THE FIVE-POINT DIFFERENCE EQUATION WITH PERIODIC COEFFICIENTS

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The five-point difference equation described in § 1 has most of the important second order partial difference equations as special cases and as limiting forms of these the more important partial differential equations of the second order. In the present paper all coefficients are assumed periodic in the same one of the two independent variables. The purpose of the paper is the study of the form of the general solution as affected by the periodic character of the coefficients. This study centers around the roots of the characteristic equation and so-called semi-periodic solutions. The reader is referred to the theorem of § 5 for a precise statement of results.

1. General discussion. Let us be given the five-point equation

$$k_1(i,j)y(i-1,j)+k_2(i,j)y(i+1,j)+k_3(i,j)y(i,j-1)+k_4(i,j)y(i,j+1)+k_5(i,j)y(i,j)=0 \tag{1}$$

where k_1, k_2, k_3, k_4 and k_5 are defined for integral values of i and j over the rectangle $1\leq i\leq n\omega-1$, $1\leq j\leq \omega-1$ where $n>1$ and $\omega>1$ are integers. This rectangle will be called the *defining rectangle* and will be denoted by R. We assume moreover that

$$k_\nu(i+\omega,j)=k_\nu(i,j)\ , \qquad \nu=1,2,3,4,5 \tag{2}$$

and that neither, k_1, k_2, k_3, nor k_4 is zero at any point of R.

A *solution* of (1) is a function of (i,j) defined at points of R and at the border points $(i=0,\ j=1,2,\cdots,\omega-1)$, $(i=n\omega,\ j=1,2,\cdots,\omega-1)$, $(j=0,\ i=1,2,\cdots,n\omega-1)$, $(j=\omega.\ i=1,2,\cdots,n\omega-1)$ and which satisfies (1) at all points of R. Notice that this second set of points, namely R plus the border points, form a lattice which is rectangular except that its corner points are missing. It will be referred to as the rectangle S.

A *fundamental domain* is a set of points of S such that there exists one and only one solution taking on prescribed arbitrary values at each point of the set.

All fundamental domains[1] contain the same number of points. We denote this number by L. For the rectangle S

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[1] For a detailed discussion see T. Fort, Amer. Math. Monthly, **62**, (1955), 161.

$$L=2\omega(n+1)-4\ .$$

We note that all sets of L points belonging to S do not form a fundamental domain.

A *fundamental system of solutions* consists of L solutions which are linearly independent over a fundamental domain.

If $y_1(i,j), y_2(i,j), \cdots, y_L(i,j)$ are a fundamental system of solutions then any solution $y(i,j)$ can be written

$$y(i,j)=\sum_{\mu=0}^{L} c_\mu y_\mu(i,j)$$

where the c's are constants.

We choose the following fundamental domain for S namely points where

$$\begin{array}{ll} j=0, & i=1,\ 2,\ \cdots,\ n\omega-1\ ; \\ j=1, & i=0,\ \omega,\ \omega+1,\ \cdots,\ 2\omega-1,\ 2\omega,\ 3\omega,\ \cdots,\ n\omega\ ; \\ j=2, & i=0,\ \omega,\ 2\omega,\ \cdots,\ n\omega\ ; \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ j=\omega-1, & i=0,\ \omega,\ 2\omega,\ \cdots,\ n\omega\ ; \\ j=\omega, & i=\omega,\ 2\omega,\ \cdots,\ (n-1)\omega\ . \end{array}$$

We shall refer to this particular fundamental domain as D. The domain D is pictured in figure 1 with $\omega=4$, $n=3$. The points of D have no accompanying numeral.

To prove D a fundamental domain simply assign values for y at each of the points of D and fill in the rectangle by means of (1). This can be done in a variety of ways. For example, the value of y at the points of S not in D can be determined in the order indicated by the accompanying numerals in Figure 1.

