ON THE INEQUALITY $\Delta u \geqq f(u)$

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We are interested in solutions of the non-linear differential inequality

$$\Delta u \geqq f(u) \tag{1}$$

where $u(x_1, \cdots, x_n)$ is to be defined in some region of Euclidean n-space and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u. Wittich [5] considered the corresponding equation

$$\Delta u = f(u) \tag{1a}$$

in two dimensions and found conditions on $f(u)$ which guarantee that (1a) has no solution valid in the whole plane. Haviland [1] found a slightly weaker result in 3 dimensions, and Walter [4] generalized Wittich's theorem to n-dimensions. The method is essentially the same in all three papers, resulting on the one hand in the requirement that the function $f(u)$ be convex, and on the other hand in a rather involved argument for the n-dimensional case. The proofs do extend immediately to the inequality (1).

In the present paper we deal directly with (1), and obtain in particular a simple proof of a stronger theorem (Theorem 1 below) where the convexity of $f(u)$ is no longer required. Our method also yields much more precise information on the behavior of solutions.

Recently Redheffer [3] has obtained in the two-dimensional case an improvement of our Theorem 1, where the monotonicity of $f(u)$ is not needed. Although Redheffers's theorem may very likely be extendable to n dimensions, it does not seem possible by his method to obtain the more precise results mentioned in the remarks following Theorem 1.

The present investigation resulted from an attempt to determine the type of a class of Riemann surfaces. One result, Theorem 2, is given here as an application of Theorem 1.

We should like to mention finally that the method presented here has been developed independently by Keller, who, in a paper to be published, derives further information on the behavior of solutions of (1a), and applies his results to an interesting physical problem described in [2].

Notation. Throughout this paper we shall reserve r for the polar

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distance, $r=\sqrt{x_1^2+\cdots+x_n^2}$, in space of some fixed dimension $n\geqq 1$. We note that if $\varphi(r)$ is considered as a function in this space depending only on r, then

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}=\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial\varphi}{\partial r}\right). \tag{2}$$

LEMMA 1. *Let $f(t)$ be a (weakly) monotone increasing continuous function defined for all t. Suppose that there exists a function $\varphi(x)$ satisfying*

$$\varphi''(x)+\frac{n-1}{x}\varphi'(x)=f(\varphi) \tag{3}$$

for $0\leqq x<R$, with $\varphi'(0)=0$ and $\varphi(x)\to+\infty$ as $x\to R$. Then if u is any solution of (1) for $r\leqq R$, we have $u(x_1,\cdots,x_n)\leqq\varphi(r)$ at each point.

Proof. By (2) the function $\varphi(r)$ satisfies $\Delta\varphi=f(\varphi)$ for $r<R$. We let $v=u-\varphi$ and wish to show that $v\leqq 0$ for $r<R$. But suppose $v>0$ at some point. Since $v\to-\infty$ as $r\to R$ it would follow that v would take on its maximum at some point P with $r<R$. Then $v>0$ in some neighborhood N of P, that is $u>\varphi$ throughout N. This implies $\Delta v=\Delta u-\Delta\varphi\geqq f(u)-f(\varphi)\geqq 0$, so that Δv would be subharmonic in N, contradicting that v had a maximum at P.

LEMMA 2. *If $f(t)>0$, $f'(t)$ continuous, and $f'(t)\geqq 0$ for all t, then equation (1) has a solution u valid for all $(x_1,\cdots,x_n)$ if and only if there is a solution of (3) valid for all $x\geqq 0$, with $\varphi'(0)=0$.*

Proof. If such a function φ exists, then $\varphi(r)$ is the desired solution of (1).

