ON SEMI-NORMAL OPERATORS

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1. A bounded linear operator A in a Hilbert space will be called semi-normal if

$$(1) \qquad H=AA^*-A^*A\geqq 0 \quad (\text{or } \leqq 0)\,.$$

If A is a finite matrix, for instance, then relation (1) implies $H=0$, so that A is even normal; cf., e.g., [4]. That (1) may hold with $H\neq 0$ is seen if one chooses, for instance, A to to the isometric matrix defined by $A=D=(d_{ij})$ where $d_{i+1,i}=1$ and $d_{ij}=0$ otherwise. The purpose of this note is to investigate the spectrum of the semi-normal operator A and of the associated self-adjoint operators J_θ defined by

$$(2) \qquad J_\theta=\frac{A_\theta+A_\theta^*}{2}\,, \quad A_\theta=Ae^{-i\theta}\ (\theta \text{ real})\,.$$

It is seen that, in particular, J_θ becomes the real or the imaginary part of A according as $\theta=0$ or $\theta=\pi/2$.

A number λ belonging to the spectrum of A (sp (A)) will be called accessible if there exists a sequence of numbers λ_n not belonging to sp(A) for which $\lambda_n\to\lambda$ as $n\to\infty$. If M is any self-adjoint operator, $\max M$ and $\min M$ will denote the greatest and the least points respectively of the set sp(M).

The following theorems will be proved:

THEOREM 1. *Let A be semi-normal with $H\geqq 0$ and let $\lambda=re^{i\theta}$ (r real, $\geqq 0$) be an accessible point of the spectrum of A. Then*

$$(3) \qquad (\max J_\theta)^2\geqq \min AA^*$$

and

$$(4) \qquad |r-\max J_\theta|\leqq((\max J_\theta)^2-\min AA^*)^{1/2}\,,$$

where J_θ is defined by (2).

THEOREM 2. *Let A be semi-normal and let $J=J_\theta$ have the spectral resolution $J=\int\lambda dE$. Then, if $S=S_\theta$ is any measurable set for which*

$$(5) \qquad \int_S dE=I\,,$$

Received March 6, 1957. This research was supported in part by the United States Air Force under Contract No. AF 18 (603)-139.

there holds the inequality

$$\|H\| \leqq 4\|A\| \text{ meas } S . \tag{6}$$

The proof of Theorem 1 will be given in § 2 below. The assertion of Theorem 2 can be considered as a supplement to Corollary 3 of [5]. The proof follows readily from the Lemma, *loc. cit.*, p. 1027 if one notes that $H/2=J_\theta A_\theta^* - A_\theta^* J_\theta$ and that $\|A_\theta\|=\|A\|$.

Various corollaries can be obtained from the two theorems. For instance, as a consequence of Theorem 1, one has the

COROLLARY 1. *If V is isometric and not unitary, then its spectrum is the disk $|\lambda| \leqq 1$ of the complex plane.*

Actually it is possible to deduce this result from a normal form for such operators; cf., e.g., [8, p. 351 ff]. It should be noted that the spectrum of the isometric matrix D defined earlier in this paper, and which occurs in the normal form, is the disk $|\lambda| \leqq 1$; cf. [9, p. 279].

The proof of the corollary as a consequence of Theorem 1 however is as follows. Put $A^*=V$ so that $AA^*=I$; clearly V is semi-normal and $H \geqq 0$. Let $\lambda=re^{i\theta}$ $(r \geqq 0)$ be an accessible point in the spectrum of A (that is, of A^* or V). Then, by (3), $|\max J_\theta| \geqq 1$. On the other hand, $\|A\|=1$, and hence $|\max J_\theta| \leqq 1$. Thus $|\max J_\theta|=1$ and (4) implies $r=1$; consequently, the only possible accessible points of the spectrum of an isometric operator lie on the circle $|\lambda|=1$. However, if the operator is not unitary, then $\lambda=0$ lies in its spectrum. Hence, the entire disk $|\lambda| \leqq 1$ is in the spectrum and the proof is complete.

