SOME REMARKS ON A PAPER OF ARONSZAJN AND PANITCHPAKDI

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In the paper of the title [1], a number of problems are posed. Negative solutions of two of them (Problems 2 and 3) are derived in a straightforward way from a paper of L. Gillman and the present author [2].

Motivation will not be supplied since it is given amply in [1], but enough definitions are given to keep the presentation reasonably self-contained.

1. A Hausdorff space X is said to satisfy $(Q_{\mathfrak{m}})$, where $\mathfrak{m}$ is an in finite cardinal, if, whenever U and V are disjoint open subsets of X such that each is a union of the closures of *less* than $\mathfrak{m}$ open subsets of X, then U and V have disjoint closures. In particular, a normal (Hausdorff) space X satisfies $(Q_{\aleph_1})$ if and only if disjoint open F_σ-subsets of X have disjoint closures. (For, an open set that is the union of less than $\aleph_1$ closed sets is a fortiori an F_σ. Conversely if U is the union of countably many closed subsets F_n, then since X is normal, for each n there is an open set U_n containing F_n whose closure is contained in U. Thus U is the union of the closures of the open sets U_n.) In Problem 3 of [1], it is asked if every compact (Hausdorff) space satisfying $(Q_{\mathfrak{m}})$ for some $\mathfrak{m}>\aleph_0$ is necessarily totally disconnected, and it is remarked that this is the case if the first axiom of countability is also assumed.

If X is a completely regular space, let $C(X)$ denote the ring of all continuous real-valued functions on X, and let $Z(f)=\{x\in X: f(x)=0\}$, let $P(f)=\{x\in X: f(x)>0\}$, and let $N(f)=P(-f)$. As usual, let βX denote the Stone-Čech compactification of X. If every finitely generated ideal of $C(X)$ is a principal ideal, then X is called an *F-space*. The following are equivalent.

(i) X is an F-space.

(ii) If $f\in C(X)$, then $P(f)$ and $N(f)$ are completely separated [2, Theorem 2.3].

(iii) If $f\in C(X)$, then every bounded $g\in C(X-Z(f))$ has an extension $\bar{g}\in C(X)$ [2, Theorem 2.6].

A good supply of compact F-spaces is provided by the fact that if X is locally compact and σ-compact, then $\beta X-X$ is an F-space [2, Theorem 2.7].

Received April 1, 1957. In revised form April 29, 1957. This paper was written while the author was an Alfred P. Sloan fellow.

We remark first that *a normal (Hausdorff) space X satisfies* $(Q_{\aleph_1})$ *if and only if it is an F-space.*

For, suppose first that X is an F-space, and let U, V be disjoint open F_σ-subsets of X. Since $X-(U\cup V)$ is a closed G_δ in a normal space, there is a bounded $f\in C(X)$ such that $Z(f)=X-(U\cup V)$. Hence by (iii), there is a $\bar{g}\in C(X)$ such that $\bar{g}[U]=0$ and $\bar{g}[V]=1$. In particular, U and V have disjoint closures, so X satisfies $(Q_{\aleph_1})$. Conversely let X satisfy $(Q_{\aleph_1})$, and let $f\in C(X)$. Then $P(f)$ and $N(f)$ are disjoint open F_σ-subsets of X, which by $(Q_{\aleph_1})$ have disjoint closures. So, by Urysohn's lemma, $P(f)$ and $N(f)$ are completely separated. Thus X is an F-space by (ii).

Compact connected F-spaces exist. In particular it is known that if R^+ denotes the space of nonnegative real numbers, then βR^+-R^+ is such a space [2, Example 2.8]. Hence Problem 3 of [1] has a negative solution.

We remark finally that if the first axiom of countability holds at a point of an F-space, then the point is isolated [2, Corollary 2.4]. In particular, every compact F-space satisfying the first axiom of countability is finite.

2. In Problem 2 of [1], it is asked (in different but equivalent language) if for every totally disconnected compact space X satisfying $(Q_{\mathfrak{m}})$ for some $\mathfrak{m}>\aleph_0$, the Boolean algebra $B(X)$ of open and closed subsets of X has the property that every subset of *less* than $\mathfrak{m}$ elements has a least upper bound. A lattice is said to be (conditionally) *σ-complete* if every bounded countable subset has a least upper bound and a greatest lower bound. In view of the above (and since every subset of $B(X)$ is bounded), in case $\mathfrak{m}=\aleph_1$, the problem asks if for every compact totally disconnected F-space X, the Boolean algebra $B(X)$ is σ-complete.

In [3, Theorem 18], it is shown that if X is compact and totally disconnected, then $B(X)$ is σ-complete if and only if $C(X)$ is σ-complete (as a lattice). It is noted in [2, Theorem 8.3, f.f.] that for a completely regular space Y, the lattice $C(Y)$ is σ-complete if and only if $f\in C(Y)$ implies $\overline{P}(f)$ and $\overline{N}(f)$ are disjoint open and closed subsets of Y ($\overline{P}(f)$ denotes the closure of $P(f)$). It is easily seen that Y has this latter property if and only if βY has [2, Lemma 1.6].

In [2, Example 8.10], an example is given of a completely regular space X such that βX is a totally disconnected F-space, and such that $C(X)$ is not σ-complete. By the above, it follows that $B(\beta X)$ yields a negative solution to Problem 2.

We remark also (as was pointed out by J. R. Isbell) that if N denotes the countable discrete space, then $\beta N-N$ is also a totally disconnected compact F-space such that $B(\beta N-N)$ is not σ-complete. The former assertion follows easily from the remarks in § 1, and the latter follows from the fact that $B(\beta N-N)$ is isomorphic to the Boolean algebra of all subsets of N modulo the ideal of finite subsets of N (under the correspondence induced by sending a subset of N to the intersection of its closure in βN with $\beta N-N$). It is easily verified that this latter Boolean algebra is not σ-complete.

REFERENCES

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