Conversely, suppose that no such function $\varphi(x)$ exists. Given an arbitrary real number a, there exists[1] in any case a solution of (3) with initial values $\varphi(0)=a$, $\varphi'(0)=0$, valid in some interval $0\leqq x\leqq x_0$. Then there is a maximal interval $0\leqq x<R$ in which this solution exists. Further, we have by (2) that $\frac{d}{dx}(x^{n-1}\varphi')=x^{n-1}f(\varphi)>0$ for $x>0$, so that $x^{n-1}\varphi'$ is increasing, hence positive for $x>0$ since $\varphi'(0)=0$. Under these conditions we must have $\varphi(x)\to+\infty$ as $x\to R$. Then by Lemma 1 any solution u of (1) would satisfy $u\leqq\varphi$ for $r<R$. In particular we would

[1] The existence does not follow immediately from classical theorems, but may be proved by writing equation (3) in the integral form $\varphi(x)=a+\int_0^x\frac{1}{s^{n-1}}\int_0^s t^{n-1}f(\varphi)\,dt\,ds$ and applying standard iteration procedure.

have $u(0) \leqq \varphi(0) = a$. But since a was arbitrary there could be no solution u valid in $r < R$ for arbitrarily large R.

LEMMA 3. *If $f(t) > 0$, $f'(t)$ continuous, and $f'(t) \geqq 0$ for all t, then equation* (3) *has a solution φ with $\varphi'(0) = 0$ valid for all $x \geqq 0$ if and only if*

$$\int^{\infty} \left(\int_0^t f(s)\, ds \right)^{-1/2} dt = \infty \ . \tag{4}$$

Proof. Suppose first that there does not exist a solution of (3) valid for all $x \geqq 0$. Then we have seen that if $\varphi(x)$ satisfies (3) in some interval, with $\varphi(0) = 0$ and $\varphi'(0) = 0$, then for some $R > 0$ we will have $\varphi(x) \to +\infty$ as $x \to R$. Further we noted that for $x > 0$, $\varphi'(x) > 0$, and hence from equation (3), $\varphi'' < f(\varphi)$. Thus $\varphi'\varphi'' < f(\varphi)\varphi'$ and integrating from $x = 0$ to $x = t$ gives

$$[\varphi'(t)]^2 < 2\int_0^t f(\varphi)\varphi'\, dx = 2\int_0^{\varphi(t)} f(\varphi)\, d\varphi \ .$$

Hence

$$\left(\int_0^{\varphi} f(s)\, ds \right)^{-1/2} d\varphi < \sqrt{2}\ dt$$

and integration from $t = 0$ to $t = R$ gives

$$\int_0^{\infty} \left(\int_0^{\varphi} f(s)\, ds \right)^{-1/2} d\varphi < \sqrt{2} R \ .$$

Suppose conversely that

$$\int_0^{\infty} \left(\int_0^t f(s)\, ds \right)^{-1/2} dt < \infty \ .$$

Then $t \cdot \left(\int_0^t f(s)\, ds \right)^{-1/2} \to 0$ as $t \to \infty$ since $\left(\int_0^t f(s)\, ds \right)^{-1/2}$ is monotone decreasing. Hence $t^{-2} \cdot \int_0^t f(s)\, ds \to \infty$ and $f(t)/t \to \infty$ since $f(t)$ is monotone increasing. Thus for an arbitrary fixed a, $f(t) > t - a$, for $t > t_0$. Further, if φ is the solution of (3) with $\varphi(0) = a$, $\varphi'(0) = 0$, then $\varphi(x) \geqq a$ for $x \geqq 0$, and $f(\varphi) \geqq f(a)$. Hence $(x^{n-1}\varphi')' \geqq f(a) \cdot x^{n-1}$, and integrating twice we find $x^{n-1}\varphi' \geqq \frac{f(a)}{n} x^n$, $\varphi \geqq \frac{f(a)}{2n} x^2$. Thus $\varphi(x) > t_0$ for $x > x_0$. As above we note that

$$\varphi'^2 < 2\int_a^{\varphi} f(\varphi)\, d\varphi \leqq 2(\varphi - a) f(\varphi) < 2[f(\varphi)]^2 \qquad \text{for } \varphi > t_0 \ .$$

Hence $\frac{n-1}{x}\varphi' < \frac{f(\varphi)}{2}$ for $x > x_1$, and consequently $\varphi'' > \frac{1}{2} f(\varphi)$ for $x > x_1$.