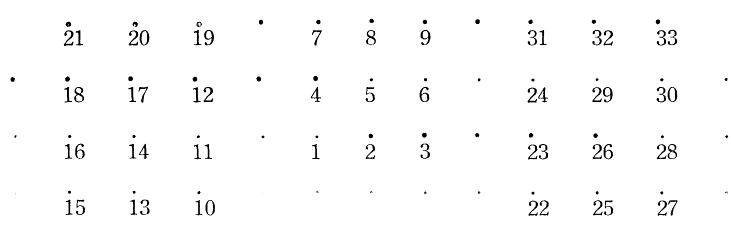

Figure 1.

We now define all coefficients to the right of $i=n\omega-1$ by formula (2). We then define every solution $y(i,j)$ at the points $(n\omega, 0)$, $(n\omega+1, 0)$, $\cdots$, $((n+1)\omega-1, 0)$ by the formula

$$y(n\omega+\nu,\, 0)=y(\nu+1,\, 0)\ , \qquad \nu=0, 1, \cdots, \omega-1\ ;$$

and at the points $((n+1)\omega, j)$, $j=1, 2, \cdots, \omega-1$ by the formula

$$y((n+1)\omega, j)=y(0, j) \quad \text{also} \quad y(n\omega,\, \omega)=y(\omega,\, \omega)\ .$$

This definition serves to determine y over a longer rectangle than S, $n\omega$ being replaced by $(n+1)\omega$, the rectangles being in every other way the same. We call this the rectangle T.

BASIC THEOREM. *If $y(i, j)$ is a solution over S then $y(i+\omega, j)$ is also a solution over S.*

This theorem follows immediately from the periodic character of the coefficients in (1).

THEOREM. *If $y_1(i, j)$, $y_2(i, j)$, $\cdots$, $y_L(i, j)$ are a fundamental system of solutions for S then so are $y_1(i+\omega, j)$, $y_2(i+\omega, j)$, $\cdots$, $y_L(i+\omega, j)$.*

This theorem follows from the fact that $y_1(i, j), y_2(i, j), \cdots, y_L(i, j)$ considered at the points of D constitute L sets of L constants linearly independent over D and that, due to the extension of each solution over T described above, $y_1(i+\omega, j), y_2(1+\omega, j), \cdots, y_L(i+\omega, j)$ at the points of D are precisely the same sets of constants as $y_1(i, j), y_2(i, j), \cdots, y_L(i, j)$ although the order may be different.

2. Semiperiodic solutions. We ask the question : Does there exist a solution of (1) not identically zero over S and satisfying the relation

$$y(i+\omega, j)=\rho y(i, j) \tag{3}$$

where $\rho\neq 0$ is constant? We, of course, except the case where either $(i+\omega, j)$ or (i, j) is a corner point of S since solutions are not defined at corner points.

Let us assume a solution $y_q(i, j)\not\equiv 0$ satisfying (3) and work for necessary conditions. As previously, let $y_1(i, j)$, $y_2(i, j)$, $\cdots$, $y_1(i, j)$ be a fundamental system of solutions for S. Then so are $y_1(i+\omega, j)$, $y_2(i+\omega, j)$, $\cdots$, $y_L(i+\omega, j)$. Consequently

$$y_\nu(i+\omega, j)=\sum_{\mu=1}^{L} a_{\nu\mu}y_\mu(i, j)\ , \qquad \nu=1, \cdots, L\ ,$$

where $\det(a_{\nu\mu})\neq 0$. Moreover

$$y_q(i, j)=\sum_{\mu=1}^{L}\alpha_\mu y_\mu(i, j)$$

where not all the α's are zero. Then

$$y_q(i+\omega, j)=\sum_{\mu=1}^{L}\alpha_\mu y_\mu(i+\omega, j)=\sum_{\mu=1}^{L}\alpha_\mu \sum_{\nu=1}^{L}a_{\mu\nu}y_\nu(i, j)$$

$$=\sum_{\nu=1}^{L}\left[\sum_{\mu=1}^{L}\alpha_\mu a_{\mu\nu}\right]y_\nu(i, j) .$$

