CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE (II)

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1. Introduction. Nearly two years ago, Alder [1] established the following generalizations of the well-known Rogers-Ramanujan identities:

$$(1)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-(M+1)})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{G_{M,t}(x)}{(x)_t}\ ,$$

$$(2)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}x^t\frac{G_{M,t}(x)}{(x)_t}\ ,$$

where $G_{M,t}(x)$ are polynomials which reduce to x^{t^2} for $M=2$ and

$$(x)_t=(1-x)(1-x^2)\cdots(1-x^t)\ ,\qquad (x)_0=1\ .$$

In a recent paper [6] I gave a simple alternative proof of (1) and (2). We used the result

$$(3)\qquad 1+\sum_{s=1}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}(1-kx^{2s})\frac{(kx)_{s-1}}{(x)_s}$$
$$=\prod_{n=1}^{\infty}(1-kx^n)\sum_{t=0}^{\infty}\frac{k^tG_{M,t}(x)}{(x)_t}\ ,\qquad M=2,3,\cdots$$

Alder in his paper states that identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(M-r)(\bmod 2M+1)$, where $0\leqq r\leqq M-1$, can be obtained by his method and indicates the result for $r=1$.

In the present paper I give a simple method of obtaining the M identities for each modulus $(2M+1)$. In § 4 identities for which $r\geqq\frac{1}{2}M$ have been deduced and in § 5 those for which $r\leqq\frac{1}{2}M$ have been obtained for any r such that $0\leqq r\leqq M-1$. The identities given in § 5 have not been mentioned by Alder. As a corollary, an interesting identity between two infinite series is given.

2. Notations. Assuming $|x|<1$, let

$$(\alpha)_n\equiv(\alpha)_{x,n}=(1-\alpha)(1-\alpha x)\cdots(1-\alpha x^{n-1})\ ,\qquad (\alpha)_0=1\ ,$$

$$(\alpha)_{-n}=(-1)^n x^{\frac{1}{2}n(n+1)}/\alpha^n(x/\alpha)_n\ ,$$

$$x_n=1+x+x^2+\cdots+x^{n-1}\ .$$

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$$(4) \qquad P_{m,t}(x) \equiv x^{\frac{1}{2}(t+1-m)t} \frac{(x)_m}{(x)_t(x)_{m-t+1}}(1-x^{m-2t+1}) ,$$

and let

$$(5) \qquad \phi(M, x^r) = 1 + \sum_{s=1}^{\infty} (-1)^s x^{Mrs} x^{\frac{1}{2}s\{(2M+1)s-1\}}(1-x^{2s+r})(x^{s+1})_{r-1} ,$$

so that (3) can be written as

$$(6) \qquad \phi(M, x^r) = \prod_{n=1}^{\infty}(1-x^n) \sum_{t=0}^{\infty} \frac{x^{rt} G_{M,t}(x)}{(x)_t} .$$

3. The polynominals $u_n(x)$. Before proceeding to deduce the generalized identities, we first give a few properties of a sequence of polynomials with the help of an operator. These we will need in later sections. Let us define a sequence $\{u_n(x)\}$ of polynomials by the relations

$$\text{(i)} \qquad u_0(x) = 0$$

$$\text{(ii)} \qquad u_n(x) = u_{n-1}(x) + x^{n-1}x_n , \qquad n \geqq 1 .$$

Let $\mathscr{R}$ be an operator which replaces x_m by $u_m(x)$ in any $u_n(x)$, that is,

$$\mathscr{R} u_n(x) = \mathscr{R} u_{n-1}(x) + x^{n-1} u_n(x) .$$

Also

$$\mathscr{R}^n u_m(x) = \mathscr{R}^{n-1}\{\mathscr{R} u_m(x)\} .$$

Then we have

$$(7) \qquad (\alpha)_n = 1 - \alpha x_n + \sum_{s=2}^{n} (-\alpha)^s x^{\frac{1}{2}s(s-1)} \mathscr{R}^{s-2} u_{n-s+1}(x) .$$

As can be easily shown

$$(8) \qquad \mathscr{R}^{s-2} u_{n-s+1}(x) = \frac{(x)_n}{(x)_s(x)_{n-s}} .$$

The above polynomials (8) have also recently occurred in a paper by Carlitz [3].

