ON THE CASIMIR OPERATOR

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The Casimir operator is an important tool in the study of associative [4], Lie [4] and alternative algebras [7]. However its use has been for algebras of characteristic 0. We give a new definition of the Casimir operator for associative, Lie and alternative algebras, which keeps desirable properties of the usual Casimir operator and which is useful for arbitrary characteristic.

We show that under certain conditions our Casimir operator is the identity transformation and for non-degenerate alternative (or associative) algebras we show that it is the transformation into which the identity element of the algebra maps. We apply our results to obtain the first Whitehead lemma for non-degenerate alternative algebras of arbitrary characteristic. We also obtain a special case of the Levi theorem for Lie algebras of prime characteristic.

1. The Casimir Operator. Let $\mathfrak{A}$ be an associative, Lie or alternative algebra with basis $e_1, e_2, \cdots, e_n$ over an arbitrary field $\mathfrak{F}$. For uniformity we use the notation $x \to S_x$ for a representation of $\mathfrak{A}$, where if $\mathfrak{A}$ is alternative we mean the S_x part of a representation $x \to (S_x, T_x)$. If $\mathfrak{A}$ is a Lie or associative algebra, $f(x, y) = t(S_xS_y)$ where t is the trace function, is an invariant symmetric bilinear form. In [7, p. 444] it is shown that if $\mathfrak{A}$ is alternative this form is invariant if $\mathfrak{F}$ is not of characteristic 2. For arbitrary characteristic we have

$$t(S_xS_{yz}) = t(S_xS_yS_z + S_xT_yS_z - S_xS_zT_y)$$
$$= t(S_xS_yS_z + S_xT_yS_z - T_yS_xS_z) = t(S_{xy}S_z)\ .$$

Similarly $t(T_xT_y)$ is invariant.

We call $\mathfrak{A}$ *non-degenerate* if $t(R_xR_y)$ is non-degenerate where R is the representation of right multiplications. It can be shown that this is equivalent to the non-degeneracy of the bilinear form $t(L_xL_y)$ of the left multiplications. It is well known that if $\mathfrak{A}$ is a non-degenerate alternative (or associative) algebra it is a direct sum of simple algebras. Dieudonne [3] has shown that this is also true for Lie algebras.

If $\mathfrak{A}$ is semi-simple and $\mathfrak{F}$ is of characteristic 0, the usual Casimir operator Γ_S^* for the representation S is defined as follows: Let $\mathfrak{N}$ be the set of all x of $\mathfrak{A}$ such that $t(S_xS_y) = 0$ for all y of $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{N} \oplus \mathfrak{C}$ where $\mathfrak{N}$ and $\mathfrak{C}$ are semi-simple ideals of $\mathfrak{A}$. Let $e_1', e_2', \cdots, e_k'$ be the

Received October 8, 1956. This research was supported by the United State Air Force through the office of Scientific Research of the Air Research and Development Command.

complementary basis to a basis $e_1, e_2, \cdots, e_k$ of $\mathfrak{C}$ such that[2] $t(S_i S_j') = \delta_{ij}$ (Kroneker's delta). (Note that the complementary basis depends on the representation.) Then $\Gamma_S^* = \sum_{i=1}^{k} S_i S_i'$.

For arbitrary $\mathfrak{F}$ we define a new Casimir operator Γ_S for each non-degenerate $\mathfrak{A}$. This will include every semi-simple $\mathfrak{A}$ of characteristic 0, since $\mathfrak{A}$ is non-degenerate in this case. We use the same complementary basis $e_1', e_2', \cdots, e_n'$ such that $t(R_i R_i') = \delta_{ij}$ for every representation (or anti-representation) and define

$$\Gamma_S = \sum_{i=1}^{n} S_i S_i' \,. \tag{1}$$

If $\mathfrak{A}$ is alternative we also define $\Gamma_T = \sum_{i=1}^{n} T_i T_i'$.

Unlike Γ_S^*, Γ_S does not automatically reduce to zero when $t(S_x S_y) = 0$ for all x, y of $\mathfrak{A}$. In fact it follows from Corollary 3.1 below that for alternative algebras $\Gamma_S \neq 0$ if $S \neq 0$. We note also that for the representation $x \to R_x$ we have $\Gamma_R^* = \Gamma_R$.

Analogous to the corresponding result for Γ_S^* for Lie and associative algebras [4, p. 682] and for alternative algebras [7, p. 445] we have the following theorem.