Also

$$y_q(i+\omega, j)=\rho y_q(i, j)=\rho\sum_{\nu=1}^{L}\alpha_\nu y_\nu(i, j) .$$

We can equate coefficients since $y_1(i, j), \cdots, y_L(i, j)$ are linearly independent over D. We get

$$\begin{matrix}(a_{11}-\rho)\alpha_1+ & a_{21}\alpha_2 & +\cdots\cdots+ & a_{L1}\alpha_L & =0\\ a_{12}\alpha_1 & +(a_{22}-\rho)\alpha_2+ & \cdots\cdots+ & a_{L2}\alpha_L & =0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{1L}\alpha_1 & + \quad a_{2L}\alpha_2 & +\cdots\cdots+ & (a_{LL}-\rho)\alpha_L & =0 .\end{matrix}$$

But the α's are not all zero. Hence

$$(4) \qquad \begin{vmatrix}(a_{11}-\rho) & a_{21} & \cdots & a_{L1}\\ a_{12} & (a_{22}-\rho) & \cdots & a_{L2}\\ \cdot & \cdot & \cdot & \cdot\\ a_{1L} & a_{2L} & \cdots & (a_{LL}-\rho)\end{vmatrix}=0 .$$

This condition is not only necessary but it is also sufficient as is seen by retracing steps.

Equation (4) is the *characteristic equation* for the problem and its roots are the *characteristic values*.

THEOREM. *The characteristic equation is independent of the particular fundamental system of solutions chosen.*

Consider a second fundamental system, $y_1^{(2)}(i, j), y_2^{(2)}(i, j), \cdots, y_L^{(2)}(i, j)$. Then

$$y_\nu^{(2)}(i+\omega, j)=\sum_{\mu=1}^{L}b_{\nu\mu}y_\nu^{(2)}(i, j) , \qquad \nu=1, \cdots, L .$$

The characteristic equation is

$$\begin{vmatrix}(b_{11}-\rho) & b_{21} & \cdots & b_{L1}\\ b_{12} & (b_{22}-\rho) & \cdots & b_{L2}\\ \cdot & \cdot & \cdot & \cdot\\ b_{1L} & b_{2L} & \cdots & (b_{LL}-\rho)\end{vmatrix}=0 .$$

But

$$y_\nu^{(2)}(i, j) = \sum_{\mu=1}^{L} h_{\nu\mu} y_\mu(i, j) , \qquad \nu = 1, \cdots, L ,$$

where $\det(h_{\nu\mu}) \neq 0$. Hence

$$y_\nu^{(2)}(i+\omega, j) = \sum_{\mu=1}^{L} b_{\nu\mu} y_\mu^{(2)}(i, j) = \sum_{\mu=1}^{L} b_{\nu\mu} \sum_{\eta=1}^{L} h_{\mu\eta} y_\mu(i, j)$$

$$= \sum_{\eta=1}^{L} \left[\sum_{\mu=1}^{L} b_{\nu\mu} h_{\mu\eta} \right] y_\eta(i, j) .$$

On the other hand

$$y_\nu^{(2)}(i+\omega, j) = \sum_{\mu=1}^{L} h_{\nu\mu} y_\mu(i+\omega, j) = \sum_{\mu=1}^{L} h_{\nu\mu} \sum_{\eta=1}^{L} a_{\mu\eta} y_\eta(i, j)$$

$$= \sum_{\eta=1}^{L} \left[\sum_{\mu=1}^{L} h_{\nu\mu} a_{\mu\eta} \right] y_\eta(i, j) .$$

We can equate coefficients, as already explained, because $y_1, y_2, \cdots, y_L$ are linearly independent. We have

$$(5) \qquad \sum_{\mu=1}^{L} b_{\nu\mu} h_{\mu\eta} = \sum_{\mu=1}^{L} h_{\nu\mu} a_{\mu\eta} , \quad \eta = 1, \cdots, L ; \quad \nu = 1, \cdots, L .$$