Comparing the coefficients of α^{s-1} in

$$(9) \qquad (\alpha)_n = (-1)^n \alpha^n x^{\frac{1}{2}n(n-1)} (1/\alpha x^{n-1})_n ,$$

we get the relation

$$(10) \qquad \mathscr{R}^{s-3} u_{n-s+2}(x) = \mathscr{R}^{n-s-1} u_s(x) , \qquad s = 1, \cdots, (n+1) .$$

We can thus write (7) as

$$(\alpha)_n=\sum_{s=0}^{n}(-\alpha)^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{n-s+1}(x)\ , \tag{11}$$

where negative indices of $\mathscr{R}$ are defined by (10). Again comparing the coefficients of α^s in

$$(\alpha/x^{n-1})_{x,n}=(\alpha)_{1/x,n}\ ,$$

we get with the help of (11),

$$\mathscr{R}^{s-2}u_{n-s+1}(x)=x^{s(n-s)}\mathscr{R}^{s-2}u_{n-s+1}(x^{-1})\ . \tag{12}$$

In particular

$$u_n(x)=x^{2n-2}u_n(x^{-1})\ . \tag{13}$$

The following values of $\mathscr{R}^m\ u_n(x)$ will also be required:

$$\mathscr{R}^{n-3}u_2(x)=x_n\ , \qquad \text{from (10)}$$

$$\begin{aligned}
&u_1(x)=1\\
&u_3(x)=1+x+2x^2+x^3+x^4\\
&u_4(x)=1+x+2x^2+2x^3+2x^4+x^5+x^6=\mathscr{R}u_3(x)\ .
\end{aligned}$$

4. Now we proceed to deduce identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(M-r)(\bmod 2M+1)$. From (11), we have

$$\begin{aligned}
&(x^{n-r+1})_{2r-1}(1-x^{2n})\\
&\quad=\left[1+\sum_{s=1}^{2r-2}\{(-x^{n-r+1})^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{2r-s}(x)\}-x^{(2r-1)n}\right](1-x^{2n})\ ,
\end{aligned}$$

whence

$$\begin{aligned}
&1+x^{(2r+1)n}\\
&\quad=\left[x^{2n}+x^{(2r-1)n}-\left\{\sum_{s=1}^{2r-2}x^{ns}(-1)^s x^{\frac{1}{2}s(s+1)-rs}\mathscr{R}^{s-2}u_{2r-s}(x)\right\}(1-x^{2n})\right]\\
&\qquad\qquad+(1-x^{2n})(x^{n-r+1})_{2r-1}\ .
\end{aligned} \tag{14}$$

And since, because of (8) or (10), the terms equidistant from the two ends in the sum on the right of (14) have equal coefficients of powers of x^n, the expression in square brackets can be written as

$$\sum_{t=1}^{r}(-1)^{t-1}x^{tn}\{1+x^{(2r-2t+1)n}\}U_{r,t}(x) \tag{15}$$

where

$$\begin{aligned}
U_{r,t}(x)&=x^{\frac{1}{2}t(t+1)-tr}\mathscr{R}^{t-2}u_{2r-t}(x)-x^{\frac{1}{2}(t-2)(t-1)-(t-2)r}\mathscr{R}^{t-4}u_{2r-t+2}(x)\\
&=P_{2r,t}(x)\ , \qquad \text{using (4) and (8).}
\end{aligned} \tag{16}$$

The polynomials $U_{r,t}(x)$ may be called "reciprocal" since they are such that the terms equidistant from the two ends have equal coefficients. Taking $n=0$ in (14) we see that

$$\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)=1 \ . \tag{17}$$

Also, with the help of (12), we have

$$U_{r,t}(x)=U_{r,t}(x^{-1}) \ . \tag{18}$$

Now from (15)

$$\begin{aligned}\sum_{n=-\infty}^{\infty}&(-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}}\\ &=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}}\{1+x^{(2r+1)n}\}\\ &=1+\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{(r-t)n}}\{1+x^{(2r-2t+1)n}\}\\ &\quad+\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}}(1-x^{2n})(x^{n-r+1})_{2r-1} \ .\end{aligned} \tag{19}$$

For $n=s+r$, the last series on the right-hand side of (19) becomes

$$(-1)^r x^{Mr^2-\frac{1}{2}r(r+1)}\phi(M,\, x^{2r})$$

since the first $(r-1)$ terms of the series vanish because of the factor $(x^{n-r+1})_{2r-1}$. Then using (17) and writing

$$F(M,\, r)=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}}$$

we obtain for (19) the form

$$F(M,\, r)=\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)F(M,\, r-t)+(-1)^r x^{Mr^2-\frac{1}{2}r(r+1)}\phi(M,\, x^{2r}) \ . \tag{20}$$

Thus, using Jacobi's classical identity

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}z^n=\prod_{n=1}^{\infty}(1-x^{2n-1}z)(1-x^{2n-1}/z)(1-x^{2n}) \tag{21}$$

to express the infinite series in (20) as infinite products, we could find for any given r, such that $0\leqq r\leqq M-1$, an expression for the generating function for the number of partitions into parts not congruent to $0,\ \pm(M-r)(\text{mod}\ 2M+1)$ in terms of generating functions for the number of partitions into parts not congruent to $0,\ \pm(M-s)(\text{mod}\ 2M+1)$,

$(s=0, 1, 2, \cdots, r-1)$. Since $F(M, 0)=\phi(M, 1)$, the F-series can be successively expressed in terms of ϕ-series and, with the help of (6), we get

THEOREM 1.