Thus

$$[\varphi'(x)]^2-[\varphi'(x_1)]^2>\int_{\varphi(x_1)}^{\varphi(x)} f(s)\,ds$$

or

$$[\varphi'(x)]^2>\int_0^{\varphi(x)} f(s)\,ds-C$$

whence

$$\int_0^{\varphi(x)}\left(\int_0^t f(s)\,ds-C\right)^{-1/2} dt>x-x_1\ .$$

Since the constant C does not affect the convergence of the integral we have that x must be bounded, which completes the proof of the lemma.

We may note that the proof of Lemma 3 is essentially that of Haviland [1]. The assumption made by Haviland that $f(t)\geqq c>0$ is seen to be unnecessary, but it is interesting to note that the theorem is no longer true in $n\geqq 3$ dimensions if we weaken the requirement to $f(t)\geqq 0$. (If we allow $f(t)=0$ we must speak of *non-constant* solutions of (3) for all x.) The reason for this is that a non-constant subharmonic function in one or two dimensions cannot be bounded above, while in three or more dimensions it can. Thus if we set $f(t)=0$ for $t\leqq 0$ and $f(t)=t^2$ for $t>0$, we see that any negative subharmonic function φ (such as $\varphi(r)=-1/(1+r^2)$ in 4 dimensions) satisfies $\Delta\varphi\geqq f(\varphi)$ throughout space, although the integral in (4) converges.

Combining these three lemmas we obtain the desired result:

THEOREM 1. *Let $f(t)$ be positive, continuous, and monotone increasing for $t\geqq t_0$, and suppose*

$$\int^\infty\left(\int_0^t f(s)\,ds\right)^{-1/2} dt<\infty\ .$$

Then a twice continuously differentiable function u cannot satisfy $\Delta u>0$ throughout space and $\Delta u\geqq f(u)$ outside of some sphere S.

Proof. Suppose such a function u exists. Then it has a maximum t_1 on S, and Δu has a minimum $m>0$ on S. Define $g(t)$ to be continuously differentiable for all t, and such that

a) $g'(t)\geqq 0$ for all t

b) $g(t)\leqq m$ for $t\leqq t_1$

c) $g(t)\leqq f(t)$ for all t

d) $g(t)\geqq f(t)-1$ for $t\geqq t_2$.

Then $\Delta u\geqq g(u)$ throughout space, so that by Lemma 2 there exists a solution of (3) with f replaced by g, and by Lemma 3 we would have

$$\int^{\infty}\left(\int_0^t g(s)\,ds\right)^{-1/2}dt=\infty$$

which, in view of d), contradicts the hypothesis.

Remarks 1. That the integral condition on $f(t)$ is the best possible can be seen most easily, as was pointed out by Walter [4], by noting that for an arbitrary continuous positive function $f(t)$ we can define $u(x_1)$ for $x_1\geqq 0$ as the inverse of

$$x_1(u)=\frac{1}{\sqrt{2}}\int_0^u\left(\int_0^t f(s)\,ds\right)^{-1/2}dt$$

and for $x_1<0$ by $u(x_1)=u(-x_1)$. Then $\Delta u=\dfrac{\partial^2 u}{\partial x_1^2}=f(u)$ in any number of dimensions, and if the integral diverges this will hold for all x_1, and hence throughout space.

2. We may note that in the proof of Lemma 3 we have obtained somewhat more than the non-existence of a solution for all x. Namely, we have an upper bound on the values of x for which (3) can hold. However, the expression obtained is not a very convenient one, and in any case does not give the best possible bound. The advantage of Lemma 1 is that it allows us to give the best bound whenever we can find the function φ explicitly. For example, if we have the inequality $\Delta u\geqq\varepsilon e^{2u}$, $\varepsilon>0$, in two dimensions, then we can easily verify that

$$\varphi=\log\frac{2R}{\sqrt{\varepsilon}\,(R^2-r^2)}$$

satisfies the hypotheses of Lemma 1, so that $u(0)\leqq\varphi(0)=\log\dfrac{2}{R\sqrt{\varepsilon}}$.