Now let us form the products

$$\begin{vmatrix} h_{11} & \cdots & h_{1L} \\ \cdot & \cdots & \cdot \\ h_{L1} & \cdots & h_{LL} \end{vmatrix} \cdot \begin{vmatrix} (a_{11}-\rho) & \cdots & a_{1L} \\ \cdot & \cdots & \cdot \\ a_{L1} & \cdots & (a_{LL}-\rho) \end{vmatrix}$$

and

$$\begin{vmatrix} (b_{11}-\rho) & \cdots & b_{1L} \\ \cdot & \cdots & \cdot \\ b_{L1} & \cdots & (b_{LL}-\rho) \end{vmatrix} \cdot \begin{vmatrix} h_{11} & \cdots & h_{1L} \\ \cdot & \cdots & \cdot \\ h_{L1} & \cdots & h_{LL} \end{vmatrix} .$$

If we perform the indicated multiplication and then use (5) we get identical determinants. This establishes the theorem.

THEOREM. *No characteristic value is zero.*

This theorem follows from the fact that $\det(a_{ij}) \neq 0$.

If it were zero then $y_1(i+\omega, j), y_2(i+\omega, j), \cdots, y_L(i+\omega, j)$ would be linearly independent over a fundamental domain which they are not.

3. Roots distinct. Let the roots of the characteristic equation be $\rho_1, \rho_2, \cdots, \rho_L$ and assume that no two are equal. Let corresponding semiperiodic solutions be $y_1(i, j), \cdots, y_L(i, j)$, that is $y_\nu(i+\omega, j)=\rho_\nu(i, j)$, $\nu=1, 2, \cdots, L$ and assume, as we can, that no one of these is identically zero.

THEOREM. *The solutions $y_1, \cdots, y_L$ constitute a fundamental system of solutions.*

To prove this theorem we assume first $y_1, \cdots, y_{L-k}$ are linearly dependent over D but that $y_1, \cdots, y_{L-k-1}$ are linearly independent over D. Then

$$\sum_{\nu=0}^{L-k} \mu_\nu y_\nu(i, j)=0 \tag{6}$$

over D with at least one $\mu \neq 0$. Replace i by $(i+\omega)$. Then

$$\sum_{\nu=0}^{L-k} \mu_\nu y \ (i+\omega, j)=0 \tag{7}$$

over D. This is true because solutions linearly dependent over D are linearly dependent over all of T.

From (7)

$$\sum_{\nu=0}^{L-k} \mu_\nu \rho_\nu y_\nu(i, j)=0 \ . \tag{8}$$

But $\mu_L - k \neq 0$ else $y_1, \cdots, y_{L-k-1}$ would be linearly dependent. Eliminate y_{L-k} between (6) and (8). We get

$$\sum_{\nu=1}^{L-k-1} \mu_\nu(\rho_\nu - \rho_{L-k}) y_\nu(i, j)=0 \ .$$

The only way that this can be true with the linear independence of $y_1, \cdots, y_{L-k-1}$ is that

$$\mu_1(\rho_1-\rho_{L-k})=\mu_2(\rho_2-\rho_{L-k})=\cdots=\mu_{L-k-1}(\rho_{L-k-1}-\rho_{L-k})=0 \ .$$

If $\mu_1=\mu_2=\cdots=\mu_{L-k-1}=0$, then $\mu_{L-k} y_{L-k}(i, j)\equiv 0$. This is not the case since $\mu_{L-k}\neq 0$ and $y_{L-k}(i, j)\not\equiv 0$ over D. Consequently ρ_{L-k} must equal some other ρ. This contradicts our simple root hypothesis. Hence, $y_1, \cdots, y_L$, are linearly independent over D and the theorem is proved.

