BOUNDS FOR THE PRINCIPAL FREQUENCY OF THE NONHOMOGENEOUS MEMBRANE AND FOR THE GENERALIZED DIRICHLET INTEGRAL

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Introduction. In § § 1 and 2 of this paper we consider an arbitrarily shaped membrane of variable density and uniform tension. We assume that this nonhomogeneous membrane is stretched in a given frame and obtain bounds for its principal frequency (fundamental tone). Before describing our results we quote the analogous result for the nonhomogeneous string proved in a paper by P. R. Beesack and the author [1, Theorem 2].

Let $p(x)$ be continuous and not identically zero for $-x_0 \leqq x \leqq x_0$, $0 < x_0 < \infty$, and let $p^+(x)$ and $p^-(x)$ be the rearrangement of $p(x)$ in symmetrically increasing respectively decreasing order. Consider the three differential systems

$$y''(x) + \lambda p(x) y(x) = 0 , \qquad y(\pm x_0) = 0 ;$$

$$u''(x) + \lambda^+ p^+(x) u(x) = 0 , \qquad u(\pm x_0) = 0 ;$$

$$v''(x) + \lambda^- p^-(x) v(x) = 0 , \qquad v(\pm x_0) = 0 :$$

denote their least positive eigenvalues also by λ, λ^+ and λ^- respectively. Then $\lambda^- \leqq \lambda$ even if $p(x)$ changes sign finitely often while $\lambda \leqq \lambda^+$ holds if $p(x) \geqq 0$.

For the nonhomogeneous membrane we consider a domain D bounded by a Jordan curve C. The differential system (for the original density) is given by

$$\Delta u(x, y) + \lambda p(x, y) u(x, y) = 0$$

for (x, y) in D and $u(C) = 0$. We base the existence of the first eigenfunction and its minimum property on the classical treatment of Courant-Hilbert [3, vol. 2, Chapter VII]. We assume therefore that $p(x, y)$ is positive and continuous in $\bar{D}$ and has continuous first derivatives in D. Together with $p(x, y)$ we consider its rearrangements in symmetrically increasing respectively decreasing order. The symmetrization is with respect to a point: $p^+(x, y) = p^+(r)$ and $p^-(x, y) = p^-(r)$ are defined in a closed disk $\bar{D}^*$ of the same area as D. The properties of $p(x, y)$ imply that $p^+(x, y)$ and $p^-(x, y)$ are positive and continuous in $\bar{D}^*$. However,

Received December 28, 1956. The author wishes to thank Dr. P. R. Beesack of McMaster University, Hamilton and Professor E. Netanyahu of the Technion, Haifa for many helpful remarks in the preparation of this paper.

their first derivatives may be discontinuous along infinitely many concentric circles which accumulate to circles lying in the open disk D^*. λ^+ and λ^- can thus not be defined as the classical first eigenvalues of a circular membrane with the density function p^+ or p^-, but are easily defined as a generalization of this notion (see formulas (8^+) and (8^-) below). The actual statement of Theorems 1 and 2 uses only density functions with continuous first derivatives, so that all eigenvalues are in the classical sense. Here we summarize these results as follows: In § 1 it is shown that if the original domain D is a disk, then $\lambda \leqq \lambda^+$ (Theorem 1). In § 2 we prove that for any domain D (bounded by a Jordan curve) $\lambda^- \leqq \lambda$. This Theorem 2 is a generalization of the theorem of Rayleigh, Faber and Krahn and it implies (essentially) a result of Szegö on the principal frequency of nonhomogeneous membranes [10, § V]. In Theorem 2′ we formulate these results in complete analogy to [1, Theorem 2], using generalized first eigenvalues.

Following Szegö([10] and [9, Note D]), we consider in § 3 a ring-shaped domain D and the class of the admissible functions $\varphi(x,y)$ in D. These admissible functions satisfy a smoothness condition, vanish on the inner boundary of D and are equal to 1 on its outer boundary. $p(x, y)$ is defined in $\bar{D}$ and satisfies the same conditions as in § § 1 and 2; p^+ and p^- are now defined in a closed annulus $\bar{D}^*$. We denote the minimum of the generalized Dirichlet integral

$$\iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma$$

in the above class by $4\pi\gamma$ and define γ^+ and γ^- in a similar way. Theorem 3 states that for any ring-shaped domain D (bounded by two Jordan curves) $\gamma^- \leqq \gamma$. After restating this theorem in terms of Szegö's—slightly different—definition of the generalized Dirichlet integral, we show that it implies (essentially) Szegö's result on this integral. Theorem 4 states that if the original domain is an annulus, then $\gamma \leqq \gamma^+$. We conclude with two theorems which are one-dimensional analogues of the results on the generalized Dirichlet integral.

Throughout this paper, symmetrization which respect to a point is the main tool. We rely in § 2 on Krahn's paper [7] and in § 3 on Szegö's paper [10], and we use their results with regard to the behavior of the (ordinary) Dirichlet integral under this symmetrization (see (11^-) and (11^+) below). In addition, we use a well known theorem of Hardy, Littlewood and Pólya on the rearrangements of functions ([5, Theorem 378] and [9, p. 153]).

1. The nonhomogeneous membrane I.

We start with the definition of the symmetrical rearrangements of a function $p(x,y)$ (cf. [5], [6] and [9]). Let D be a simply connected bounded domain in the x, y-plane and let $p(x, y)$ be defined and continuous in the closure $\bar{D}$ of D and be positive in D. We denote by D^* the open disk with the same area as D. R is the radius of D^* and $r=(x^2+y^2)^{1/2}$ the distance from its center. (By using x, y-coordinates for the planes containing D and D^* we do not imply that these planes have to coincide). The rearrangements of $p(x, y)$ in symmetrically increasing and decreasing order will be denoted by $p^+(x, y)$ and $p^-(x, y)$ respectively. They are uniquely defined in the closure $\bar{D}^*$ of D^* by the following three requirements: First, both functions have circular symmetry, $p^+(x, y)=p^+(r)$, $p^-(x, y)=p^-(r)$, $0\leqq r\leqq R$, and $p^+(r)$ is a nondecreasing, $p^-(r)$ a nonincreasing function of r. Secondly, both functions are equimeasurable to $p(x, y)$; that is denoting by $A(z)$ the area of the open set in D for which $p(x, y)>z$ and similarly by $A^+(z)$ and $A^-(z)$ the area of the set in D for which $p^+(x, y)>z$ and $p^-(x, y)>z$ respectively, then we require that for each $z\geqq 0$ $A(z)=A^+(z)$ $=A^-(z)$. Finally, at the center $r=0$ of D^* we let $p^+(p^-)$ be equal to the minimum (maximum) of p in $\bar{D}$ and we complete $p^+(p^-)$ to the closure $\bar{D}^*$ of D^* by assuming that its value on the boundary circle C^* is equal to the maximum (minimum) of p in $\bar{D}$.

The two rearrangements are connected by the formula $p^-(r)=p^+((R^2-r^2)^{1/2})$, $0\leqq r\leqq R$. If p is positive in $\bar{D}$ then, clearly, the same holds for p^+ and p^- in $\bar{D}^*$. Moreover, the continuity of p in $\bar{D}$ implies the continuity of its rearrangements in $\bar{D}^*$ (cf. [6, Theorem 5]). Indeed, the continuity of $p(x, y)$ implies that $A(z)$ is a strictly decreasing function of z (for the z-interval bounded by the minimum and maximum of $p(x, y)$ in $\bar{D}$). As $p^+(r)$ and $p^-(r)$ are monotonic functions their only possible discontinuities would be jumps. Such a jump would imply that $A^+(z)$ or $A^-(z)$ had to be constant for the corresponding z-interval. But, as $A^+(z)\equiv A^-(z)\equiv A(z)$, this possibility is excluded.

