$. If $\Omega_s^m(P)$ is expressed in terms of Kummer functions, then (6) can be rewritten as

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi}=\sum_{j=n-m}^{\infty}\frac{(n+m)!\, i^{n+m}e^{-m\phi}[(m+j)!]^2(2k\sqrt{\xi\eta})^{-m}e^{ik(\xi-\eta)}}{(n-m)!\qquad (j-n+m)!(m+n+j+1)!}$$
$$\cdot\frac{1}{(2\pi i)^2}\int_C\int_{C'}\frac{e^{2ik(\eta\zeta-\xi z)}}{(z\zeta)^{m+1}}\left[\frac{1}{z}+\frac{1}{\zeta}-\frac{1}{z\zeta}\right]^j dz\, d\zeta\,.$$

On sufficiently large circles the quantity $\left[\frac{1}{z}+\frac{1}{\zeta}-\frac{1}{z\zeta}\right]$ becomes sufficiently small so that an interchange of summation and integrations is permissible and the series converges. One then obtains the double integral

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi}=\frac{(n+m)!}{(n-m)!}\frac{e^{-im\phi}e^{ik(\xi-\eta)}i^{n+m}n!\, n!}{(2k\sqrt{\xi\eta})^{m+1}(2n+1)!}$$
$$\cdot\frac{1}{(2\pi i)^2}\int_C\int_{C'}\frac{e^{2ik(\eta\zeta-\xi z)}}{(\zeta z)^{m+1}}\left[\frac{1}{z}+\frac{1}{\zeta}-\frac{1}{z\zeta}\right]^{n-m}{}_2F_1\Big(n+1,\, n+1;\, 2n+2;$$
$$\cdot\frac{1}{\zeta}+\frac{1}{z}-\frac{1}{\zeta z}\Big)dz\, d\zeta\,.$$

As consequences of Theorem 2 and the integral relations [4]

$$\int_0^{\pi}P_n^m(\cos\theta)P_{n'}^m(\cos\theta)\sin\theta\, d\theta=\frac{2(n+m)!}{(2n+1)(n-m)!}\delta_{n,n'}$$

$$\int_0^\pi \frac{[P_n^m(\cos\theta)]^2\,d\theta}{\sin\theta}=\frac{(n+m)!}{m(n-m)!}$$

one can state the following.

COROLLARY 1.

$$\int_0^\pi [\Omega_s^m(P)]^2 \sin\theta\,d\theta=\sum_{n=m}^\infty [a(n;\,m,\,s)j_n(kr)e^{-im\phi}]^2\frac{2(n+m)!}{(2n+1)(n-m)!}\,,$$

$$\int_0^\pi \Omega_s^m(P)P_n^m(\cos\theta)\sin\theta\,d\theta=a(n;\,m,\,s)j_n(kr)\frac{2(n+m)!}{(2n+1)(n-m)!}e^{-im\phi}$$

$$\int_0^\pi \frac{\Omega_s^m(P)P_n^m(\cos\theta)\,d\theta}{\sin\theta}=\sum_{n=m}^\infty a(n;\,m,\,s)j_n(kr)e^{-im\phi}\frac{(n+m)!}{m(n-m)!}$$

$$\int_0^\pi \Omega_s^m(P)\Omega_\sigma^m(P)\sin\theta\,d\theta=\sum_{n=m}^\infty a(n;\,m,\,s)a(n;\,m,\,\sigma)j_n^2(kr)e^{-2im\phi}\frac{2(n+m)!}{(2n+1)(n-m)!}\,,$$

3. The addition theorem resulting from a translation of the axes along the axis of symmetry.

Since z is the axis of symmetry one can introduce the translated coordinates

$$x'=x\,,\qquad y'=y\,,\qquad z'=z-\xi_0\,.$$

It follows from Theorem 1 that

$$(7)\quad G_\mu(P,\,t)=\frac{\exp\left[ikz\dfrac{1-t}{1+t}\right]J_\mu\left(\dfrac{2kp\sqrt{t}}{1+t}\right)e^{-i\mu\phi}}{t^{\mu/2}(1+t)}=\exp\left[ik\xi_0\frac{1-t}{1+t}\right]G_\mu(P',\,t)\,.$$

In particular, for $\mu=\eta=0$, $\xi=\xi_0$ Theorem 1 yields

$$\exp\left[ik\xi_0\frac{1-t}{1+t}\right]=(1+t)\sum_{n=0}^\infty m_{n+1/2}^0(-2ik\xi_0)(-t)^n\,.$$

Using this expression in (7), expanding and multiplying the power series in t and comparing coefficients, we obtain the following.