4. Multiple roots; special discussion. Let us assume that ρ_1 is a double root of the characteristic equation but that all other roots are simple. Let $y_1(i, j)$ be as before; namely $y_1(i+\omega, j)=\rho_1 y_1(i, j)\not\equiv 0$ and let $\tilde{y}_2(i, j)$, $\tilde{y}_3(i, j)$, $\cdots$, $\tilde{y}_L(i, j)$ be so chosen that $y_1, \tilde{y}_2, \tilde{y}_3, \cdots, \tilde{y}_L$ form a fundamental system. We have the relations

$$(9)\qquad y_1(i+\omega, j)=\rho_1 y_1(i, j)\ ,$$

$$\tilde{y}_\nu(i+\omega, j)=c_{\nu 1}y_1(i, j)+\sum_{\mu=2}^{L} c_{\nu\mu}\tilde{y}_\mu(i, j)\ , \qquad \nu=2, \cdots, L\ .$$

The characteristic equation is

$$(10)\qquad \begin{vmatrix} (\rho_1-\rho) & 0 & 0 & \cdots & 0 \\ c_{21} & (c_{22}-\rho) & c_{23} & \cdots & c_{2L} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{L1} & c_{L2} & c_{L3} & \cdots & (c_{LL}-\rho) \end{vmatrix}=0\ .$$

Since ρ_1 is a double root of (10),

$$\begin{vmatrix} (c_{22}-\rho_1) & c_{23} & \cdots & c_{2L} \\ \cdot & \cdot & \cdots & \cdot \\ c_{L2} & c_{L3} & \cdots & (c_{LL}-\rho_1) \end{vmatrix}=0.$$

Hence from (9) the solutions $\tilde{y}_\nu(i+\omega, j)-c_{\nu 1}y_1(i, j)-\rho_1\tilde{y}_\nu(i, j)$, $\nu=2, \cdots, L$ are linearly dependent.

This means that

$$\sum_{\nu=2}^{L} C_\nu\tilde{y}_\nu(i+\omega, j)=y_1(i, j)\sum_{\nu=2}^{L} C_\nu c_{\nu 1}+\rho_1\sum_{\nu=2}^{L} C_\nu\tilde{y}_\nu(i, j)\ .$$

Let $\sum_{\nu=2}^{L} C_\nu c_{\nu 1}=\kappa$ and $Y_2(i, j)=\sum_{\nu=2}^{L} C_\nu\tilde{y}_\nu(i, j)$. We note that $Y_2(i, j)\not\equiv 0$ since $y_1, \tilde{y}_2, \cdots, \tilde{y}_L$ are linearly independent over D. Then

$$(11)\qquad Y_2(i, +\omega, j)=\rho_1 Y_2(i, j)+\kappa y_1(i, j)\ .$$

This is a difference equation in Y_2 as a function of i with difference interval ω. We shall solve[2] for $Y_2(i+\mu\omega, j)$. Let $U(i+\mu\omega, j)$ be a solution of the difference equation

$$(12)\qquad U(i+\omega, j)=\rho_1 U(i, j)\ .$$

Then $U(i+\mu\omega, j)=\rho_1^\mu U(i, j)$. Moreover $U(i, j)$ is arbitrary so we assume it different from zero. Then

$$(13)\qquad Y_2(i+\mu\omega, j)=U(i+\mu\omega, j)\left[\sum_{\nu=0}^{\mu-1}\frac{\kappa y_1(i+\nu\omega, j)}{U(i+(\nu+1)\omega, j)}+c_1(i, j)\right].$$

We note that $y_1(i+\nu\omega, j)=\rho_1^\nu y_1(i, j)$. With this in mind (13) yields

$$Y_2(i+\mu\omega, j)=\left[\frac{\kappa}{\rho_1}\mu y_1(i, j)+U(i, j)\,c_1(i,j)\right]\rho_1^\mu$$

[2] T. Fort, *Finite differences and difference equations in the real domain.* Clarendon, 1948, p. 117.