Though not necessary for the following proofs, we wish to justify our above statement concerning the discontinuities of the first derivatives of $p^+(x, y)$ and $p^-(x, y)$. We assume therefore that $p(x, y)$ has continuous partial derivatives of first order—or, indeed, of any desired order—and we consider the surface $z=p(x, y)$ lying above D. Let us perform the transition from $p(x, y)$ to $p^-(x, y)=p^-(r)$ in the direction of decreasing z-values. The absolute maximum of $p(x, y)$ in $\bar{D}$ becomes $p^-(0)$ and every z-value, smaller than this absolute maximum, for which $p(x, y)$ has a local extremum induces a jump of dp^-/dr at the corresponding value $p^-(r)=z$. Clearly, the values of the local extrema of $p(x, y)$ may accumulate to one or more values lying in the open interval bounded by the absolute extrema of $p(x, y)$ in $\bar{D}$. This case generates the situation mentioned in the introduction with respect to the discontinuities of the first derivatives of $p^-(x, y)$ and $p^+(x, y)$. We shall return to this question in a special case (for the function $u^-(x, y)$ appearing in the proof of Theorem 2).

We state now the following.

THEOREM 1. *Let D be the disk $0 \leqq x^2+y^2 < R^2$, $0 < R < \infty$, and denote its boundary by C. Let the function $p(x, y)$ be positive and continuous in $\bar{D}$ $(=D \cup C)$ and have continuous first derivatives in D. Let $p^+(x, y) = p^+(r)$ $(r^2 = x^2+y^2, 0 \leqq r \leqq R)$ be the rearrangement of $p(x, y)$ in symmetrically increasing order defined in $\bar{D}$ $(=\bar{D}^*)$. Further let $m(x, y)$ be a function which is positive and continuous in $\bar{D}$, has continuous first derivatives in D and satisfies for each $(x, y) \in \bar{D}$*

$$(1^+) \qquad m(x, y) \leqq p^+(x, y) .$$

Consider the differential systems

$$(2) \qquad \Delta u(x, y) + \lambda p(x, y) u(x, y) = 0 \quad \textit{for} \quad (x, y) \in D , \quad u(C) = 0 ,$$

and

$$(3^+) \qquad \Delta v(x, y) + \mu m(x, y) v(x, y) = 0 \quad \textit{for} \quad (x, y) \in D , \quad v(C) \equiv 0 ,$$

and denote their first eigenvalues by λ and $\mu = \mu(m)$ respectively. Then

$$(4^+) \qquad \lambda \leqq \mu(m) .$$

For the proof we need the properties of the first eigenfunction. As mentioned, we rely on the last chapter of Courant-Hilbert [3, Vol. 2, Chapter VII]. In our § § 1 and 2 we deal with the eigenvalue problem for vanishing boundary values. (See their § 3 ; and put in their notation $p \equiv 1$, $a \equiv b \equiv q \equiv 0$, and replace their k—in case of our system (2)—by p). Throughout this paper we use the result of their § 4 ; this implies that if the domain D is bounded by a Jordan curve C, then a function belonging to their classes $\mathring{\boldsymbol{D}}$ and $\boldsymbol{F}$ is continuous in the closure $\bar{D}$ of D and vanishes on the boundary C. We state now all the needed properties, e.g. for system (2).

A first eigenfunction $u(x, y)$ of the system (2) is defined as a (nontrivial) solution of this system corresponding to the first eigenvalue $\lambda (\lambda > 0)$. $u(x, y)$ is continuous in $\bar{D}$, vanishes on C, has continuous derivatives of first and second order in D and the integral

$$\iint_D |\operatorname{grad} u|^2 d\sigma = \iint_D (u_x^2 + u_y^2) d\sigma$$

exists. $d\sigma$ denotes the area element of D and throughout this paper all area integrals are improper Riemann integrals [3, Vol. 2, p. 478]. Moreover, $u(x, y) \neq 0$ in D [3, Vol. 1, Chapter VI, § 6] and the first eigenfunction is therefore essentially unique (i.e. except for a multiplicative constant). The Rayleigh ratio

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma \Big/ \iint_D p\varphi^2 \, d\sigma$$

attains its minimum λ in the class of all admissible functions $\varphi(x, y)$ for $\varphi \equiv u$. Here a function $\varphi(x, y)$ is called admissible in D if it is continuous in $\overline{D}$, vanishes on C, has piecewise continuous[1] first derivatives in D and if the integral

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma$$

exists.

To prove Theorem 1 assume first that $m(x, y)$ has circular symmetry in $D, m(x, y) = m(r)$. Let $v(x, y)$ be a fixed first eigenfunction of (3^+). As the first eigenfunction is essentially unique, it follows from the circular symmetry of $m(r)$ that $v(x, y)$ too has circular symmetry, $v(x, y) = v(r)$. (3^+) becomes therefore

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\} + \mu m(r)v(r) = 0 \quad \text{for} \quad 0 < r < R\,, \quad v(R) = 0\,.$$

As $v \neq 0$ in D, we may assume that $v(r) > 0$, $0 \leqq r < R$, and it follows that

$$\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\} < 0 \quad \text{for} \quad 0 < r < R\,.$$

This inequality and

$$\lim_{r=0}\left\{r\frac{d}{dr}v(r)\right\} = 0$$

imply

$$\frac{d}{dr}v(r) < 0 \quad \text{for} \quad 0 < r < R\,.$$

[1] A function is called piecewise continuous in a domain D if it is continuous there except for arbitrary discontinuities at isolated points and discontinuities of the first kind (jumps) along smooth arcs; and it is required that each closed subdomain of D has a nonempty intersection with only a finite number of these arcs [**3**, Vol. 2, p. 473].

$v(x, y)=v(r)$ is therefore symmetrically decreasing in D and, as $v>0$, the same holds for v^2. We have now

$$\mu(m)=\frac{\iint |\operatorname{grad} v|^2\, d\sigma}{\iint m v^2 d\sigma} \geqq \frac{\iint |\operatorname{grad} v|^2\, d\sigma}{\iint p^+ v^2 d\sigma} \tag{5}$$

$$\geqq \frac{\iint |\operatorname{grad} v|^2\, d\sigma}{\iint p v^2 d\sigma} \geqq \min \frac{\iint |\operatorname{grad} \varphi|^2\, d\sigma}{\iint p \varphi^2 d\sigma} = \lambda\ .$$

All the integrals are taken over the disk D. The first inequality sign follows from (1^+). The second inequality sign is justified by the above mentioned theorem on the rearrangements of functions [9, p. 153]. To apply this theorem, we note that p and p^+ are equimeasurable and that p^+ and v^2 are oppositely ordered. The minimum in (5) is taken over the class of the admissible functions φ, and v clearly belongs to this class. We proved thus (4^+) under the additional assumption that $m(x, y)$ has circular symmetry.

We define now

$$\lambda^+ = \text{g.l.b. } \mu(m)\ ; \tag{6$^+$}$$

here the g.l.b. is taken over all functions $m(x, y)$ fulfilling the requirements stated in the theorem and having, in addition, circular symmetry. Hence, we have until now established that

$$\lambda \leqq \lambda^+\ . \tag{7$^+$}$$

λ^+ is connected with the function p^+ in a more direct way ; that is, we show that

$$\lambda^+ = \text{g.l.b.} \frac{\iint |\operatorname{grad} \varphi|^2\, d\sigma}{\iint p^+ \varphi^2 d\sigma}\ , \tag{8$^+$}$$

where the g.l.b. is taken over all admissible functions $\varphi(x, y)$. To prove (8^+) let us denote its right hand side by λ_+. (6^+) implies that for every $\varepsilon>0$ there exists a circular symmetric function $m(x, y)=m(r)$, fulfilling all our above requirements, for which $\mu(m) \leqq \lambda^+ + \varepsilon$. Denoting the corresponding first eigenfunction by v and using (1^+) we obtain

$$\lambda^+ + \varepsilon \geqq \mu(m) = \frac{\iint |\operatorname{grad} v|^2\, d\sigma}{\iint m v^2\, d\sigma} \geqq \frac{\iint |\operatorname{grad} v|^2\, d\sigma}{\iint p^+ v^2 d\sigma} \geqq \lambda_+\ .$$

