MONOTONE COMPLETENESS OF NORMED SEMI-ORDERED LINEAR SPACES

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Introduction. Let R be a continuous semi-ordered linear space, namely, a semi-ordered linear space where, for any sequence $x_\nu \geqq 0$ $(\nu=1,2,\cdots)$, $\bigcap_{\nu=1}^{\infty} x_\nu$ exists.[1] R is said to be a normed semi-ordered linear space, if a norm $\|x\|(x \in R)$ is defined and satisfies the condition:

$$|x| \leqq |y| \quad \text{implies} \quad \|x\| \leqq \|y\|$$

in addition to the usual conditions.

A norm $\|x\|(x \in R)$ on a normed semi-ordered linear space is said to be *monotone complete*, if, when $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$, there exists $\bigcup_{\nu=1}^{\infty} x_\nu$.

A norm on R is said to be *continuous*, if $x_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu\to\infty} \|x_\nu\| = 0$ and *semi-continuous*, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\sup_{\nu \geqq 1} \|x_\nu\| = \|x\|$. It is clear that continuity implies semi-continuity.

Kantorovitch [4] has proved that, if a norm on R is monotone complete and continuous, then it is complete, namely, R is a Banach lattice. Nakano [5; Theorem 31.7] has proved that, if a norm on R is monotone complete and semi-continuous, then the norm is complete, and, recently, Amemiya [1] has proved that, if a norm on R is monotone complete, it is complete.[2] In this connection, see also [2].

In this paper, we will consider several problems concerning monotone completeness and completeness of normed semi-ordered linear spaces and Nakano spaces.

1. Monotone completeness of normed semi-ordered linear spaces. In this section, we will consider two problems.

As usual, let (c_0) be the set of all null-sequences of real numbers. This is a normed semi-ordered linear space by the usual ordering and

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[1] Namely, a conditionally σ-complete vector lattice. In this paper we use the terminology and notation of [5].

[2] In this paper, Amemiya also proved the following lemma: Let R be a monotone complete normed semi-ordered linear space. Then there exists a number $\gamma > 0$ such that $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\gamma\|x\| \leqq \sup_{\nu \geqq 1} \|x_\nu\|$.

the norm: $\|x\|=\sup_{\nu=1}|\xi_\nu|$ for $x=(\xi_\nu)\in(c_0)$. The fact that this norm is complete is well known. But, it is not monotone complete, because, for the sequence of elements:

$$e_1=(1,0,0,\cdots),\quad e_2=(1,1,0,\cdots),\quad e_3=(1,1,1,0,\cdots),\ \cdots,$$

we have $0\leqq e_\nu\uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}\|e_\nu\|\leqq 1$, but $\bigcup_{\nu=1}^{\infty} e_\nu$ does not exist in the space (c_0).

Among function spaces, we can also find an example of this type. Let $L_{1/t}^f$ be the set of all measurable functions $x(t)$ $(0\leqq t\leqq 1)$ such that

$$\int_0^1|\xi x(t)|^{1/t}\,dt<+\infty \qquad \text{for all} \qquad \xi>0\,.$$

Then $L_{1/t}^f$ is a Banach lattice by the norm:

$$\|x\|=\inf_{m(\xi x)\leqq 1}\frac{1}{|\xi|} \qquad \text{where} \qquad m(x)=\int_0^1|x(t)|^{1/t}\,dt\,,$$

but this norm is not monotone complete.

In § 1.1, we will state a necessary and sufficient condition in order that a complete norm be monotone complete.

It is well known that every (norm) closed subset of a Banach lattice is also complete. But, we have a monotone complete semi-ordered linear space which contains a closed, but not monotone complete subspace. Namely, let $L_{1/t}$ be the set of all measurable functions $x(t)$ $(0\leqq t\leqq 1)$ such that

$$\int_0^1|\xi x(t)|^{1/t}\,dt<+\infty \qquad \text{for some} \qquad \xi>0\,.$$

This is a monotone complete normed semi-ordered linear space and $L_{1/t}^f$ is a (norm) closed subspace of $L_{1/t}$.

In § 1.2, we will state a necessary and sufficient condition in order that every closed subspace of a monotone complete semi-ordered linear space be monotone complete.