We rewrite this[3]

$$(14)\qquad Y_2(i+\mu\omega, j)=\left[\frac{\kappa}{\rho_1}\mu y_1(i, j)+Y_2(i, j)\right]\rho_1^{\mu} .$$

This is an interesting form for $Y_2(i+\mu\omega, j)$. We note particularly the μ in the first term of the bracket.

THEOREM. *The solution $y_1(i, j)$, $Y_2(i, j)$, $y_3(i, j)$, $\cdots$, $y_L(i, j)$ form a fundamental system*

To prove this theorem assume the contrary, namely linear dependence:

$$(15)\qquad c_1y_1(i, j)+c_2Y_2(i, j)+c_3y_3(i, j)+\cdots+c_Ly_L(i, j)=0 .$$

Then increasing i by ω yields

$$(16)\qquad c_1\rho_1y_1(i, j)+c_2\kappa y_1(i, j)+c_2\rho_1Y_2(i, j)+c_3\rho_3y_3(i, j)+\cdots+c_L\rho_Ly_L(i, j)=0 .$$

Now c_2 is not zero else $y_1, y_3, \cdots, y_L$ would be linearly dependent which they are not. We eliminate $Y_2(i, j)$ from (15) and (16). We get

$$c_2\kappa y_1(i, j)+c_3(\rho_3-\rho_1)y_3(i, j)+\cdots+c_L(\rho_L-\rho_1)y_L(i, j)=0 .$$

But $c_3, \cdots, c_L$ are not all zero. If they were we would have $y_1(i, j)$ and $Y_2(i, j)$ linearly dependent. They are not since $Y_2(i. j)$ is linearly dependent upon $\tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ and by hypothesis $y_1(i, j)$ is not. It results that ρ_1 must equal at least one of $\rho_3, \cdots, \rho_L$. This contradicts our hypothesis.

We now assume ρ_1 a triple root but that other roots are distinct.

We consider $y_1(i, j)$ and $Y_2(i, j)$ of the double root discussion and note that they are not linearly dependent. We then define $\tilde{y}_3(i, j), \cdots$, $\tilde{y}_L(i, j)$ so that $y_1(i, j)$, $Y_2(i, j)$, $\tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ form a fundamental system. The characteristic equation takes the form

$$\begin{vmatrix} (\rho_1-\rho) & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & (\rho_1-\rho) & 0 & 0 & \cdots & 0 \\ c_{31} & c_{32} & (c_{33}-\rho) & c_{34} & \cdots & c_{3L} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{L1} & c_{L2} & c_{L3} & c_{L4} & \cdots & (c_{LL}-\rho) \end{vmatrix}=0 .$$

Since ρ_1 is a triple root of this equation we have

[3] The convention used in (13) is $\sum_{\nu=0}^{-1} f(\nu)=0$.

$$\begin{vmatrix} (c_{33}-\rho_1) & c_{34} & \cdots & c_{3L} \\ \cdot & \cdot & \cdot & \cdot \\ c_{L3} & c_{L4} & \cdots & (c_{LL}-\rho_1) \end{vmatrix} = 0 \tag{17}$$

It follows from (17) that $\tilde{y}_\nu(i+\omega, j) - c_{\nu 1}y_1(i, j) - c_{\nu 2}Y_2(i, j) - \rho_1\tilde{y}_\nu(i, j)$, $\nu=3, \cdots, L$ are linearly dependent. Let constants determining the linear dependence be $\gamma_3, \gamma_4, \cdots, \gamma_L$. Then let $\kappa_\nu = \sum_{\mu=3}^{L} \gamma_\mu c_{\mu\nu}$, $\nu=1, 2$. Let

$$Y_3(i, j) = \sum_{\mu=3}^{L} \gamma_\mu \tilde{y}_\mu(i, j) .$$

Then

$$Y_3(i+\omega, j) = \kappa_1 y_1(i, j) + \kappa_2 Y_2(i, j) + \rho_1 Y_3(i, j) .$$

This is a difference equation in Y_3 as a function of i with difference interval ω. We solve precisely as we solved (11).