It follows that

$$\lambda^{+} \geqq \lambda_{+} . \tag{9}$$

On the other hand, given any ε, $0<\varepsilon<1$, there exists an admissible function $\varphi(x, y)$ such that

$$\lambda_{+}+\varepsilon \geqq \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p^{+}\varphi^2 d\sigma} .$$

Furthermore, by using the Weierstrass approximation theorem with respect to $p^{+}(r)$, we can find a function $m(x, y)=m(r)$ which, in addition to all our former requirements, fulfills also $p^{+}(r)(1-\varepsilon)\leqq m(r)$ for $0\leqq r\leqq R$. Hence,

$$\lambda_{+}+\varepsilon \geqq \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p^{+}\varphi^2 d\sigma} \geqq (1-\varepsilon)\frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint m\varphi^2 d\sigma}$$

$$\geqq (1-\varepsilon)\mu(m) \geqq (1-\varepsilon)\lambda^{+} .$$

This implies

$$\lambda_{+} \geqq \lambda^{+} \tag{10}$$

(9) and (10) give (8^{+}).

Let us interpret the g.l.b. $\mu(m)$ in a less restrictive way than in (6^{+}) ; that is, we take now this g.l.b. over all functions $m(x, y)$ fulfilling the requirements stated in the theorem (and drop the additional requirement of circular symmetry). By a proof entirely analogous to the one given just now, it follows that also this g.l.b. $\mu(m)$ (for the wider class) is equal to the right hand side of (8^{+}). This and (8^{+}) imply that λ^{+}, that is, the g.l.b. $\mu(m)$ for the restricted class (of circular symmetric functions), is equal to the g.l.b. $\mu(m)$ for the wider class of functions $m(x, y)$ (not necessarily having circular symmetry). (7^{+}) establshes therefore (4^{+}) for any function $m(x, y)$ fulfilling the requirements stated in the theorem. This concludes the proof of Theorem 1.

In the special case of $p^{+}(x, y)$ having continuous first derivatives in D, λ^{+} is the first eigenvalue (in the classical sense) of the differential system

$$\Delta v(x, y)+\lambda^{+}p^{+}(x, y)\, v(x, y)=0 \quad \text{for} \quad (x, y)\varepsilon D,\ v(C)=0 .$$

In any case we shall call λ^{+} the generalized first eigenvalue of this system.

2. The nonhomogeneous membrane II.

THEOREM 2. *Let D be a domain in the x, y-plane bounded by a Jordan curve C. Let the function $p(x, y)$ be positive and continuous in $\bar{D}\,(=D \cup C)$ and have continuous first derivatives in D. Let $p^-(x, y)=p^-(r)$ ($r^2=x^2+y^2$, $0 \leqq r \leqq R$) be the rearrangement of $p(x, y)$ in symmetrically decreasing order defined in the closed disk $\bar{D}^*$ (whose boundary we denote by C^*). Further let $k(x, y)$ be a function which is positive and continuous in $\bar{D}^*$, has continuous first derivatives in the open disk D^* and satisfies for each $(x, y) \in \bar{D}^*$*

$$k(x, y) \geqq p^-(x, y) \,. \tag{1^-}$$

Consider the differential systems

$$\Delta u(x, y)+\lambda p(x,y)\, u(x, y)=0 \quad \textit{for} \quad (x, y) \in D,\ u(C)=0 \,, \tag{2}$$

and

$$\Delta w(x, y)+\kappa k(x, y)\, w(x, y)=0 \quad \textit{for} \quad (x, y) \in D^*,\ w(C^*)=0 \,, \tag{3^-}$$

and denote their first eigenvalues by λ and $\kappa=\kappa(k)$ respectively. Then

$$\lambda \geqq \kappa(k) \,. \tag{4^-}$$

For the proof set

$$\lambda^- = \text{l.u.b.}\ \kappa(k) \,, \tag{6^-}$$

where the l.u.b. is taken over all functions $k(x, y)$ satisfying the just stated conditions. The theorem will be proved if we show that

$$\lambda \geqq \lambda^- \,. \tag{7^-}$$

Similar to (8^+), it follows that

$$\lambda^- = \text{g.l.b.} \frac{\displaystyle\iint_{D^*} |\operatorname{grad} \varphi|^2 d\sigma}{\displaystyle\iint_{D^*} p^- \varphi^2 d\sigma} \,; \tag{8^-}$$

here the g.l.b. is taken over all admissible functions $\varphi(x, y)$ in D^*. We shall use (8^-) for the proof of (7^-).

In the proof we make use of the first eigenfunction $u(x,y)$ of (2) and of its rearrangement in symmetrically decreasing order $u^-(x, y)=u^-(r)$. In particular, we have to show that u^- is an admissible function in D^* (see (12) below). u^- is continuous in $\bar{D}^*$ and vanishes on C^* ; it is, however, doubtful whether in the case of a general $p(x, y)$, satisfying the conditions of the theorem, the first derivatives of $u^-(x, y)$ are piecewise continuous in D^*. But this is true, as we shall see presently, in the case in which the function $p(x, y)$ is analytic.

We therefore prove (7^-) first under the assumption that $p(x, y)$ is positive and continuous in $\bar{D}$ and analytic in D. The first eigenfunction $u(x,y)$ of (12) is then also analytic in D [8, p. 162]. We assume $u(x, y)$ fixed so that $u(x, y)>0$ for $(x, y) \in D$. Following Krahn [7], we consider the planes z=constant which touch the surface $z=u(x, y)$, $(x, y) \in D$, and we claim that this (finite or infinite) set of horizontal planes can be enumerated $z=z_i$, $i=1, 2, \cdots$, in such a way that $z_1>z_2>\cdots, z_i>0$, and that (in case of infinitely many such planes) $\lim z_i=0$. Indeed, as $u(x, y)$ is continuous in $\bar{D}$, positive in D and vanishes on C, if this were not so then we could find a sequence $(x_n, y_n) \in D$, $n=1, 2, \cdots$, with the following properties:

(a) $\lim_{n=\infty} (x_n, y_n)=(x_0, y_0) \in D$;

(b) $\operatorname{grad} u(x_n, y_n)=0$, $n=1, 2, \cdots$;

(c) $u(x_m, y_m)\neq u(x_n, y_n)$ for $m\neq n$, $m, n=1, 2, \cdots$. We show now that the existence of such a sequence (x_n, y_n) is impossible. Let us consider the two sets of points (x, y) in D given by $u_x(x, y)=0$ and $u_y(x, y)=0$ respectively. u_x and u_y are together with u analytic functions in D. As u is a solution of (2) the identically vanishing of u_x or u_y is excluded. Hence, both these sets consist of analytic curves (or arcs) and we consider these curves near (x_0, y_0). Using $\Delta u<0$ and, if necessary, rotating the coordinate system of the plane, we may assume that both $u_{xx}\neq 0$ and $u_{yy}\neq 0$ at (x_0, y_0). The curve $u_x(x, y)=0$ is thus near (x_0, y_0) represented by a power series of the form $x-x_0=P_1(y-y_0)$. Similary, $u_y(x, y)=0$ is there represented by $y-y_0=P_2(x-x_0)$. The expansion for $u_x(x, y)=0$ may be solved by $y-y_0=P_3((x-x_0)^{1/k})$, where $k\geqq 1$ is given by the index of the first nonvanishing coefficient of P_1. By the above properties (a) and (b) of the sequence (x_n, y_n) it follows that $P_2(x_n-x_0)=P_3((x_n-x_0)^{1/k})$, $n=1, 2, \cdots$. As infinitely many of these last equalities hold for a fixed branch of $(x-x_0)^{1/k}$, it follows that $P_2(x-x_0)\equiv P_3((x-x_0)^{1/k})$ and that $k=1$. u_x and u_y vanish along this analytic curve which contains all the points (x_n, y_n). This gives the desired contradiction to property (c) and we have justified the enumeration of the horizontal tangential planes $z=z_i$.