1.1. Let R be a continuous semi-ordered linear space. A sequence x_ν $(\nu=1,2,\cdots)$ is said to be *bounded*, if there exists an element $x\in R$ such that $x_\nu\leqq x$ $(\nu=1,2,\cdots)$. If $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}$ and this sequence is not bounded, then we write $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}+\infty$.

DEFINITION. R is said to be *K-bounded* (bounded in the sense of Kantorovitch), if $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}+\infty$ implies we can find a sequence of real numbers ξ_ν ($\nu=1, 2, \cdots$) such that $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$ and the sequence $\xi_\nu x_\nu$ ($\nu=1, 2, \cdots$) is not bounded.

DEFINITION. R is said to be *K^2-bounded,* if $0 \leqq x_{\mu,\nu} \uparrow_{\mu=1}^{\infty} +\infty$ for every ν implies we can find a sequence of indices μ_ν ($\nu=1, 2, \cdots$) such that the sequence $x_{\mu_\nu,\nu}$ ($\nu=1, 2, \cdots$) is not bounded.

These concepts were introduced by Kantorovitch [4]. It is easily seen that K^2-boundedness implies K-boundedness. If R is reflexive in the sense of [5] § 24, then it is easily seen that R is K-bounded. Therefore, for any R, its conjugate space is always K-bounded.

The K-boundedness can be expressed in other ways, namely, *the following three conditions are mutually equivalent*:

(1) *R is K-bounded*;

(2) *if* $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ *and* $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ *is order-convergent for all sequences* (ξ_ν) *with* $\sum_{\nu=1}^{\infty} |\xi_\nu| < +\infty$, *then the sequence* x_ν ($\nu=1, 2, \cdots$) *is bounded*;

(3) *if* $x_\nu \geqq 0$ *and* $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ *is order-convergent for all sequences* (ξ_ν) *with* $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$, *then* $\sum_{\nu=1}^{\infty} x_\nu$ *is order-convergent.*

For example, we will prove that (1) implies (2). Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} +\infty$. Then there exists a sequence of real numbers $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$ such that $\xi_\nu x_\nu$ ($\nu=1, 2, \cdots$) are not bounded. Since

$$x_\nu = \sum_{\mu=2}^{\nu} (x_\mu - x_{\mu-1}) + x_1 \uparrow_{\nu=1}^{\infty} +\infty ,$$

and

$$\xi_\nu x_\nu \leqq \sum_{\mu=2}^{\nu} \xi_\mu (x_\mu - x_{\mu-1}) + \xi_1 x_1 ,$$

the sequence:

$$\sum_{\mu=1}^{\nu} (\xi_\mu - \xi_{\mu+1}) x_\mu \qquad (\nu=1, 2, \cdots)$$

is not bounded and

$$\sum_{\mu=1}^{\infty} |\xi_\mu - \xi_{\mu+1}| < +\infty .$$

This is inconsistent with the hypothesis of (2).

THEOREM 1.1. *Let R be a normed semi-ordered linear space. Then the following three conditions are mutually equivalent;*

(1) *The norm on R is monotone complete;*

(2) *the norm is complete and R is K-bounded;*

(3) *the norm is complete and R is K^2-bounded.*

Proof. We have only to prove that (2) implies (1).
Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$. Then, for any sequence of numbers $\xi_\nu > 0$ ($\nu = 1, 2, \cdots$) such that $\sum_{\nu=1}^{\infty} \xi_\nu < +\infty$, we have $\sum_{\nu=1}^{\infty} \xi_\nu \|x_\nu\| < +\infty$. Since the norm is complete by assumption, $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ is convergent in norm, and so, in order convergence. Therefore, x_ν ($\nu = 1, 2, \cdots$) is bounded, because R is K-bounded.

1.2. Let R be a continuous semi-ordered linear space. For any element $p \geqq 0$ and for all $x \geqq 0$, the *projector* $[p]$ is defined as

$$[p]x = \bigcup_{\nu=1}^{\infty} (x \cap \nu p) .$$

$[p] \geqq [q]$ means $[p]x \geqq [q]x$ for any $x \geqq 0$.

Let R be a normed semi-ordered linear space. *A norm $\|x\|$ on R is continuous if and only if $x \geqq 0$ and $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu \to \infty} \|[p_\nu]x\| = 0$* ([Nakano] Theorem 30.8) We will call a subset A of R *monotone complete*, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$ for $x_\nu \in A$ implies $\bigcup_{\nu=1}^{\infty} x_\nu \in A$.