As previously we choose $U(i, j)$ not equal to zero then

$$Y_3(i+\mu\omega, j) = U(i+\mu\omega, j)\left[\sum_{\nu=1}^{\mu-1} \frac{\kappa_1 y_1(i+\nu\omega, j) + \kappa_2 Y_2(i+\nu\omega, j)}{U(i+(\nu+1)\omega, j)} + c(i, j)\right] .$$

Now substitute $y_1(i+\nu\omega, j) = \rho_1^\nu y_1(i, j)$ and

$$Y_2(i+\nu\omega, j) = \rho_1^\nu\left[\frac{\kappa}{\rho_1^\nu} y_1(i, j) + Y_2(i, j)\right] .$$

we get

$$Y_3(i+\mu\omega, j) = \rho_1^{\mu-1}\mu\kappa y_1(i, j) + \kappa\kappa_2\rho_1^{\mu-2}\frac{\mu(\mu-1)}{3\,!} y_1(i, j)$$
$$+ \kappa_2\rho_1^{\mu-1}\mu Y_2(i, j) + \rho_1^\mu Y_3(i, j) .$$

5. Multiple roots ; general discussion. The work that we have just done is easily generalized. Details are ommitted but can be readily supplied by one who has read § 4.

THEOREM. *If ρ_1 is an α_1-fold root of the characteristic equation there exist solutions of* (1) *which we call $Y_1^{(1)}(i, j)$, $Y_2^{(1)}(i, j), \cdots, Y_{\alpha_1}^{(1)}(i, j)$, were*

$$Y_1^{(1)}(i+\mu\omega, j) = \rho_1^\mu Y_1^{(1)}(i, j) ,$$
$$Y_2^{(1)}(i+\mu\omega, j) = \rho_1^\mu[c_1^{(1)}\mu Y_1^{(1)}(i, j) + Y_2^{(1)}(i, j)] ,$$
$$Y_3^{(1)}(i+\mu\omega, j) = \rho_1^\mu\left[\left\{c_1^{(2)}\mu + c_2^{(2)}\frac{\mu(\mu-1)}{2\,!}\right\}Y_1^{(1)}(i, j) + c_3^{(2)}\mu Y_2^{(1)}(i, j) + Y_3(i, j)\right] ,$$

. . . .

$$
\begin{aligned}
&Y^{(1)}_{\alpha_1}(i+\mu\omega, j)\\
&=\rho_1^{\mu}\Big[\Big\{c_1^{(\alpha_1)}\mu + c_2^{(\alpha_1)}\frac{\mu(\mu-1)}{2\,!} + \cdots + c^{(\alpha_1)}_{\alpha_1-1}\frac{\mu(\mu-1)\cdots(\mu-\alpha_1+2)}{(\alpha_1-1)\,!}Y_1^{(1)}(i,j)\Big\}\\
&\quad+\Big\{c^{(\alpha_1)}_{\alpha_1}\mu+\cdots+c^{(\alpha_1)}_{2\alpha_1-3}\frac{\mu(\mu-1)\cdots(\mu-\alpha_1+3)}{(\alpha_1-2)\,!}\Big\}Y_2^{(1)}(i,j)+\cdots\\
&\quad+c^{(\alpha_1)}_{(\alpha_1^2-\alpha_1)/2}\,\mu\, Y^{(1)}_{\alpha_1-1}(i,j)+Y^{(1)}_{\alpha_1}(i,j)\Big].
\end{aligned}
$$

If the roots are ρ_1 of order α_1, ρ_2 of order α_2, $\cdots$, ρ_t of order α_t; then the solutions $Y_1^{(1)}$, $Y_2^{(1)}$, $\cdots$, $Y^{(2)}_{\alpha_1}$, $Y_1^{(1)}$, $\cdots$, $Y^{(2)}_{\alpha_2}$, $\cdots$, $Y_1^{(t)}$, $\cdots$, $Y^{(t)}_{\alpha_t}$ form a fundamental system of solutions.

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