Using $\Delta u<0$—which excludes the existence of minima of $u(x, y)$—it follows that there are no closed curves along which $\operatorname{grad} u=0$. Arcs, ending at the boundary C of D, along which $\operatorname{grad} u=0$ are clearly excluded. This implies that no sequence (x_n, y_n) having the above properties (a) and (b) exists. Hence, each critical plane $z=z_i$ touches the surface $z=u(x, y)$ only in a finite number of points (and, for $i=2, 3, \cdots$, cuts the surface along certain analytic curves).

For any z, $0<z<z_1$, denote by $C(z)$ the level set $u(x, y)=z$ and let $A(z)$ be the area of the open set in D for which $u(x, y)>z$. $C(z)$ consists of the boundary of this open set and contains for $z=z_i$ $(i=2, 3, \cdots)$ perhaps an additional finite number of points. For $z\neq z_i$ $C(z)$ separates into a finite number of simple closed analytic curves and it follows that for each z, $0<z<z_1$, $C(z)$ is of finite positive length. We consider now the open intervals $z_i>z>z_{i+1}$ $(i=1, 2, \cdots)$, where in the case of only a finite number n of critical values z_i the last interval is $z_n>z>0$. For each z in one of these open intervals we have ([7, formula (10)] and [10, § II])

$$\frac{dA}{dz}=-\int_{C(z)}\frac{ds}{|\operatorname{grad} u|},$$

where ds denotes the length element of $C(z)$. Clearly $dA/dz<0$ $(z\neq z_i)$.

Let k_i denote the number of simple closed analytic curves into which $C(z)$ separates for z in the open interval (z_{i+1}, z_i), $i=1, 2, \cdots$. k_i is a function of i only, and it follows from the last formula that dA/dz is continuous for (z_{i+1}, z_i). The same consideration implies the existence of the one-sided limits of dA/dz as z tends to $z_{i+1}+0$ and z_i-0. These limits may conceivably be equal to $-\infty$[2], but are different from zero. Indeed, as $C(z_{i+1})$ is of positive length $(i=1, 2, \cdots)$, it follows that for $z\to z_{i+1}+0$ at least one of the k_i families of simple closed curves, into which the level sets $C(z)$ separate for z in (z_{i+1}, z_i), converges to a part of positive length of $C(z_{i+1})$. The same argument holds for $z\to z_i-0$ $(i=2, 3, \cdots)$.

As remarked in § 1, $A(z)$ is a strictly decreasing function of z, $0\leq z\leq z_1$. In the present case $A(z)$ is also continuous in this interval. This follows from the fact that $u(x, y)$ achieves a fixed z-value only for finitely many curves and (perhaps) points in D and not for a set of positive area. The definition of $u^-(x, y)=u^-(r)$ and the continuity of $A(z)$ imply that $A(z)=\pi r^2$ for $u^-(r)=z$ $(0\leq z\leq z_1,\ 0\leq r\leq R)$. Hence, $u^-(r)=A^{-1}(\pi r^2)$ and $u^-(r)$ is not only continuous (see § 1) but also strictly decreasing. The critical z-values $z_1, z_2, \cdots$ correspond to the critical r-values $r_1, r_2, \cdots$ with $r_1=0<r_2<r_3\cdots$, $r_i<R$ and (in case of infinitely many critical values) $\lim_{i=\infty} r_i=R$. As

$$\frac{du^-(r)}{dr}=\frac{dA^{-1}(\pi r^2)}{d(\pi r^2)}2\pi r=2\pi r\Big/\left(\frac{dA}{dz}\right),$$

[2] This, indeed, cannot occur. We do not have to bring the argument which excludes this case, as we may allow that the one-sided limits of du^-/dr at $r_i (i=2, 3, \cdots)$ are equal to 0. Similary, it can be shown that dA/dz tends to a (finite) negative value as $z\to z_1-0$ so that $du^-/dr\to 0$ as $r\to 0$. This again will not be needed as an arbitrary singularity of u_x^- and u_y^- at (0, 0) does not invalidate their being piecewise continuous in D^*. (See below).

it follows that du^-/dr is continuous for each open interval $r_i < r < r_{i+1}$ $(i=1, 2, \cdots)$ and that its discontinuities at the values r_i $(i=2, 3, \cdots)$ are of the first kind (jumps). Every interval $0 \leqq r \leqq \rho$, $0 < \rho < R$, contains only a finite number of critical values r_i and every closed subdomain of D^* intersects therefore with only a finite number of critical circles $x^2+y^2=r_i^2$. The continuity of du^-/dr at $r \neq r_i$ implies the continuity of u_x^- and u_y^- at all points of D^* different from the center and not lying on these critical circles. At the critical circles $x^2+y^2=r_i^2$ $(i=2, 3, \cdots)$ u_x^- and u_y^- have (at most) jumps and it follows that these first derivatives of $u^-(x, y)$ are piecewise continuous in D^*. Moreover, as was shown by Faber [4] and Krahn [7] in their proofs of Rayleigh's conjecture, the Dirichlet integral

$$\iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma$$

exists and fulfills the inequality.

$$\iint_D |\operatorname{grad} u|^2 d\sigma \geqq \iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma\,, \tag{11$^-$}$$

which we shall use presently. All this, together with the previously established continuity of $u^-(x, y)$ in $\overline{D}^*$ and its vanishing on C^*, prove finally that the function $u^-(x, y)$ is admissible in D^*.

We have now

$$\lambda = \frac{\iint_D |\operatorname{grad} u|^2 d\sigma}{\iint_D p u^2 d\sigma} \geqq \frac{\iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma}{\iint_{D^*} p^-(u^-)^2 d\sigma} \tag{12}$$

$$\geqq \text{g.l.b.} \frac{\iint_{D^*} |\operatorname{grad} \varphi|^2 d\sigma}{\iint_{D^*} p^- \varphi^2 d\sigma} = \lambda^-\,.$$

To justify the first inequality sign in (12) we use (11$^-$) for the numerators and for the denominators we apply again the theorem on the rearrangements of functions. (As $u > 0$, $(u^-)^2$ is together with u^- symmetrically decreasing, and p^- and $(u^-)^2$ are therefore similarly ordered.)[3]

[3] The integrals in the theorem on the rearrangements of functions [**9**, p. 153] are taken over the same bounded region. Our case, integrating once over D and the other time over D^*, can easily be reduced to that case of the same region of integration. We embed D and D^* into the same plane and take all integrals over a bounded region G containing both D and D^*, after having completed p, p^-, u and u^- into G by steting $p \equiv u \equiv 0$ in $G-\overline{D}$ and $p^- \equiv u^- \equiv 0$ in $G-\overline{D}^*$.

The g.l.b. appearing in (12) is taken over all admissible functions φ in D^* and is thus by (8^-) equal to λ^-. We proved (7^-), and hence the theorem, under the additional assumption of $p(x, y)$ being analytic in D.

This special case implies now (7^-) for any function $p(x, y)$ satisfying the conditions stated in the theorem. Indeed, as $p(x, y)$ is positive and continuous in $\overline{D}$, the Weierstrass approximation theorem assures that for every $\delta > 0$ there exists a polynomial $p_\delta(x, y) = p_\delta$, so that

$$0 < p(x, y) \leqq p_\delta(x, y) \leqq p(x, y)(1 + \delta) \tag{13}$$

holds for all points (x, y) of $\overline{D}$. Denoting by $\lambda(\delta)$ the (classical) first eigenvalue of the differential system with the density function p_δ, the minimum property of the first eigenvalue implies

$$\lambda(\delta) \leqq \lambda \leqq \lambda(\delta)(1 + \delta) . \tag{14}$$

Let $p_\delta^-(x, y) = p_\delta^-(r)$ be the rearrangement of p_δ in symmetrically decreasing order defined in $\overline{D}^*$. (13) gives

$$0 < p^-(r) \leqq p_\delta^-(r) \leqq p^-(r)(1 + \delta) \tag{13$^-$}$$

for $0 \leqq r \leqq R$. For the corresponding generalized first eigenvalues it follows by (8^-) and the analogous definition of $\lambda^-(\delta)$ that

$$\lambda^-(\delta) \leqq \lambda^- \leqq \lambda^-(\delta)(1 + \delta) . \tag{14$^-$}$$

For each polynomial $p_\delta(x, y)$ we proved

$$\lambda(\delta) \geqq \lambda^-(\delta) .$$

As δ tends to 0, we obtain from (14), (14^-) and the last inequality

$$\lambda \geqq \lambda^- . \tag{7$^-$}$$

Theorem 2 is therefore established.