If a norm on R is monotone complete and continuous, then every (norm) closed subset is monotone complete in the sense described above. Here, we will prove the converse. A subset A is said to be *semi-normal*, if $x \in A$, $|y| \leqq |x|$ implies $y \in A$.

THEOREM 1.2. *Let R be a normed semi-ordered linear space and suppose every* (*norm*) *closed, semi-normal subset of R is monotone complete. Then the norm is continuous.*

Proof. Let us assume that there exist $[p_\nu]$ ($\nu = 1, 2, \cdots$) and $x_0 \in R$ such that $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ and $\lim_{\nu \to \infty} \|[p_\nu]x_0\| \geqq \varepsilon$ for some $\varepsilon > 0$. Then the least closed set A containing all $x \in R$ such that $\lim_{\nu \to \infty} \|[p_\nu]x\| = 0$ is semi-normal and $(1 - [p_\nu])x_0 \in A$. On the other hand,

$$0 \leqq (1 - [p_\nu])x_0 \uparrow_{\nu=1}^{\infty} x_0 \qquad \text{and} \qquad \|(1 - [p_\nu])x_0\| \leqq \|x_0\| .$$

Therefore, since A is monotone complete, $x_0 \in A$. This is inconsistent with the definition of A.

2. Monotone completeness of Nakano spaces. It will be necessary to state here the definition and several properties of Nakano spaces.

A semi-ordered linear space is said to be *universally continuous*, if for any system of positive elements x_λ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} x_\lambda$. A *Nakano space* is a universally continuous semi-ordered linear space where a functional $m(x)$ $(x \in R)$ is defined and satisfies the following conditions:

(1) $0 \leqq m(x) \leqq +\infty \, (x \in R)$;

(2) for any $x \in R$ we can find a number $\xi > 0$ such that $m(\xi x) < +\infty$;

(3) if $m(\xi x) = 0$ for every $\xi > 0$, then $x = 0$;

(4) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$;

(5) $m\left(\frac{\xi+\eta}{2} x\right) \leqq \frac{1}{2}\{m(\xi x) + m(\xi y)\}$ for numbers $\xi, \eta > 0$ and for every element $x \in R$;

(6) $|x| \frown |y| = 0$ implies $m(x+y) = m(x) + m(y)$;

(7) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda)$.

This functional $m(x)$ is called a *modular* on the Nakano space R. In the Nakano space R, we can define two kinds of norms:

$$\text{the first norm:} \qquad \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \, ;$$

$$\text{the second norm:} \quad |||x||| = \inf_{m(\xi x) \leqq 1} \frac{1}{|\xi|} \, .$$

It is easily seen that $|||x||| \leqq \|x\| \leqq 2|||x|||$. The modular is said to be complete or monotone complete, if these norms are complete or monotone complete. Namely, a modular m on R is said to be monotone complete, if, when $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} m(x_\nu) < +\infty$, then there exists $\bigcup_{\nu=1}^{\infty} x_\nu$.

A modular m is said to be *simple*, if $m(x) = 0$ implies $x = 0$. If m is simple, we can define in R a convergence by this modular. Namely, a sequence x_ν $(\nu = 1, 2, \cdots)$ is said to be *modular-convergent* to $x \in R$, if $\lim_{\nu \to \infty} m(x_\nu - x) = 0$. If a sequence x_ν $(\nu = 1, 2, \cdots)$ is convergent to $x \in R$ by the norms defined above, then it is modular-convergent to the same limit. But the converse is not always true. In order that the modular-convergence be equivalent to the norm convergence, it is necessary and sufficient that the modular is *uniformly simple*: $\inf_{0 \neq x \in R} m\left(\xi \frac{x}{|||x|||}\right) > 0$ for any $\xi > 0$ ([5] Theorem 48.1)

The norms defined above are not always continuous. If the modular is *finite*, namely, $m(x) < +\infty$ for every $x \in R$, then the norms are continuous ([5] Theorem 44.4)

A modular m is said to be *uniformly finite*, if $\sup_{0 \neq x \in R} m\left(\xi \frac{x}{\|x\|}\right) < +\infty$ for every $\xi > 0$. It is clear that uniform finiteness is stronger than finiteness.[3]

2.1. In this section, we will consider the relations between monotone completeness and completeness of Nakano spaces. In the sequel, let R be a Nakano space and $m(x)$ $(x \in R)$ be its modular.