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-r)})(1-x^{(2M+1)n-(M+r+1)})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{A_r(x,t)G_{M,t}(x)}{(x)_t} \tag{22}$$

where

$$A_r(x, t)=\sum_{s=0}^{r}(-1)^s x^{Ms^2-\frac{1}{2}s(s+1)+2st}U'_{r,s+1}(x) .$$

The polynomials $U'(x)$ *are of the "reciprocal" kind, with*

$$U'_{r,r+1}(x)=1$$

$$U'_{r,s+1}(x)=\sum_{m=1}^{r-s}(-1)^{m-1}U_{r,m}(x)U'_{r-m,s+1}(x) , \qquad s\neq r .$$

so that

$$U'_{r,1}(x)=1 , \qquad \textit{because of } (17)$$

and

$$U'_{r,m}(x)=U'_{r,m}(x^{-1}) , \qquad \textit{because of } (18).$$

As an example of Theorem 1, taking the case $r=1$, we have

$$1+x^{3n}=x^n(1+x^n)+(1-x^n)(1-x^{2n}) .$$

Therefore

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n\{(2M+1)n+3\}}=\phi(M, 1)-x^{M-1}\phi(M, x^2) \tag{23}$$

which is equivalent to equation (23) of Alder [1].

From (23), using (21) and (6), we get the identity

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-1)})(1-x^{(2M+1)n-(M+2)})(1-x^{(2M+1)n})}{(1-x^n)} =\sum_{t=0}^{\infty}\frac{(1-x^{M+2t-1})}{(x)_t}G_{M,t}(x) . \tag{24}$$

For $r=2$,

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-2)})(1-x^{(2M+1)n-(M+3)})(1-x^{(2M+1)n})}{(1-x^n)} =\sum_{t=1}^{\infty}\frac{\{1-U'_{2,2}(x)x^{M+2t-1}+x^{4M+4t-3}\}}{(x)_t}G_{M,t}(x) , \tag{25}$$

where

$$U'_{2,2}(x)=x^{-1}+1+x\ .$$

Similarly for $r=3$ we get

$$U'_{3,2}(x)=x^{-2}+x^{-1}+2+x+x^2$$
$$U'_{3,3}(x)=x^{-2}+x^{-1}+1+x+x^2\ ,$$

and so on for any r such that $0\leqq r\leqq M-1$.

5. In this section identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm r\ (\mathrm{mod}\ 2M+1)$ are obtained.

From (11) we have

$$(x^{n-r+2})_{2r-2}(1-x^{2n+1})$$
$$=\left[1+\left\{\sum_{s=1}^{2r-3}(-x^{n-r+2})^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{2r-s-1}(x)\right\}+x^{(2n+1)(r-1)}\right](1-x^{2n+1})\ ,$$

whence

$$1-x^{(2n+1)r}$$
$$=\left[x^{2n+1}-x^{(2n+1)(r-1)}-\left\{\sum_{s=1}^{2r-3}x^{ns}(-1)^s x^{\frac{1}{2}s(s+3)-rs}\mathscr{R}^{s-2}u_{2r-s-1}(x)\right\}(1-x^{2n+1})\right]$$
$$+(1-x^{2n+1})(x^{n-r+2})_{2r-2}\ . \tag{26}$$

In the expression in square brackets in (26), the terms containing x^{nr} cancel and the other terms can again be grouped in pairs to give

$$1-x^{(2n+1)r}=\sum_{t=1}^{r-1}(-1)^{t-1}V_{r,t}(x)x^{tn}\{1-x^{(2n+1)(r-t)}\}+(1-x^{2n+1})(x^{n-r+2})_{2r-2}\ , \tag{27}$$

where

$$V_{r,t}(x)=x^{\frac{1}{2}t(t+3)-rt}\mathscr{R}^{t-2}u_{2r-t-1}(x)-x^{\frac{1}{2}t(t-1)-r(t-2)}\mathscr{R}^{t-4}u_{2r-t+1}(x)$$
$$=x^{\frac{1}{2}t}P_{2r-1,t}(x)\ . \tag{28}$$

The polynomials $V(x)$ are less symmetric than $U(x)$. In particular, corresponding to (17) and (18), they satisfy the relations

$$\sum_{t=1}^{r-1}(-1)^{t-1}V_{r,t}(x)x_{r-t}=x_r\ , \tag{29}$$

and

$$V_{r,t}(x)=x^t V_{r,t}(x^{-1})\ . \tag{30}$$

Now

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)(n+1)\}-rn}$$
$$=\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{1}{2})n}}{x^{(r-1)n}} \{1-x^{(2n+1)r}\} .$$