Inequality (11^-), i.e. the fact that the Dirichlet integral of the first eigenfunction decreases under symmetrization, was an essential step in our proof. On the other hand, this inequality constitutes Faber's and Krahn's proof of Rayleigh's conjecture. It is thus by no means surprising that Theorem 2 includes the theorem of Rayleigh, Faber and Krahn as the special case $p(x, y) \equiv 1$. However, Theorem 2 implies only a weakened from of their theorem, since with regard to inequality (11^-) Faber and Krahn proved more than we used. They showed that equality in (11^-) can occur only if D is a circle. Their theorem thus states that *for all homogeneous membranes with constant area the minimum of the principal frequency is achieved for the disk and only for the disk.* As for any homogeneous membrane $\lambda^+ = \lambda^-$, it follows that if $p \equiv 1$ and D

is not a disk then $\lambda > \lambda^+$. Hence, *Theorem* 1 *can not be extended to any noncircular domain.* For any such domain there exist functions $p(x, y)$, for example, all the positive constants, so that $\lambda > \lambda^+$, and at least for nearly circular domains there exist functions so that $\lambda < \lambda^+$. This last fact follows from the continuity of the first eigenvalue as a function of the domain [3, Vol. 1, Chapter VI, Theorem 11] (and we assume that for some functions $p(x, y)$ in the disk the proper inequality sign holds in (7^+)).

A lower bound for the principal frequency of nonhomogeneous membranes was obtained by Szegö in his paper on the generalized Dirichlet integral [10]. In this case the density function $p(x, y)$ is given in the whole x, y-plane (except at the origin) and satisfies there the following conditions:

(a) $p(x, y)$ is positive in the whole x, y-plane (with the exception of the origin);

(b) $p(x, y)$ has circular symmetry, $p(x, y) = p(r)$, and $p(r)$ is a nonincreasing function of r, $r > 0$;

(c) $rp(r)$ is integrable in a neighborhood of $r = 0$. Considering membranes lying in this plane, Szegö's result is that *for all membranes with given area the minimum of the principal frequency is achieved for the disk whose center coincides with the origin of the plane.* [10, § V]. While keeping Szegö's condition (b), we replace his conditions (a) and (c) by the following more restrictive assumptions: (a′) $p(x, y)$ is positive and continuous in the whole x, y-plane; (c′) $p(x, y)$ has continuous first derivatives in the whole x, y-plane. Under these more restrictive conditions (a′), (b) and (c′), Szegö's result follows from Theorem 2. Indeed, let D be a domain in the x, y-plane with the given density function $p(x, y)$. Let D^* and $p^-(x, y)$ be defined as in Theorem 2, but put the center of D^* into the origin of the given x, y-plane. As $p(r)$ is a nonincreasing function of r, $r \geqq 0$, it follows that for each $(x, y) \in D^*$

$$p(x, y) \geqq p^-(x, y) . \tag{15}$$

(a′), (c′) and (15) imply that $p(x, y)$ in D^* satisfies all the conditions which were in Theorem 2 required of $k(x, y)$. (4^-) is thus the desired conclusion. (For a one-dimensional analogue of Szegö's theorem see [1, Lemma 3].)

We state now our results on the nonhomogeneous membrane in a form involving only generalized first eigenvalues. We drop therefore the requirement that the original density function $p(x, y)$ has continuous first derivatives.

THEOREM 2′. *Let D be a domain in the x, y-plane bounded by a Jordan curve C and let $p(x, y)$ be positive and continuous in $\overline{D}$. Let*

$p^+(x, y)=p^+(r)$ and $p^-(x, y)=p^-(r)$ be the rearrangements of $p(x, y)$ in symmetrically increasing respectively decreasing order defined in the closed disk $\bar{D}^*$. Consider the three differential systems

$$\Delta u(x, y)+\lambda p(x, y)\, u(x, y)=0 \quad \text{for} \quad (x, y)\in D, \quad u(C)=0\,;$$

$$\Delta v(x, y)+\lambda^+ p^+(x, y)\, v(x, y)=0 \quad \text{for} \quad (x, y)\in D^*, \quad v(C^*)=0\,;$$

$$\Delta w(x, y)+\lambda^- p^-(x, y)\, w(x, y)=0 \quad \text{for} \quad (x, y)\in D^*, \quad w(C^*)=0\,;$$

and denote their generalized first eigenvalves by λ, λ^+ and λ^- respectively. λ is defined by

$$\lambda=\text{g.l.b.}\frac{\iint_D |\operatorname{grad}\varphi|^2 d\sigma}{\iint_D p\varphi^2 d\sigma}, \tag{8}$$

where the g.l.b. *is taken over all admissible functions $\varphi(x, y)$ in D, and λ^+ and λ^- are analogously defined by* (8^+) *and* (8^-). *Then $\lambda^-\leqq\lambda$. In the special case of D being a disk ($D=D^*$) we have in addition $\lambda\leqq\lambda^+$.*

To prove this let us again approximate $p(x, y)$ by polynomials $p_\delta(x, y)$ satisfying (13). This implies (14), with λ now being defined by (8); (14) and (14^-) give as before (7^-), that is, $\lambda^-\leqq\lambda$. The additional result for the disk follows, by the same approximation, from Theorem 1.

We conclude the treatment of the nonhomogeneous membrane with the following remarks. It is known that the second proper frequency of a homogeneous membrane of given area does not attain its minimum for the disk [**9**, p. 168]. This implies that Theorem 2 cannot be extended to the second proper frequency; i.e. under its assumptions the relation $\lambda_2^-\leqq\lambda_2$ cannot be proved. Even for the circular nonhomogeneous membrane we are not able to establish any inequality—or equality—between λ_2, λ_2^+ and λ_2^-. It is thus of some interest to note that for the one-dimensional case (see [**1**, Theorem 2]) $\lambda_2^-=\lambda_2^+$. This follows easily from the relation $p^-(x)=p^+(x_0-x)$, $0\leqq x\leqq x_0$.

Finally, an intuitive proof gives the following analogue of Theorem 2. *The principal frequency of a nonhomogeneous membrane of arbitrary shape decreases (i.e. does not increase) under Steiner symmetrization or under Pólya (circular) symmetrization.* (cf. [**9**, Note A] and [**6**, Chapter I]). Indeed, formula (12) holds also for these symmetrizations. The Dirichlet integral of the first eigenfunction decreases and we apply the one-dimensional case of the theorem on the rearrangements of functions for each member of an (obvious) one parameter family of straight or circular segments respectively. (Note that if D is not convex with respect to this family, then p^- is in general not continuous in D^*.

On the other hand, u^- is always continuous in D^*.) It is easily seen that Steiner and Pólya symmetrizations are weaker than Schwarz symmetrization used in Theorem 2; the lower bounds obtained by the first two kinds of symmetrization are not smaller than λ^- of Theorem 2.