Another consequence of Theorem 1 is

COROLLARY 2. *If A is semi-normal, if 0 lies in the spectrum of A, and if* $\min AA^*>0$, *then for any θ the circular disk*

$$|\lambda| \leqq \max J_\theta - ((\max J_\theta)^2 - \min AA^*)^{1/2}$$

lies in the spectrum of A (where, of course, $\max J_\theta > 0$).

The proof follows from the observation that $\lambda=0$ is in $\operatorname{sp}(A)$ but no accessible points of the spectrum can lie in the disk in question.

It can be remarked that if A is an arbitrary bounded linear operator (not necessarily semi-normal), and if the conditions that 0 be in $\operatorname{sp}(A)$ and $\min AA^*>0$ are fulfilled, then there surely exists *some* circular disk $|\lambda| \leqq$const. in the spectrum of A; sec, e.g., [7, pp. 76–78]. If however A is semi-normal, the radius of the corollary can even be specified.

An immediate consequence of Theorem 2 is the

COROLLARY 3. *If A is semi-normal but not normal, then the spectrum* *of* J_θ *(in particular, of the real or imaginary part of* A*) has a positive measure not less than* $\|H\|/4\|A\|$.

It should be noted that (5) surely holds if S is the spectrum of J although it may hold for a set of measure less than that of the spectrum (but whose closure would, of course, contain the spectrum).

It seems natural to conjecture that the spectrum of (say) the real part, $J=(A+A^*)/2$ of any semi-normal, but not normal, operator A must be an interval. Evidence to support the conjecture is furnished by the isometric, but not unitary, operators V, in which case the spectrum of $(V+V^*)/2$ is the interval $-1\leqq\lambda\leqq 1$. This fact also follows from the normal form for isometric operators referred to above and from the fact that the spectrum of $(D+D^*)/2$ is the interval $-1\leqq\lambda\leqq 1$ (cf., e.g., [**3**, p. 155]). Further evidence is furnished by the (bounded) matrices $A=(c_{j-i})$, where $c_n=0$ if $n<0$, for which the spectra of the associated Toeplitz materices $J=(A+A^*)/2$ are intervals, provided J is not a multiple of the unit matrix (in which case A is also); see [**1**, p. 361] and [**2**, p. 868]. It was shown in [**6**] that the matrices A are semi-normal.

The conjecture will remain unsettled. In fact, it will remain undecided whether or not the spectrum of the real part J of a semi-normal, but not normal, operator must even contain some interval. The assertion of Corollary 3 does not seem to preclude the possibility of, for instance, a nowhere dense spectrum (of positive measure).

2. Proof of Theorem 1. Let $\lambda_n=r_ne^{i\theta_n}$ be chosen so that λ_n is not in $\mathrm{sp}\,(A)$ and $\lambda_n\to\lambda$ as $n\to\infty$. Put $A_n=A-\lambda_nI$. Then $A_nA_n^*=A_nA_n^*A_nA_n^{-1}$, so that the spectra of $A_nA_n^*$ and $A_n^*A_n$, hence the spectra of $AA^*-2r_nJ_{\theta_n}$, and $A^*A-2r_nJ_{\theta_n}$, are (respectively) identical. Since $\lambda=re^{i\theta}$ is in the spectrum of A, then either $(A-\lambda)x_m\to 0$ or $(A-\lambda)^*x_m\to 0$ for some sequence of unit vectors x_m. In either case, it follows from (1) that $\limsup\,(x_m, A^*Ax_m)\leqq r^2$ as $m\to\infty$ and that $(x_m, J_{\theta_n}x_m)\to r$ as $m, n\to\infty$. Consequently, $\min\,(AA^*-2rJ_\theta)\leqq -r^2$ and hence $\min AA^*-2r\max J_\theta+r^2\leqq 0$. The desired relations (3) and (4) follow and the proof of Theorem 1 is complete.

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