The following lemma is a generalization of the essential part of Kalugyna's results [3].

LEMMA 2.1. *If m is monotone complete, simple, and its norms are continuous, then m is uniformly simple.*

Proof. If m is not uniformly simple, we can find a sequence $x_\nu \geqq 0$ $(\nu = 1, 2, \cdots)$ such that $\lim_{\nu \to \infty} m(x_\nu) = 0$ and $\|x_\nu\| \geqq \varepsilon > 0$ for all ν. Hence, we can select a subsequence x_{ν_μ} $(\mu = 1, 2, \cdots)$ such that

$$m(x_{\nu_\mu}) \leqq 1/2^\mu .$$

Then, for the elements:

$$y_{\mu,\lambda} = x_{\nu_\mu} \smile x_{\nu_{\mu+1}} \smile \cdots \smile x_{\nu_{\mu+\lambda}} ,$$

we have

$$\begin{aligned} m(y_{\mu,\lambda}) &\leqq m(x_{\nu_\mu}) + m(x_{\nu_{\mu+1}}) + \cdots + m(x_{\nu_{\mu+\lambda}}) \\ &\leqq \frac{1}{2^\mu} + \cdots + \frac{1}{2^{\mu+\lambda}} . \end{aligned}$$

Namely, we have $y_{\mu,\lambda} \uparrow_{\lambda=1}^{\infty}$ and $\sup_{\lambda \geqq 1} m(y_{\mu,\lambda}) < +\infty$. Since m is monotone complete, there exist y_μ $(\mu = 1, 2, \cdots)$ such that $y_\mu = \bigcup_{\lambda=1}^{\infty} y_{\mu,\lambda}$ and $m(y_\mu) \leqq 1/2^{\mu+1}$. It is clear that $y_\mu \downarrow_{\mu=1}^{\infty}$. On the other hand, for any $x \geqq 0$ such that $x \leqq y_\mu$ $(\mu = 1, 2, \cdots)$, we have

$$m(y_\mu - x) \leqq m(y_\mu) , \qquad \text{thus,} \qquad \lim_{\mu \to \infty} m(y_\mu - x) = 0 .$$

[3] More details of the theory of Nakano spaces are given in [**5**]. As examples of Nakano spaces, we cite two representative types. The first is an Orlicz space. The second is the space $L_{p(t)}(p(t) \geqq 1)$, namely, the set of measurable functions $x(t)(0 \leqq t \leqq 1)$ such that $\int_0^1 |\xi x(t)|^{p(t)} dt$ is finite for some $\xi > 0$. Here $p(t)$ is a measurable function on $0 \leqq t \leqq 1$.

Therefore,

$$m\left(\frac{1}{2}x\right)=m\left(\frac{1}{2}(y_\mu-x)+\frac{1}{2}y_\mu\right)$$
$$\leqq\frac{1}{2}\left\{m(y_\mu-x)+m(y_\mu)\right\}\to 0 \qquad (\mu\to\infty)\ ,$$

that is to say, $m\left(\frac{1}{2}x\right)=0$. Since m is simple, $x=0$. This means that $y_\mu\downarrow_{\mu=1}^{\infty}0$. As the norm is continuous, we have $\lim_{\mu\to\infty}\|y_\mu\|=0$, which contradicts the assumption, because

$$\|y_\mu\|\geqq\|x_{\nu_\mu}\|\geqq\varepsilon\ .$$

Therefore, m is uniformly simple.

The next two lemmas constitute the converse of the above.

LEMMA 2.2. *If m is uniformly simple, then its norms are continuous.*

Proof. Let $x_\nu\downarrow_{\nu=1}^{\infty}0$. Then there exists a number $\xi>0$ such that $m(\xi x_\nu)<+\infty$ for all ν. For the elements $y_\nu=\xi x_1-\xi x_\nu$, since $y_\nu\geqq 0$ and $\xi x_\nu\geqq 0$, we have

$$m(y_\nu+\xi x_\nu)\geqq m(y_\nu)+m(\xi x_\nu)\ ,$$

so,

$$m(\xi x_\nu)\leqq m(y_\nu+\xi x_\nu)-m(y_\nu)=m(\xi x_1)-m(y_\nu)\ .$$

On the other hand, we have $m(\xi x_1)=\sup_{\nu\geqq 1}m(y_\nu)$, because $0\leqq y_\nu\uparrow_{\nu=1}^{\infty}\xi x_1$. Therefore, $\lim_{\nu\to\infty}m(\xi x_\nu)=0$, and hence it follows that $\lim_{\nu\to\infty}\|x_\nu\|=0$, because m is uniformly simple.