3. The generalized Dirichlet integral. In this section we follow closely Szegö's treatment of the generalized Dirichlet integral ([10] and [9, Note D]); however, our definition of this integral will be somewhat simpler than Szegö's. We consider a ring-shaped domain D in the x, y-plane, that is, D is bounded by two Jordan curves C_0 and C_1 such that C_0 is completely in the interior of C_1. We call C_0 and C_1 the inner and outer boundary of D respectively and we denote the interior of C_1 by G. Let D^* be the open annulus which has the circle C_0^* of radius R_0 as inner boundary and the (concentric) circle C_1^* of radius R_1 as outer boundary ($0<R_0<R_1<\infty$). The radii are so chosen that the disk bounded by C_0^* has the same area as the interior of C_0 and that the disk G^* bounded by C_1^* has the same area as G. Hence D^* has the same area as D and we assume that the center of D^* is the origin of a (new) x, y-plane and use again $r=(x^2+y^2)^{1/2}$.

Let $p(x, y)$ be nonnegative and continuous in the closure $\bar{D}$ of D. Its rearrangements in symmetrically increasing and decreasing order are defined in complete analogy to the case of a simply connected domain: $p^+(x, y)$ and $p^-(x, y)$ are defined in $\bar{D}^*$; both functions have circular symmetry $p^+(x, y)=p^+(r)$, $p^-(x, y)=p^-(r)$ and $p^+(r)$ is a nondecreasing, $p^-(r)$ a nonincreasing function of r, $R_0\leqq r\leqq R_1$; p, p^+ and p^- are equimeasurable; finally, $p^+(R_0)$ ($p^-(R_0)$) is equal to the minimum (maximum) of p in $\bar{D}$ and $p^+(R_1)$ ($p^-(R_1)$) is equal to the maximum (minimum) of p in $\bar{D}$. Both rearrangements are nonnegative and continuous in $\bar{D}^*$.

The admissible functions are now defined as follows. A function $\varphi(x, y)$ is called admissible in D if it is continuous in $\bar{D}$, vanishes on C_0, is equal to 1 on C_1, has piecewise continuous first derivatives in D and if the integral

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma$$

exists. The admissible functions in D^* are defined analogously and will be denoted by $\psi(x, y)$. Using these definitions, we state.

THEOREM 3. *Let D be a ring-shaped domain in the x, y-plane and let the Jordan curves C_0 and C_1 be the inner and outer boundary of D respectively. Let the function $p(x, y)$ be positive and continuous in $\bar{D}$ and have continuous first derivatives in D. Let $p^-(x, y)=p^-(r)$ ($R_0\leqq r\leqq R_1$)*

be the rearrangement of $p(x, y)$ in symmetrically decreasing order defined in the closed annulus $\overline{D}$. Denote by $4\pi\gamma$ the minimum of the generalized Dirichlet integral*

$$E(\varphi)=\iint_D \{|\operatorname{grad} \varphi|^2+p\varphi^2\}\, d\sigma \tag{16}$$

in the class of all admissible functions $\varphi(x, y)$ in D. Similarly, denote by $4\pi\gamma^-$ the g.l.b. *of the generalized Dirichlet integral*

$$E^-(\psi)=\iint_{D^*} \{|\operatorname{grad} \psi|^2+p^-\psi^2\}\, d\tau \tag{16^-}$$

in the class of all admissible functions $\psi(x, y)$ in D^* which satisfy $|\psi|\leqq 1$. Then[4]

$$\gamma \geqq \gamma^- . \tag{17^-}$$

We rely again on Courant-Hilbert [3, Vol. 2, Chapter VII]. To minimize $E(\varphi)$ in the class of all admissible functions $\varphi(x, y)$ in D is a special case of their Variational Problem I corresponding to the first boundary value problem. (See their § 2 ; and put in their notation $p\equiv k\equiv 1$, $a\equiv b\equiv f\equiv 0$, and replace their q by our p. To assure that all their assumptions are satisfied, we have to show that there exists a function g which is continuous in $\overline{D}$, vanishes on C_0, is equal to 1 on C_1 and has piecewise continuous first derivatives in D which are such that

$$\iint_D |\operatorname{grad} g|^2\, d\sigma$$

exists. The existence of such a function g follows by conformal mapping. Set $z=x+iy$ and let $\zeta=\varphi(z)$ be the function which maps D onto the annulus $\rho<|\zeta|<1$. The harmonic function $g(x, y)=g(z)$,

$$g(z)=\log \frac{|\varphi(z)|}{\rho} \Big/ \log \frac{1}{\rho}$$

has all the required properties.)

We again use the result of their § 4 with an implication similar to the one stated in our § 1. With regard to the same problem for $E^-(\psi)$, the conditions of Courant-Hilbert are satisfied only if $p^-(x, y)$ has continuous first derivatives in D^*. As this is in general not true, $4\pi\gamma^-$ has to be defined as the g.l.b. $E^-(\psi)$.

[4] The words "which satisfy $|\psi|\leqq 1$" may of course be deleted. But we shall need the above given formulation of Theorem 3 to obtain Theorem 3′.

The variational problem to minimize $E(\varphi)$ in the class of all admissible functions $\varphi(x, y)$ in D has a unique solution $u(x, y)$. This admissible function $u(x, y)$ has continuous derivatives of first and second order in D and is also the unique solution of the corresponding boundary value problem ; that is, $u(x, y)$ solves the system

$$\text{(18)} \quad \Delta u(x, y) - p(x, y)\, u(x, y) = 0 \quad \text{for} \quad (x, y) \in D\,, \quad u(C_0) = 0\,, \quad u(C_1) = 1\,,$$

and is the only admissible function having continuous first and second derivatives which solves this system. (18) and $p(x, y) > 0$ imply $0 \leqq u(x, y) \leqq 1$ for $(x, y) \in \overline{D}$.

For the same reason as in § 2, we prove Theorem 3 first under the assumption that $p(x, y)$ is not only positive and continuous in $\overline{D}$ but is also analytic in D. (18) implies the analyticity of $u(x, y)$ in D and in complete analogy to § 2—using (11⁺) below— it follows that $u^+(x, y) = u^+(r)$ is an admissible function $\psi(x, y)$ in D^* which, by the above, satisfies $|\psi| \leqq 1$. We have now

$$\text{(19)} \qquad 4\pi\gamma = \iint_D \{|\operatorname{grad} u|^2 + pu^2\}\, d\sigma \geqq \iint_{D^*} \{|\operatorname{grad}(u^+)|^2 + p^-(u^+)^2\}\, d\sigma$$

$$\geqq \text{g.l.b.} \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\}\, d\sigma = 4\pi\gamma^- \,.$$

To establish (19) it remains only to justify its first inequality sign. For this purpose we use

$$\text{(11}^+\text{)} \qquad \iint_D |\operatorname{grad} u|^2 d\sigma \geqq \iint_{D^*} |\operatorname{grad}(u^+)|^2 d\sigma \,;$$

that is, the fact, proved by Szegö [10], that also in this case the Dirichlet integral decreases under symmetrization. The remaining inequality

$$\iint_D pu^2\, d\sigma \geqq \iint_{D^*} p^-(u^+)^2\, d\sigma$$

is again a consequence of the theorem on the rearrangements of functions. (See footnote 3) and complete u, u^+, p and p^- in an obvious way into a bounded region containing D and D^*.) This establishes (19) and thus proves Theorem 3 for analytic functions $p(x, y)$.