THEOREM 3.

$$\Omega_n^\mu(P)=\sum_{j=0}^n [m_{n+1/2-j}^0(-2ik\xi_0)+m_{n-1/2-j}^0(-2ik\xi_0)(\delta_{nj}-1)]\Omega_j^\mu(P')\,,$$
$$\mu\neq -1,\,-2,\,\cdots\,;\quad n=0,\,1,\,2,\,\cdots\,.$$

The case in which μ is a negative integer can be handled as a limiting case of Theorem 3. By differentiating both sides with respect to ξ_0 at $\xi_0=0$ one obtains the following.

COROLLARY 2.

$$\frac{d}{d(2ik\xi_0)}\Omega_n^\mu(P')\Big|_{\xi_0=0}=-\sum_{n=0}^{n}\Omega_j^\mu(P)\Big(1-\frac{\delta_{jn}}{2}\Big).$$

In particular for $\eta=0$ one obtains from the above

$$\begin{aligned}&\frac{1}{n!}\Gamma(1+\mu+n)\frac{d}{d(2ik\xi)}m^\mu_{n+(1+\mu)/2}(-2ik\xi)\\&=\frac{1}{4ik\xi n!}\mu m^\mu_{n+(1+\mu)/2}(-2ik\xi)\Gamma(1+\mu+n)\\&\quad+\sum_{j=0}^{n}\frac{1}{j!}\Gamma(1+\mu+j)m^\mu_{j+(1+\mu)/2}(-2ik\xi)\Big(1-\frac{\delta_{jn}}{2}\Big).\end{aligned}$$

It is possible to define a vector

$$V^\mu(P)=\begin{pmatrix}\Omega_0^\mu(P)\\ \Omega_1^\mu(P)\\ \Omega_2^\mu(P)\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}$$

and a matrix

$$T(\xi_0)=\begin{pmatrix}a_{00}&0&0&\cdots\\ a_{10}&a_{11}&0&\cdots\\ a_{20}&a_{21}&a_{22}&\cdots\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{pmatrix},$$

where

$$a_{nj}=\begin{cases}[m^0_{n+1/2-j}(-2ik\xi_0)+m^0_{n-1/2-j}(-2ik\xi_0)(\delta_{nj}-1)]\,, & n\geqq j\\ 0\,, & n<j\,,\end{cases}$$

such that Theorem 3 can be restated as follows.

THEOREM 3′.

$$V^\mu(P)=T(\xi_0)V^\mu(P')\qquad \mu\neq-1,\ -2,\ -3,\ \cdots.$$

4. The addition theorem resulting from a translation of axes perpendicular to the axis of symmetry.

The translation can be assumed to be in the x-direction without loss of generality. Introducing the new coordinates

$$x=x'-\delta\ ,\quad y=y'\ ,\quad z=z'\ ,$$

$$R=\sqrt{\rho^2+\delta^2-2\rho\delta\cos\phi'}\ ,$$

$$e^{2i(\phi-\phi')}=\frac{\rho-\delta e^{-i\phi'}}{\rho-\delta e^{i\phi'}}\ ,$$

$$P=(x,\, y,\, z)\ ,$$

$$P'=(x',\, y',\, z')\ ,$$

one obtains from Theorem 1

$$G_\mu(P,\, t)=\frac{\exp\left[ikz\dfrac{1-t}{1+t}\right]J_\mu\left(\dfrac{2kR\sqrt{t}}{1+t}\right)e^{-i\mu\phi}}{t^{\mu/2}(1+t)}\ .$$

Under the condition $\rho>\delta$ one can take advantage of the addition theorem for the Bessel functions

$$J_\mu(kR)e^{-i\mu\phi}=\sum_{-\infty}^{\infty}J_n(k\delta)J_{n+\mu}(kr)e^{-i(n+\mu)\phi'}$$

and obtain

$$\begin{aligned} G_\mu(P,\, t)&=\sum_{-\infty}^{\infty}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}\,\frac{\exp\left[ikz\dfrac{1-t}{1+t}\right]J_{n+\mu}\left(\dfrac{2k\rho\sqrt{t}}{1+t}\right)e^{-i(n+\mu)\phi'}}{t^{(\mu+n)/2}(1+t)} \\ &=\sum_{-\infty}^{\infty}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}G_{\mu+n}(P',\, t) \qquad \mu\neq\pm1,\ \pm2,\ \cdots\ . \end{aligned} \tag{8}$$

The case where μ is an integer must be handled as a limiting case. To determine the addition theorem one must expand both sides in powers of t and compare coefficients. Using

$$t^{-n/2}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)=\sum_{s=0}^{\infty}g_{s,n}t^s\ ,$$

$$g_{s,n}=\sum_{r=0}^{s}\frac{(k\delta)^{2s-2r+n}(-)^s(2s-2r+n)_{(r)}}{(s-r)!r!(n+s-r)!}\ ,$$

one obtains the following.