THEOREM 1. *Let Γ_S be the Casimir operator* (1) *for a representation $x \to S_x (x \to (S_x, T_x))$ of a non-degenerate Lie or associative (alternative) algebra $\mathfrak{A}$ over an arbitrary field. Then Γ_S commutes with S_x (and T_x) for all x of $\mathfrak{A}$.*

Except for the commutativity of Γ_S and T_x which will be proved along with Lemma 3.2, the proof is similar to those in the references.

We also have the following result which follows from the properties of the complementary basis.

THEOREM 2. *Let $\mathfrak{A}$ be a non-degenerate associative, Lie or alternative algebra over an arbitrary field. Then the Casimir operators Γ_R and Γ_L of the right and left multiplications of $\mathfrak{A}$ are both the identity transformation.*

2. Application to alternative (and associative) algebras.

Since every associative algebra is an alternative algebra, the results of this section hold for associative algebras.

In place of the identities (4) of [6] used in the definition of a representation $x \to (S_x, T_x)$ of an alternative algebra $\mathfrak{A}$, we will use the

[2] For simplification we write S_{e_i} as S_i and $S_{e_i'}$ as S_i'.

equivalent (except for characteristic 2) identities

$$(2) \qquad S_x^2 = S_{x^2}\,, \quad T_x^2 = T_{x^2} \qquad \text{for all } x \text{ of } \mathfrak{A},$$

in order to insure that the *semi-direct sum* [6, p. 3] or *split null extension* $\mathfrak{S} = \mathfrak{A} + \mathfrak{M}$ of $\mathfrak{A}$ and the representation space $\mathfrak{M}$ is an alternative algebra for arbitrary characteristic.

THEOREM 3. *For every representation S of a non-degenerate alternative algebra $\mathfrak{A}$, $\Gamma_S = S_e$ where $e = \Sigma e_i e_i'$ is the identity element of $\mathfrak{A}$.*

The proof follows from Theorem 2 and the properties of the complementary basis.

COROLLARY 3.1. *If $S \neq 0$ the matrix of Γ_S can be taken to have the form* diag$(I, 0)$. *Hence if in addition the representation is irreducible, Γ_S is the identity transformation.*

Proof. By (2), $S_e^2 = S_e$ and the result follows.

COROLLARY 3.2. *$\Gamma_S S_x = S_x$ for all x of $\mathfrak{A}$.*

Proof. Assume $S \neq 0$ and take Γ_S to have the form diag$(I, 0)$. Then the matrix of S_x must have the form diag(S_x', S_x'') where I and S_x' have the same order. By identity (4) of [6] we have $T_x\Gamma_S - \Gamma_S T_x = S_x - S_x\Gamma_S$. Hence $S_x'' = 0$ and $T_x = \text{diag}(T_x', T_x'')$ and so $S_x\Gamma_S = S_x$. This completes the proof of Theorem 1, for we also have $T_x\Gamma_S = \Gamma_S T_x$.

Evidently all of the above results also hold when S is replaced by T.

Now for a non-degenerate alternative algebra $\mathfrak{A}$ with neither S nor $T = 0$ we may apply Corollary 3.1 and Theorem 1 to take

$$(3) \qquad \begin{aligned} &\Gamma_S = \text{diag}(I^{(1)}, I^{(2)}, 0^{(3)}, 0^{(4)})\,, \quad \Gamma_T = \text{diag}(I^{(1)}, 0^{(2)}, I^{(3)}, 0^{(4)}) \\ &S_x = \text{diag}(S_x^{(1)}, S_x^{(2)}, S_x^{(3)}, 0^{(4)})\,, \quad T_x = \text{diag}(T_x^{(1)}, T_x^{(2)}, T_x^{(3)}, 0^{(4)}) \end{aligned}$$

where the superscript (i) indicates the matrix has order k_i and each I is an identity matrix and $S_x^{(3)} = 0^{(3)}$, $T_x^{(2)} = 0^{(2)}$. Also $x \to (S_x^{(i)}, T_x^{(i)})$, $(i = 1, 2, 3)$ are representations of $\mathfrak{A}$ with respective Casimir operators

$$(4) \qquad \begin{aligned} &\Gamma_S^{(1)} = \Gamma_T^{(1)} = I^{(1)}\,; \quad \Gamma_S^{(2)} = I^{(2)}\,, \quad \Gamma_T^{(2)} = 0^{(2)}\,; \\ &\Gamma_S^{(3)} = 0^{(3)}\,, \quad \Gamma_T^{(3)} = I^{(3)}\,. \end{aligned}$$

Thus the representation space $\mathfrak{M}$ can be expressed as $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$

$+\mathfrak{M}_3+\mathfrak{M}_4$ where $\mathfrak{M}_i$ is an invariant subspace of dimension k_i and hence is an ideal of the split-null extension $\mathfrak{S}=\mathfrak{A}+\mathfrak{M}$. It also follows that $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are in the *nucleus* [2] of $\mathfrak{S}$.