LEMMA 2.3. *If m is uniformly simple and its norms are complete, then m is monotone complete.*

Proof. Let $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}m(x_\nu)<+\infty$. Then

$$m(x_\nu-x_\mu)\leqq m(x_\nu)-m(x_\mu) \qquad (\nu\geqq\mu)\ ,$$

and hence, we have

$$\lim_{\nu,\mu\to\infty}m(x_\nu-x_\mu)=0\ .$$

Since m is uniformly simple, we have

$$\lim_{\nu,\mu\to\infty} \|x_\nu - x_\mu\| = 0 ,$$

so that there exists an element $x \in R$ such that $\lim_{\nu\to\infty} \|x_\nu - x\| = 0$. For this x, it is easily seen that $x = \bigcup_{\nu=1}^{\infty} x_\nu$, which shows that m is monotone complete.

From these lemmas, we obtain the following theorem:

THEOREM 2.1. *A modular on a Nakano space is monotone complete, simple, and its norms are continuous, if and only if it is uniformly simple and complete.*

Next, we will consider the case when m is finite.

DEFINITION. A modular $m(x)$ ($x \in R$) is said to be totally finite, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$ implies $\sup_{\nu \geqq 1} m(\xi x_\nu) < +\infty$ for every $\xi > 0$.

LEMMA 2.4. *If m is monotone complete and finite, then it is totally finite.*

Proof. $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$. Then, since m is monotone complete, there exists $x \in R$ such that $x = \bigcup_{\nu=1}^{\infty} x_\nu$. Therefore $\xi x = \bigcup_{\nu=1}^{\infty} \xi x_\nu$ for every $\xi > 0$. Hence it follows that $m(\xi x) = \sup_{\nu\geqq 1} m(\xi x_\nu) < +\infty$, because m is finite.

LEMMA 2.5. *If m is totally finite and complete, then it is monotone complete.*

Proof. Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$. Then, by the assumption, we have $\sup_{\nu\geqq 1} m(\xi x_\nu) < +\infty$ for every $\xi > 0$. Since

$$m(\xi x_\nu - \xi x_\mu) \leqq m(\xi x_\nu) - m(\xi x_\mu) \qquad (\nu \geqq \mu) ,$$

we have $\lim_{\nu,\mu\to\infty} m(\xi x_\nu - \xi x_\mu) = 0$ for every $\xi > 0$, therefore we have

$$\lim_{\nu,\mu\to\infty} \|x_\nu - x_\mu\| = 0 .$$

Hence, there exists an element $x \in R$ such that $\lim_{\nu\to\infty} \|x_\nu - x\| = 0$. Therefore, we have $x = \bigcup_{\nu=1}^{\infty} x_\nu$, which shows that m is monotone complete.

Thus we obtain the following theorem:

THEOREM 2.2. *A modular on a Nakano space is monotone complete and finite if and only if it is totally finite and complete.*

REMARK. It is easily seen that uniform finiteness implies total finiteness and the latter implies finiteness. The converses are not always true. In fact, $L^f_{1/t}$ is a finite Nakano space by the following modular:

$$m(x)=\int_0^1 |x(t)|^{1/t}\,dt \qquad \text{for} \qquad x(t)\in L^f_{1/t}\,.$$

But, this is not totally finite, because, if it were totally finite, then, by Theorem 2.2, it would be monotone complete, which is impossible. Next, let $f_\nu(\xi)$ $(\nu=1, 2, \cdots)$ be a sequence of convex functions such that

$$f_\nu(\xi)=\begin{cases}\xi & \text{if} \quad 0\leqq\xi\leqq 1\,;\\ \nu(\xi-1)+1 & \text{if} \quad \xi>1\,.\end{cases}$$