THEOREM 4.

$$(-)^s\Omega_s^\mu(P)=\sum_{n=1}^{s}\sum_{j=0}^{s} g_{s-j,n}\Omega_j^{\mu+n}(P')+\sum_{n=0}^{\infty}(-)^n\sum_{j=0}^{s} g_{s-j,n}\Omega_j^{\mu-n}(P')\ ,$$

for $\mu\neq\pm 1, \pm 2, \cdots$. *For* $\mu=m$, *with* m *a positive integer*,

$$(-)^s\Omega_s^m(P)=\sum_{j=0}^{s}\sum_{n=0}^{s} g_{s-j,n}\Omega_j^{n+m}(P')+\sum_{j=0}^{s}\sum_{n=m}^{j+m} g_{s-j,n}(-)^n\Omega_{j+m-n}^{n-m}(P')e^{2i(n-m)\phi'}\ .$$

For $\mu=-m$

$$\lim_{\mu\to -m}\Omega_n^\mu(P)=\begin{cases}\Omega_{n-m}^m e^{2im\phi}\ , & n\geq m\\ 0\ , & n<m\ .\end{cases}$$

Another method by which such addition theorems can be derived is to take advantage of a theorem by Friedman [3], which is an addition theorem for spherical harmonics under translations of the coordinate system. This theorem in combination with Theorem 2 will yield an addition theorem, but in a very cumbersome form. Conversely the theorem for spherical harmonics could be derived by using Theorems 2 and 4.

A similar plan will be used in the next section. The addition theorem for spherical harmonics under rotations of the coordinate system in combination with Theorem 2 yields the corresponding theorem for parabolic functions.

5. The addition theorem resulting from a rotation of coordinates.

Since a rotation about the axis of symmetry, namely the z-axis, yields trivial results, a rotation about the y-axis will be used without loss of generality. Let

$$\begin{aligned} z&=z'\cos\Psi-x'\sin\Psi\\ x&=x'\cos\Psi+z'\sin\Psi \qquad (9)\\ y&=y'\ .\end{aligned}$$

Under this rotation the following addition theorem holds for the spherical harmonics [2]:

$$P_n^m(\cos\theta)e^{-im\phi}=\sum_{l=-n}^{n} g_l\frac{(n-|l|)!}{(n+|l|)!}S_{2n}^{n+m,n+l}(\Psi)P_n^{|l|}(\cos\theta')e^{-il\phi'}\ ,$$

where

$$S_{2n}^{n+m,n+l}(\Psi)=(-\ )^{n+m}\binom{n-m}{n+l}\left(\cos\frac{\Psi}{2}\right)^{-m-l}\left(i\sin\frac{\Psi}{2}\right)^{m-l}$$
$$\cdot {}_2F_1\left(-n-l,\ n-l+1;\ 1-m-l;\ \cos^2\frac{\Psi}{2}\right)$$

for $m+l \leq 0$, and

$$S_{2n}^{n+m,n+l}(\Psi)=-\binom{n+m}{n-l}\left(\cos\frac{\Psi}{2}\right)^{m+l}\left(-i\sin\frac{\Psi}{2}\right)^{l-m}$$
$$\cdot {}_2F_1\left(l-n,\ n+l+1;\ 1+m+l;\ \cos^2\frac{\Psi}{2}\right)$$

for $m+l>0$, and where

$$g_l=\begin{cases}1\ , & l\geq 0\\ (-1)^l\ , & l\leq 0\ .\end{cases}$$

Using the above in conjunction with Theorem 2 one can state the full addition theorem.