We may therefore state the following result:

If u satisfies $\Delta u\geqq\varepsilon e^{2u}$ for $r\leqq R$ and $u(0)=a$,

$$\textit{then}\quad R\leqq\frac{2}{e^a\sqrt{\varepsilon}}\,.$$

3. We note that in the proof of Lemma 1 we need only assume that φ satisfies the inequality $\varphi''+\dfrac{n-1}{x}\varphi'\leqq f(\varphi)$. In many cases it may be possible to find an explicit solution of this inequality, but not of equation (3). For example, if $f(\varphi)=\varepsilon|\varphi|^\alpha$, $\alpha>1$, then the function

$$\varphi=\frac{cR^{2m}}{(R^2-r^2)^m}\,,\qquad c>0$$

satisfies in n dimensions

$$\begin{aligned}\Delta\varphi &= 2mR^{-4}c^{-2/m}(nR^2+(2m+2-n)r^2)\,\varphi^{1+2/m} \\ &\leqq 4m(m+1)R^{-2}c^{-2/m}\varphi^{1+2/m} \qquad \text{if } 2m+2\geqq n,\ r<R\,.\end{aligned}$$

Hence $\Delta\varphi\leqq\varepsilon u^{1+2/m}$ if $R\geqq 2(m+1)\varepsilon^{-1/2}c^{-1/m}$. We can therefore state the following:

If u satisfies $\Delta u\geqq\varepsilon|u|^{\alpha}$ for $r\leqq R$ in n-dimensions, where $\varepsilon>0$ and $\alpha>1$, and if $u(0)=a>0$, then $R\leqq 2(m+1)\varepsilon^{-1/2}a^{-1/m}$, where

$$m=\max\left\{\frac{n}{2}-1,\ \frac{2}{\alpha-1}\right\}.$$

4. The above remarks may also be viewed from the other direction. That is, if a function u is known to satisfy (1) for $r\leqq R$, then we get a pointwise upper bound on u in terms of the solution of (3). Furthermore, if we know that $u<M$ for $r=R$, then we can improve these bounds. Namely, we have $u\leqq\varphi$, where φ is the solution of (3) with $\varphi'(0)=0$ and $\varphi(R)=M$. Finally, these bounds are again the best possible since $\varphi(r)$ itself satisfies (1).

We turn next to an application of Theorem 1.

THEOREM 2. *If a simply-connected surface S has a Riemannian metric whose Gauss curvature K satisfies $K\leqq-\varepsilon<0$ everywhere, then S is conformally equivalent to the interior of the unit circle.*

Proof. Considering S as a Riemann surface, we know that it can be mapped conformally onto either the interior of the unit circle or else the whole plane. We proceed by contradiction. Suppose we could map S conformally onto the x, y-plane. The Riemannian metric on S could then be expressed as $ds^2=\lambda^2(dx^2+dy^2)$, and we have for the Gauss curvature:

$$K=-\frac{\Delta\log\lambda}{\lambda^2}\,.$$

$K\leqq-\varepsilon$ means that the function $u=\log\lambda$ would have to satisfy $\Delta u\geqq\varepsilon e^{2u}$ throughout the plane, contradicting Theorem 1.

We remark finally that the condition $K\leqq-\varepsilon$ can be weakened slightly to $K<0$, and

$$\frac{\displaystyle\iint_R K\,d\omega}{\displaystyle\iint_R d\omega}\leqq-\varepsilon<0$$

for every region R on the surface including some fixed compact set D, where $d\omega$ is the area element on the surface. The proof of this again involves assuming that the surface may be mapped conformally onto the whole x, y-plane and defining $z(r)$ as the mean value of K over the disk $x^2+y^2 \leqq r^2$. Simple inequalities yield

$$z''+\frac{3z'}{r} \geqq \varepsilon e^{2z} \qquad \text{for } r \text{ sufficiently large,}$$

and

$$z''+\frac{3z'}{r} > 0 \qquad \text{everywhere.}$$

Since $z''+\dfrac{3z'}{r}$ is just the Laplacian of z in four dimensions, we again have a contradiction to Theorem 1.

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