This special case implies (17⁻) for any function $p(x, y)$ satisfying the conditions stated in the theorem. We use the same approximation as in the analogue step in § 2. $p_\delta(x, y) = p_\delta$ is again a polynomial satisfying (13) in $\overline{D}$ and (13⁻) holds therefore for $\overline{D}^*$. Replacing in (16) p by p_δ and in (16⁻) p^- by p_δ^-, we denote the corresponding minimum and g.l.b.

by $4\pi\gamma(\delta)$ and $4\pi\gamma^-(\delta)$ respectively. By (13) and the definitions of γ and $\gamma(\delta)$ we obtain (using that $0 \leqq u \leqq 1$)

$$4\pi\gamma = \iint_D \{|\operatorname{grad} u|^2 + pu^2\} d\sigma \geqq \iint_D |\operatorname{grad} u|^2 d\sigma + \left(1 - \frac{\delta}{1+\delta}\right) \iint_D p_\delta u^2 d\sigma$$
$$\geqq 4\pi\gamma(\delta) - \delta P d ,$$

where P is the maximum of $p(x, y)$ in $\bar{D}$ and d denotes the area of D. Setting $\alpha = Pd/4\pi$ we have

$$\gamma \geqq \gamma(\delta) - \delta\alpha . \tag{20}$$

By (13^-) and the definitions of γ^- and $\gamma^-(\delta)$, there exists for each $\varepsilon > 0$ an admissible function $\psi(x, y)$ in D^*, satisfying $|\psi| \leqq 1$, so that

$$4\pi\gamma^-(\delta) + \varepsilon \geqq \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p_\delta^- \psi^2\} d\sigma \geqq \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\} d\sigma \geqq 4\pi\gamma^- ;$$

hence,

$$\gamma^-(\delta) \geqq \gamma^- . \tag{21}$$

For each polynomial $p_\delta(x, y)$ we proved

$$\gamma(\delta) \geqq \gamma^-(\delta) .$$

As δ tends to 0, we obtain from (20), (21) and the last inequality the desired conclusion (17^-) and we thus completed the proof of Theorem 3.

The assumptions of this theorem can be weakened ; that is, as in Theorem 2′, there is no need to assume the existence (and continuity) of the first derivatives of $p(x, y)$. *Theorem* 3 *remains correct if we assume with respect to* $p(x, y)$ *only its being positive and continuous in* $\bar{D}$ *and if we accordingly define* $4\pi\gamma$ *as the* g.l.b. $E(\varphi)$ *in the class of all admissible functions* $\varphi(x, y)$ *in* D *which satisfy* $|\varphi| \leqq 1$. Indeed, the just given proof remains unchanged except for a slight modification in the derivation of (20).

We mentioned that definition (16) differs from Szegö's definition of the generalized Dirichlet integral. In order to obtain his result on this integral it will be convenient to restate Theorem 3 using his definition.

THEOREM 3′ *Let* D, C_0, C_1, D^*, C_0^* *and* C_1^* *have the same meaning as in Theorem* 3 *and denote the interior of* C_1 *by* G *and the interior of* C_1^* *by* G^*. *Let* $p(x, y)$ *be positive and continuous in* $\bar{G}$ *and have continuous first derivatives in* G (*or at least in* D). *Let* $p^-(x, y) = p^-(r)$ $(0 \leqq r \leqq R_1)$ *be the rearrangement of* p *in symmetrically decreasing order* (*in the sense of* § 1) *defined in* $\bar{G}^*$. *Further let* $k(x, y)$ *be positive and* *continuous in* $\overline{G}^*$, *have continuous first derivatives in* G^* (*or at least in* D^*) *and satisfy for each* $(x, y) \in \overline{G}^*$

$$(1^-) \qquad k(x, y) \geqq p^-(x, y) .$$

Denote by $4\pi c$ *the minimum of the generalized Dirichlet integral*

$$(22) \qquad D(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma - \iint_G p\, d\sigma$$

in the class of all admissible functions $\varphi(x, y)$ *in* D. *Similarly, denote by* $4\pi c(k)$ *the minimum of the generalized Dirichlet integral*

$$(23) \qquad D_k(\psi) = \iint_{D^*} \{|\operatorname{grad} \psi|^2 + k\psi^2\}\, d\sigma - \iint_{G^*} k\, d\sigma$$

in the class of all admissible functions $\psi(x, y)$ *in* D^*. *Then*

$$(24) \qquad c \geqq c(k) .$$

For the proof let $4\pi c^-$ be the g.l.b. of the generalized Dirichlet integral

$$(22^-) \qquad D^-(\psi) = \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\}\, d\sigma - \iint_{G^*} p^- d\sigma$$

in the class of all admissible functions $\psi(x, y)$ in D^* which satisfy $|\psi| \leqq 1$. We show first that

$$(25) \qquad c \geqq c^- .$$

As

$$\iint_G p\, d\sigma = \iint_{G^*} p^- d\sigma$$

and as these two integrals are independent of φ and ψ respectively, (25) is equivalent to

$$(26) \qquad \min \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma \geqq \text{g.l.b.} \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^-\psi^2\}\, d\sigma ;$$

here the minimum is taken over all admissible functions φ in D, the g.l.b. only over those admissible functions ψ in D^* which satisfy $|\psi| \leqq 1$. p^- in (26) is obtained by rearranging — in the sense of § 1 — the in $\overline{G}$ defined function p and then considering this rearrangement only in $\overline{D}^*$. p^- in (16^-) is the rearrangement—in the sense of the beginning of this section—of the restriction of the function p to $\overline{D}$. It is easily seen that, at each point $(x, y) \in \bar{D}^*$, $p^-(x, y)$ in the sense of (26) is not larger than $p^-(x, y)$ in the sense of (16^-). Theorem 3 implies thus (26) and hence the proof of (25).

Let now $k(x, y)$ be any function satisfying the conditions stated in Theorem 3′. By the definition of c^-, there exists for each $\varepsilon > 0$ an admissible function ψ in D^* satisfying $|\psi| \leqq 1$, so that $4\pi c^- + \varepsilon \geqq D^-(\psi)$. Using ($1^-$), ($22^-$), (23) and $|\psi| \leqq 1$ we obtain

$$4\pi c^- + \varepsilon \geqq D^-(\psi) = \iint_{D^*} |\operatorname{grad} \psi|^2 d\sigma - \iint_{D^*} p^-(1-\psi^2) d\sigma - \iint_{G^*-D^*} p^- d\sigma$$

$$\geqq \iint_{D^*} |\operatorname{grad} \psi|^2 d\sigma - \iint_{D^*} k(1-\psi^2)\, d\sigma - \iint_{G^*-D^*} k d\sigma = D_k(\psi) \geqq 4\pi c(k) \,.$$

We thus obtain $c^- \geqq c(k)$ which together with (25) gives (24). Theorem 3′ is therefore established.

We state now Szegö's theorem on the generalized Dirichlet integral ([10], [9, Note D]) in the following restricted form: *Let the function $p(x, y)$ be given in the whole x, y-plane and satisfy there conditions* (a′), (b) *and* (c′) *stated in* § 2. *Let D be a ring-shaped domain in this plane bounded by the inner Jordan curve C_0 and the outer Jordan curve C_1. Denote by $4\pi c$ the minimum of the generalized Dirichlet integral*

$$D(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma - \iint_G p d\sigma \tag{22}$$

in the class of all admissible functions $\varphi(x, y)$ in D. Of all ring-shaped domains D with given area and with given area of the containing simply connected domain G, the annulus whose center coincides with the origin of the given plane has the minimum generalized capacity c.

This theorem follows from Theorem 3′ in the same way as our restricted form of Szegö's theorem on the membrane followed from Theorem 2. ((15) holds now in $\bar{G}^*$.) Szegö proves this theorem on the generalized Dirichlet integral assuming only conditions (b) and (c) stated in § 2[5] instead of our more restrictive conditions (a′), (b) and (c′).

Similarly to the final remark of § 2, it follows intuitively that *Theorem 3 and Theorem 3′ remain correct if we use Steiner or Pólya symmetrization instead of Schwarz symmetrization.* For the analogues of Theorem 3, Steiner and Pólya symmetrizations of functions given in a ring-shaped domain have to be defined in an obvious way.

Theorem 3 corresponds to Theorem 2 on the membrane. We state now a theorem on the generalized Dirichlet integral which corresponds to Theorem 1.

[5] We are unable to follow Szegö's argument allowing to drop the condition $p > 0$ (that is, condition (a) of § 2).