THEOREM 5. *Under a rotation of coordinates* (9) *the following statement holds*:

$$\Omega_s^m(P)=\sum_{n=m}^{\infty}a(n;\ m,\ s)\sum_{l=-n}^{n}g_l S_{2n}^{n+m,n+l}(\Psi)\sum_{j=n-|l|}^{\infty}\frac{i^{n+|l|}[(j+|l|)!]^2}{(j-n+|l|)!(j+n+|l|+1)!}$$
$$\cdot\sum_{s=0}^{j}(-)^s\frac{j!}{(j-s)!(m+s)!}\Omega_s^{|l|}(P')e^{i(|l|-l)\phi'}\ .$$

6. The infinitesimal transformations. It is possible to restate the addition theorems for infinitesimal transformations. The theorem for a translation along the z-axis can be rewritten from Theorem 3:

$$a_{n,j}=[m_{n+1/2-j}^0(-2ik\xi_0)+m_{n-1/2-j}^0(-2ik\xi_0)(\delta_{nj}-1)]\ ,\qquad n\geq j\ ,$$

where

$$m_k^0(z)=e^{-z/2}{}_1F_1\left(\frac{1}{2}-k;\ 1;\ z\right)\ .$$

For small values of ξ_0, namely $d\xi_0$, it follows that

$$a_{n,j}=\begin{cases}\delta_{nj}+2ikd\xi_0\left(1-\dfrac{\delta_{nj}}{2}\right)\ , & n\geq j\\ 0\ , & n<j\end{cases}$$

and that

$$T(d\xi_0)=I+ikd\xi_0\begin{pmatrix}1&0&0&\cdots\\2&1&0&\cdots\\2&2&1&\cdots\\ \cdot&\cdot&\cdot&\\ \cdot&\cdot&\cdot&\\ \cdot&\cdot&\cdot&\end{pmatrix},\tag{10}$$

where I is the identity matrix.

THEOREM 3″. *Consider an infinitesimal translation along the z-axis such that*

$$x'=x\ , \qquad y'=y\ , \qquad z'=z-d\xi_0\ .$$

Then

$$V^{\mu}(P)=T(d\xi_0)V^{\mu}(P')\ , \qquad \mu\neq -1,\ -2,\ \cdots\ ,$$

where $T(d\xi_0)$ *is given by* (10) *and* $V^{\mu}(P)$ *is as defined in Theorem* 3′.

Similarly one can find the addition theorem for translations in the x-direction from expression (8):

$$G_{\mu}(P,\ t)=\sum_{-\infty}^{\infty} J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}G_{\mu+n}(P',\ t)\ .$$

For a differential translation $d\delta$ this expression reduces to

$$G_{\mu}(P,\ t)=G_{\mu}(P',\ t)+\frac{kd\delta}{1+t}[tG_{\mu+1}(P',\ t)-G_{\mu-1}(P',\ t)]$$

from which it is possible to state

THEOREM 4′. *For an infinitesimal translation of coordinates given by*

$$x=x'-d\delta\ , \qquad y=y'\ , \qquad z=z'$$

the following holds:

$$\Omega_n^{\mu}(P)=\Omega_n^{\mu}(P')-kd\delta\left\{\sum_{l=0}^{n}\Omega_l^{\mu-1}(P')+\sum_{l=1}^{n-1}\Omega_l^{\mu+1}(P')\right\}\ , \quad \mu\neq 0,\ -1,\ -2,\ \cdots\ .$$

For negative integral values of μ one can use limit processes.

To derive the analogous theorem for a rotation of coordinates it is first necessary to derive the addition theorem for the spherical harmonics. This can be done conveniently by starting with the following definition of the spherical harmonics [2]:

$$D_1^{n-m}(D_2+iD_3)^m\frac{1}{r}=\frac{(-)^{n-m}(n-m)!}{r^{n+1}}P_n^m(\cos\theta)e^{\pm im\phi}\ , \tag{11}$$

where

$$D_1=\frac{d}{dz}\ , \quad D_2=\frac{d}{dx}\ , \quad D_3=\frac{d}{dy}\ .$$

Under the rotation

$$x' = z \sin \Psi + x \cos \Psi$$
$$y' = y$$
$$z' = z \cos \Psi - \sin \Psi$$

these differential operators are also transformed :

$$D_1 = D_1' \cos \Psi + D_2' \sin \Psi$$
$$D_2 = -D_1' \sin \Psi + D_2' \cos \Psi$$
$$D_3 = D_3' \, .$$