Then, *the space*[4] $l(f_1, f_2, \cdots)$ with the modular

$$m(x)=\sum_{\nu=1}^{\infty} f_\nu(|\xi_\nu|) \qquad \text{for} \qquad x=(\xi_\nu)$$

is totally finite, but not uniformly finite. To see this, we need only take the elements:

$$e_1=(1, 0, 0, \cdots)\,,\quad e_2=(0, 1, 0, \cdots)\,,\quad e_3=(0, 0, 1, 0, \cdots)\,,\quad \cdots\,.$$

It is easily proved that $|||e_\nu|||=1$ and $m(2e_\nu)=\nu+1\to+\infty$. But, this sequence space is uniformly simple by Theorem 2.1. The relations between uniform simplicity and uniform finiteness were considered by my colleagues. If a modular on a Nakano space is uniformly finite and simple, then, by considering the monotone completion and applying Theorem 2.1, we can prove that it is uniformly simple. On the other hand, T. Shimogaki has proved in an unpublished paper that, if a modular is uniformly simple and the space has no atomic elements, then it is uniformly finite.

2.2. In this section, we will consider relations between monotone completeness and finiteness.

An element x is said to be *finite*, if $m(\xi x)<+\infty$ for every $\xi>0$. The set of all finite elements is called a *finite manifold* of R and denoted by F. F is a (norm) closed subspace of R and the norms are continuous in F ([5] Theorem 44.5.). If the norms are continuous in R and m is monotone complete, then F is *universally monotone complete*, that is, if $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda} m(x_\lambda)<+\infty$ then there exists $\bigcup_{\lambda\in\Lambda} x_\lambda$.

m is said to be *almost finite*, if F is complete in R (that is, if $|x|\frown|y|=0$ for all $y\in F$, then $x=0$).

[4] For the definition of this sequence space, see [**6**].

THEOREM 2.3. *If m is almost finite and monotone complete, then m is finite if and only if F is, as a space, universally monotone complete.*

Proof. We need only prove the sufficiency. For any $x \in R$, since m is almost finite, there exists a system of projectors $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$ such that $[p_\lambda]x \in F$, and there exists a number $\xi>0$ such that $m(\xi x)<+\infty$. Therefore we have

$$\bigcup_{\lambda\in\Lambda} \xi[p_\lambda]x=\xi x \in F ,$$

since m is monotone complete. Hence it follows that m is finite.

THEOREM 2.4. *If m is almost finite, monotone complete and separable in its norm topology, then m is finite.*

Proof. It is well known that if m is almost finite and norms are continuous, then m is finite. Therefore, we need only prove that if m is monotone complete and separable, then its norms are continuous.

For this purpose, let us suppose that there exists an element $x\geqq 0$ and a sequence of projectors $[p_\nu]\downarrow_{\nu=1}^{\infty} 0$ such that

$$\inf_{\nu\geqq 1} \|[p_\nu]x\| >\varepsilon \qquad \text{for some} \qquad \varepsilon>0 .$$

Then, by Amemiya's lemma, we can find a number $\xi>0$ such that

$$\lim_{\nu\to\infty} \|[p_\mu]a-[p_\nu]a\| \geqq \xi\|[p_\mu]a\| >\xi\varepsilon >0 ,$$

and here, we can select μ_ν $(\nu=1, 2, \cdots)$ such that

$$\|[p_{\mu_\nu}]x-[p_{\mu_{\nu+1}}]x\| >\xi\varepsilon .$$

Putting $p_\nu=[p_{\mu_\nu}]x-[p_{\mu_{\nu+1}}]x$ $(\nu=1, 2, \cdots)$, we see easily that

$$p_\nu\geqq 0 , \qquad p_\nu \cap p_\lambda=0(\nu\neq\lambda) \qquad \text{and} \qquad \|p_\nu\| >\xi\varepsilon ,$$

and, for any subsequence p_{ν_λ} $(\lambda=1, 2, \cdots)$, we have

$$\sum_{\lambda=1}^{\infty} p_{\nu_\lambda}\leqq x .$$

Moreover, the set of all such sequences is not denumerable and

$$\| \sum_{\lambda=1}^{\infty} p_{\nu_\lambda}- \sum_{\rho=1}^{\infty} p_{\nu_\rho}\| >\xi\varepsilon$$

for different sequences $\{p_{\nu_\lambda}\}$ and $\{p_{\nu_\rho}\}$. This contradicts the separability. Therefore, norms are continuous and the proof is established.

REMARK. In order that m be finite, it is necessary and sufficient that its norms be continuous and all atomic elements belong to F.

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