We are now able to obtain the following generalization of the first Whitehead lemma (see [8]) for alternative algebras of characteristic zero [6, Theorem 3].

THEOREM 4. *Let $\mathfrak{A}$ be a non-degenerate alternative algebra over an arbitrary field and let $x\rightarrow(S_x, T_x)$ be a representation of $\mathfrak{A}$ acting in a space M. Let $\mathfrak{S}$ be the split null extension $\mathfrak{S}=\mathfrak{A}+\mathfrak{M}$ and let $h(x)$ be a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}$ such that*

$$h(xy)=xh(y)+h(x)y=h(x)S_y+h(y)T_x \tag{5}$$

for all x, y of $\mathfrak{A}$. Then $h(x)$ is an inner derivation of $\mathfrak{S}$. If $\mathfrak{A}$ is not of characteristic 2 then[3]

$$h(x)=[x, g]+\frac{x}{2}\sum_{i=1}^{n}\{[R_i', R_{h(e_i)}]+[L_i', L_{h(e_i)}]\} \tag{6}$$

where g is in the nucleus of $\mathfrak{S}$; R, L are right and left multiplications in $\mathfrak{S}$ and $e_1', e_2', \cdots, e_n'$ are a complementary basis to a basis $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$.

Proof. If either S or T is zero the theorem follows similarly to the associative characteristic zero case, so assume neither is. Since $\mathfrak{M}_i$ is invariant,

$$h(x)=h_0(x)=h_1(x)+h_2(x)+h_3(x)$$

where $h_j(x)$ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}_j$ ($\mathfrak{M}_0=\mathfrak{M}$) such that

$$h_j(xy)=xh_j(y)+h_j(x)y=h_j(x)S_y+h_j(y)T_x\ .$$

Then we have

$$h_j(x)\Gamma_S=\sum_{i=1}^{n}\{h_j(xe_i)e_i'-xh_j(e_i)\cdot e_i'\}=\sum_{i=1}^{n}\{h_j(e_i)(e_i'x)-xh_j(e_i)\cdot e_i'\}\ .$$

Consequently for $j=0, 1, 2, 3$

$$h_j(x)\Gamma_S=x\sum_{i=1}^{n}\{L_i'L_{h_j(e_i)}-R_{h_j(e_i)}R_i'\}\ . \tag{7}$$

Similarly

$$h_j(x)\Gamma_T=x\sum_{i=1}^{n}\{R_i'R_{h_j(e_i)}-L_{h_j(e_i)}L_i'\}\ . \tag{8}$$

[3] We use $[P, Q]$ to denote the commutator $PQ-QP$.

By (3) and (4) we have

$$h(x)=h_1(x)\Gamma_S+h_2(x)\Gamma_S+h_3(x)\Gamma_T\ .$$

Hence by (7) and (8) $h(x)=xD$ where

$$D=\sum_i \{L_i'L_{h_1(e_i)}-R_{h_1(e_i)}R_i'\}+\sum_i \{L_i'L_{h_2(e_i)}-R_{h_2(e_i)}R_i'\}$$
$$+\sum_i \{R_i'R_{h_3(e_i)}-L_{h_3(e_i)}L_i'\}\ .$$

To show that D is inner it suffices to show that for x, y in $\mathfrak{S}$, $L_xL_y-R_yR_x$ is in the Lie algebra $\mathfrak{L}(\mathfrak{S})$ of linear transformations generated by the right and left multiplications of $\mathfrak{S}$. This is true since $L_xL_y-R_yR_x=2[R_y, L_x]+L_{yx}-R_{yx}$.