Denoting the left-hand side of the last equation by $\psi(M, r)$ and using (27) and (5), we get, after slight simplification,

$$(31) \quad \psi(M, r)=\sum_{t=1}^{r-1}(-1)^{t-1}\psi(M, r-t)V_{r,t}(x)+(-1)^{r-1}x^{\frac{1}{2}(2M-1)r(r-1)}\phi(M, x^{2r-1}) .$$

Using (21), the generating function for the number of partitions into parts not congruent to 0, $\pm r$ (mod $2M+1$) can now be expressed in terms of the generating function for the number of partitions into parts not congruent to 0, $\pm s$ (mod $2M+1$), ($s=1, 2, \cdots, r-1$). Thus, we finally have

THEOREM 2.

$$(32) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-r})(1-x^{(2M+1)n-(2M+1-r)})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{B_r(x, t)G_{M,t}(x)}{(x)_t} ,$$

where

$$B_r(x, t)=\sum_{s=1}^{r}(-1)^{s-1}x^{\frac{1}{2}(2M-1)s(s-1)+(2s-1)t}V'_{r,s}(x) ,$$

and $V'_{r,s}(x)$ *are polynomials with*

$$V'_{r,r}(x)=1$$
$$V'_{r,s}(x)=\sum_{m=1}^{r-s}(-1)^{m-1}V_{r,m}(x)V'_{r-m,s}(x) , \qquad s\neq r ,$$

so that

$$V'_{r,1}(x)=x_r$$

and

$$V'_{r,t}(x)=x^{r-t}V'_{r,t}(x^{-1}) .$$

As an illustration, for $r=2$ in Theorem 2, we have

$$1-x^{4n+2}=x^n(1+x)(1-x^{2n+1})+(1-x^{2n+1})(1-x^n)(1-x^{n+1}) .$$

Therefore

$$\sum_{n=-\alpha}^{\infty} (-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{3}{2})n}=(1+x)\phi(M, x)-x^{2M-1}\phi(M, x^3) ,$$

which gives us the identity

$$(33)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-2})(1-x^{(2M+1)n-(2M-1)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty}\frac{\{(1+x)x^t-x^{2M+3t-1}\}}{(x)_t}G_{M,t}(x)\,.$$

COROLLARY. *If r is replaced by $M-r$ in Theorem 2 then the left-hand sides of* (22) *and* (32) *become the same and we have*

$$(34)\qquad \sum_{t=0}^{\infty}\frac{A_r(x,\,t)G_{M,t}(x)}{(x)_t}=\sum_{t=0}^{\infty}\frac{B_{M-r}(x,\,t)G_{M,t}(x)}{(x)_t}\,,\quad r=0,\,1,\,\cdots,\,M-1\,,\ M=2,\,3,\,\cdots.$$

For $M=2$ and $r=0$ and 1 we get respectively the relations

$$\sum_{t=0}^{\infty}\frac{x^{t^2}}{(x)_t}=\sum_{t=0}^{\infty}\frac{\{(1+x)x^t-x^{3(t+1)}\}}{(x)_t}x^{t^2}$$

$$\sum_{t=0}^{\infty}\frac{(1-x^{2t+1})}{(x)_t}x^{t^2}=\sum_{t=0}^{\infty}\frac{x^{t+t^2}}{(x)_t}$$

the truth of which can easily be verified.

Some time ago, Slater ([4] and [5]) gave a very large number of identities of the Rogers-Ramanujan type using Bailey's summation theorem [2] for a well-poised ${}_6\Psi_6$. It is interesting to note that, as special cases of our identities, we get some of those given by Slater, differing only in form as can be easily verified. To mention an example, let us take equation (90) of Slater [5]:

$$(35)\qquad \prod_{n=1}^{\infty}\frac{(1-x^{27n-3})(1-x^{27n-24})(1-x^{27n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{(x^3)_{x^3,t}x^{t(t+3)}}{(x)_t(x)_{2t+2}}\,,$$

If we put $M=13$, $r=3$ in Theorem 2, we obtain another series for the product on the left of (35). I propose to study the equivalence of identities (22) and (32) above and those of Slater in a subsequent paper, as also identities involving products in which the powers increase by $2M$.

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Corrigenda. In [6] the following corrections may be noted:
p. 1011. The series for $T_{n,m}$ runs up to

$$t_n=\left[\frac{M-n-1}{M-n}t_{n-1}\right].$$

p. 1012. In the line immediately preceding (3.3), a_{2Mn-1} should be a_{2M+1}.

In the right hand side of (3.4) a factor $(kn;t)$ should be inserted in the denominator of the outer series.

p. 1014. In the right hand side of the last identity of the paper, we should have Π instead of π.

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