THEOREM 4. *Let D be the annulus $R_0^2<x^2+y^2<R_1^2$, $0<R_0<R_1<\infty$ and denote its inner boundary by C_0 and its outer boundary by C_1. Let $p(x,y)$ be positive and continuous in $\overline{D}$ and have continuous first derivatives in D. Let $p^+(x,y)=p^+(r)$ $(R_0\leqq r\leqq R_1)$ be the rearrangement of $p(x,y)$ in symmetrically increasing order defined in $\overline{D}$ $(=\overline{D}^*)$. Let γ have the same meaning as in Theorem 3 and denote by $4\pi\gamma^+$ the* g.l.b. *of the generalized Dirichlet integral*

$$E^+(\varphi)=\iint_D\{|\operatorname{grad}\varphi|^2+p^+\varphi^2\}\,d\sigma \tag{16$^+$}$$

in the class of all admissible functions $\varphi(x,y)$ in D which satisfy $|\varphi|\leqq 1$. Then

$$\gamma\leqq\gamma^+ . \tag{17$^+$}$$

For the proof let $m(x.\,y)=m(r)$ be a function having circular symmetry in $\overline{D}$ and assume that m is continuous in $\overline{D}$ and has continuous first derivatives in D. Moreover, for each $(x,y)\in\overline{D}$ let $m(x,y)\geqq p^+(x,y)$. Denote by $4\pi\gamma(m)$ the minimum of

$$E_m(\varphi)=\iint_D\{|\operatorname{grad}\varphi|^2+m\varphi^2\}\,d\sigma$$

in the class of all admissible functions $\varphi(x,y)$ in D. Then it is easily proved that

$$\gamma^+=\text{g.l.b. }\gamma(m)\ ,$$

where the g.l.b. is taken over all functions $m(x,y)=m(r)$ satisfying the above conditions. Let now m be such a function and let $v(x,y)$ be the uniquely given admissible function for which $E_m(v)=4\pi\gamma(m)$. The uniqueness of v and the circular symmetry of m imply that v too has circular symmetry, $v(x,y)=v(r)$. As $v(r)$ solves the differential system

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\}-m(r)\,v(r)=0\quad\text{for } R_0<r<R_1\ ,\quad v(R_0)=0\ ,\quad v(R_1)=1\ ,$$

and as $m(r)>0$ and $v(r)\geqq 0$ for $R_0\leqq r\leqq R_1$, it follows that $v(r)$ is a nondecreasing function of r in this interval. We thus obtain

$$\begin{aligned}4\pi\gamma(m)&=\iint_D\{|\operatorname{grad}v|^2+mv^2\}\,d\sigma\geqq\iint_D\{|\operatorname{grad}v|^2+p^+v^2\}\,d\sigma\\&\geqq\iint_D\{|\operatorname{grad}v|^2+pv^2\}\,d\sigma\geqq 4\pi\gamma\ .\end{aligned}$$

This proves Theorem 4. The last step of this proof shows that *the* *(italicized) statement following the proof of Theorem 3 holds also true with respect to Theorem 4.*

We started this paper with quoting the one-dimensional analogue of the results on the nonhomogeneous membrane. With regard to the generalized Dirichlet integral we state now the one-dimensional analogue of Theorem 3. It will be convenient to exchange the boundary conditions. We thus require the vanishing of the admissible functions at the outer endpoints of the two disjoint segments, so that p^+ (instead of p^- of Theorem 3) appears in our statement. Moreover, we let the inner endpoints of the two segments coincide and thus obtain

THEOREM 5. *Let $p(x)$ be positive and continuous for $-x_0 \leqq x \leqq x_0$, $0 < x_0 < \infty$, and let $p^+(x)$ be the rearrangement of $p(x)$ in symmetrically increasing order. Let $u(x)$ be the unique solution of the differential system*

$$u''(x) - p^+(x)u(x) = 0 \quad for \quad -x_0 \leqq x \leqq 0 , \quad u(-x_0) = 0 , \quad u(0) = 1 ,$$

and set $\alpha = 2u'(0)$. Let $\varphi(x)$ be any function of class D' in $-x_0 \leqq x \leqq x_0$[6] *which satisfies $\varphi(-x_0) = \varphi(x_0) = 0$ and denote the maximum of $|\varphi(x)|$ in this interval by ϕ. Then*

$$\int_{-x_0}^{x_0} (\varphi'^2 + p\varphi^2)dx \geqq \alpha\phi^2 .$$

Equality is obtained in the case $p(x) = p^+(x)$ and $\varphi(x) = Cu(x)$ for $-x_0 \leqq x \leqq 0$, $\varphi(x) = \varphi(-x)$ for $0 \leqq x \leqq x_0$.

For the proof let x_1 be a point in $\langle -x_0, x_0 \rangle$ such that $|\varphi(x_1)| = \phi$ and assume that $\varphi(x_1) = \phi$. Let us minimize the integral

$$\int_{-x_0}^{x_1} (y'^2 + py^2)dx$$

under the boundary conditions $y(-x_0) = 0$ and $y(x_1) = \phi$. The Euler equation $y'' - py = 0$ has (by $p > 0$) a unique solution satisfying the boundary conditions and it follows by standard criteria of the calculus of variations[7] that this unique extremal satisfying the boundary conditions gives the absolute (strong) minimum of the variational problem. Considering also the analogue problem for $\langle x_1, x_0 \rangle$ with the boundary conditions $y(x_1) = \phi$ and $y(x_0) = 0$ we finally obtain

$$\int_{-x_0}^{x_0} (\varphi'^2 + p\varphi^2)\, dx \geqq \int_{-x_0}^{x_0} (y'^2 + py^2)dx ,$$

[6] See [**2**, p. 7].

[7] See Bolza [**2**; pp. 101, 102] and use his conditions (I), (IIb′) and (III′).

where $y(x)$ is the unique solution of $y''-py=0$ for $-x_0\leqq x\leqq x_1$ and $x_1\leqq x\leqq x_0$ which satisfies $y(-x_0)=y(x_0)=0$, $y(x_1)=\phi$. (Note that $0\leqq y\leqq\phi$ follows). We have now

$$\int_{-x_0}^{x_0}(y'^2+py^2)dx\geqq\int_{-x_0}^{x_0}\{(y^-)'^2+p^+(y^-)^2\}\,dx=2\int_{-x_0}^{0}\{(y^-)'^2+p^+(y^-)^2\}\,dx$$

$$\geqq 2\phi^2\int_{-x_0}^{0}(u'^2+p^+u^2)dx=\alpha\phi^2\ .$$

Here we used again the calculus of variations to justify the last inequality sign and we obtained the last equality by partial integration of u'^2. This completes the proof of Theorem 5.

In case of $p(x)$ being a monotonic function we obtain

THEOREM 6. *Let $p(x)$ be positive, continuous and non-decreasing for $a\leqq x\leqq b$ $(-\infty<a<b<\infty)$. Let $y_1(x)$ and $y_2(x)$ be any (nontrivial) solutions of*

$$y''(x)-p(x)y(x)=0\ ,\qquad a\leqq x\leqq b\ ,$$

which satisfy $y_1(a)=y_2(b)=0$. Then

$$\frac{y_1'(b)}{y_1'(a)}\geqq\frac{y_2'(a)}{y_2'(b)}(>0)\ .$$

For the proof we may assume $y_1(b)=y_2(a)=1$. As the Wronskian of the two solutions $y_1(x)$ and $y_2(x)$ is constant, (and using $p>0$) we obtain

$$y_1'(a)=-y_2'(b)>0\ .$$

Setting $p^*(x)=p(a+b-x)$, $a\leqq x\leqq b$, we have

$$y_1'(b)=\int_a^b(y_1'^2+py_1^2)dx\geqq\int_a^b(y_1'^2+p^*y_1^2)\,dx$$

$$\geqq\int_a^b(y_2'^2+py_2^2)dx=-y_2'(a)(>0)\ .$$

Dividing $y_1'(b)\geqq-y_2'(a)$ by $y_1'(a)=-y_2'(b)$ we obtain the assertion of the theorem.

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