Now let $\mathfrak{A}$ have characteristic $\neq 2$ and use (7) and (8) to get

$$h(x)(\Gamma_S+\Gamma_T)=x\left\{\sum_i [R_i', R_{h(e_i)}]+\sum_i [L_i', L_{h(e_i)}]\right\}\ .$$

Then by (7) and the nucleus property of $\mathfrak{M}_2$ we have[4] $h_2(x)\Gamma_S=[x, v_2]$ where $v_2=\sum_i h_2(e_i)e_i'$ is in $\mathfrak{M}_2$. Similarly $h_3(x)\Gamma_T=[x, v_3]$ where v_3 is in $\mathfrak{M}_3$. But

$$h(x)(\Gamma_S+\Gamma_T)+h_2(x)\Gamma_S+h_3(x)\Gamma_T=2h(x)$$

hence

$$h(x)=[x, g]+x\left\{\frac{1}{2}\sum_i [R_i', R_{h(e_i)}]+\frac{1}{2}\sum_i [L_i', L_{h(e_i)}]\right\}$$

where $g=\frac{1}{2}(v_2+v_3)$ is in the nucleus of $\mathfrak{S}$.

As is the case for similar theorems, the first part of Theorem 4 can be stated in the following form.

THEOREM 5. *Let $\mathfrak{A}$ be a non-degenerate subalgebra of an alternative algebra $\mathfrak{B}$ over an arbitrary field. Then any derivation of $\mathfrak{A}$ into $\mathfrak{B}$ can be extended to an inner derivation of $\mathfrak{B}$.*

3. Application to Lie algebras.

We obtain the following special case of the generalization of the Levi theorem to algebras of prime characteristic.

THEOREM 6. *Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field with radical $\mathfrak{R}\neq\mathfrak{L}$ such that $\mathfrak{L}\mathfrak{R}=0$ and $\mathfrak{L}/\mathfrak{R}$ is non-degenerate. Then there is an algebra $\mathfrak{S}$ (which is isomorphic to $\mathfrak{L}/\mathfrak{R}$ and is a direct sum of*

[4] This actually $=-v_2x$ since $xv_2=0$.

simple algebras) such that $\mathfrak{L}$ *is the direct sum* $\mathfrak{L}=\mathfrak{S}\oplus\mathfrak{N}$.

Proof. Let $e_1, e_2, \cdots, e_n$ be a basis for $\mathfrak{L}$ such that $e_1, e_2, \cdots, e_k$ are a basis for a subspace $\mathfrak{B}$ and $e_{k+1}, \cdots, e_n$ are a basis for $\mathfrak{N}$. Then the right multiplication of each x of $\mathfrak{L}$ has the form

$$R_x=\begin{bmatrix} P_x & Q_x \\ 0 & 0 \end{bmatrix} \tag{9}$$

where $P_x=Q_x=0$ if x is in $\mathfrak{N}$ and P_x is the right multiplication of the image $\bar{x}$ of x in $\mathfrak{L}/\mathfrak{N}$. Now if $\Gamma_P=\sum_{i=1}^{k} P_iP'_i$ is the Casimir operator (1) for the representation P of $\mathfrak{L}/\mathfrak{N}$, then by Theorem 2, Γ_P is the identity I and hence

$$\Gamma=\sum_{i=1}^{k} R_iR'_i=\begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix}.$$

By using the properties of the complementary basis of $\mathfrak{L}/\mathfrak{N}$ and the fact that the Lie algebra of right multiplications of the elements of $\mathfrak{B}$ is isomorphic to $\mathfrak{L}/\mathfrak{N}$ it can be shown that Γ commutes with R_x for all x of $\mathfrak{L}$.

We now show that the associative algebra $\mathfrak{L}^*$ generated by the R_x for all x of $\mathfrak{L}$ is isomorphic to the associative algebra $\mathfrak{P}^*$ generated by the P_x. Certainly by (9) there is a homomorphism of $\mathfrak{L}^*$ onto $\mathfrak{P}^*$ which maps any polynomial $p(R_x, R_y, \cdots)$ into $p(P_x, P_y, \cdots)$. Now if $p(R_x, R_y, \cdots)=0$ then $p(P_x, P_y, \cdots)=0$ since Γ commutes with $p(R_x, R_y, \cdots)$. Hence $\mathfrak{L}^*\cong\mathfrak{P}^*$.

Now $\mathfrak{L}/\mathfrak{N}$ is a direct sum of simple algebras and therefore [**1**, Lemma 2], $\mathfrak{P}^*$ (and hence $\mathfrak{L}^*$) is semi-simple. Consequently [**1**, Lemma 2] $\mathfrak{L}$ is a direct sum of an algebra $\mathfrak{S}$, which is a direct sum of simple algebras, and an abelian algebra $\mathfrak{N}_1$. But we must have $\mathfrak{N}_1=\mathfrak{N}$ completing the proof.

It is to be noted that it is easy to give examples of prime characteristic where all but the non-degeneracy of $\mathfrak{L}/\mathfrak{N}$ of the hypothesis is satisfied but for which the conclusion is false.

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