Let

$$D_2 - iD_3 = Q \, , \qquad D_2 + iD_3 = \overline{Q} \, .$$

Then it follows that

$$D_1^{n-m} Q^m = \left[D_1' \cos \Psi + \frac{1}{2} \sin \Psi (Q' + \overline{Q}') \right]^{n-m} \left[-D_1' \sin \Psi + \frac{1}{2} \cos \Psi (Q' + \overline{Q}') + \frac{1}{2} (Q' - \overline{Q}') \right]^m . \tag{12}$$

The existence of the operational equivalence

$$Q\overline{Q} \frac{1}{r} \equiv -D_1^2 \frac{1}{r}$$

follows from

$$(D_1^2 + Q\overline{Q}) \frac{1}{r} \equiv \Delta \frac{1}{r} = 0 \, .$$

If Ψ is taken to be a differential angle $d\Psi$ in (12), then one obtains from (11)

$$e^{-im\phi} P_n^m(\cos \theta) = e^{-im\phi'} P_n^m(\cos \theta') - \frac{d\Psi}{2} [e^{-i(m+1)\phi'} P_n^{n+1}(\cos \theta') - (n+m)(n-m+1) e^{-i(m-1)\phi'} P_n^{m-1}(\cos \theta')] \, . \tag{13}$$

Equation (2) written in the form

$$G_m(P, t) = \sum_{n=m}^{\infty} i^{n+m}(2n+1) j_n(kr) P_n^m(\cos \theta) e^{-im\phi} \frac{{}_2F_1\left(m-n, \, m+n+1; \, m+1; \, \frac{t}{1+t}\right)}{m! \, (1+t)}$$

combined with (13) yields

$$G_m(P, t)=G_m(P', t)-\frac{d\Psi}{2}\sum_{n=m}^{\infty} i^{n+m}(2n+1)j_n(kr)P_n^{m+1}(\cos\theta')e^{-i(m+1)\phi'}$$

$$\cdot\frac{{}_2F_1\left(m-n,\ m+n+1;\ m+1;\ \frac{t}{1+t}\right)}{m!\,(1+t)^m}$$

(14)

$$+\frac{d\Psi}{2}\sum_{n=m}^{\infty} i^{n+m}(2n+1)j_n(kr)P_n^{m-1}(\cos\theta')e^{-i(m-1)\phi'}$$

$$\cdot\frac{(n+m)(n-m+1)\,{}_2F_1\left(m-n,\ m+n+1;\ m+1;\ \frac{t}{1+t}\right)}{m!\,(1+t)^m}\ .$$

In order to be able to rewrite the above as generating functions one can make use of the differentiation formulas [2]

$$\frac{d}{dz}[z^{m+1}(1-z)^{m+1}\,{}_2F_1(m-n+1,\ m+n+2;\ m+2;\ z)]$$

$$=(m+1)z^m(1-z)^m\,{}_2F_1(m-n,\ m+n+1;\ m+1;\ z)\ ,$$

$$\frac{d}{dz}[{}_2F_1(m-n-1,\ m+n;\ m;\ z)]$$

$$=\frac{-(n+m)(n-m+1)}{m}\,{}_2F_1(m-n,\ m+n+1;\ m+1;\ z)\ .$$

Using these in (14) one obtains

$$G_m(P, t)=G_m(P', t)+\frac{id\Psi}{2}\left\{\frac{(1+t)^{m+2}}{t^m}\frac{d}{dt}\left[\left(\frac{t}{1+t}\right)^{m+1}G_{m+1}(P', t)\right]\right.$$

$$\left.-(1+t)^{2-m}\frac{d}{dt}[(1+t)^{m-1}G_{m-1}(P', t)]\right\}$$

from which one derives

$$G_m(P, t)=G_m(P', t)+\frac{id\Psi}{2}\left[(m+1)G_{m+1}(P', t)+t(1+t)\frac{d}{dt}G_{m+1}(P', t)\right.$$

$$\left.-(m-1)G_{m-1}(P', t)-(1+t)\frac{d}{dt}G_{m-1}(P', t)\right]\ .$$

One can now state the following.

THEOREM 5′. *Under the infinitesimal rotation*

$$x' = x + z d\Psi \;, \qquad y' = y \;, \qquad z' = z - x d\Psi$$

one has the formula

$$\Omega_n^m(P) = \Omega_n^m(P') + \frac{i d\Psi}{2} [(m+1+n)\Omega_n^{m+1}(P') - (n-1)\Omega_{n-1}^{m+1}(P') - (m+n-1)\Omega_n^{m-1}(P') + (n+1)\Omega_{n+1}^{m-1}(P')] \;.$$

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