(γ, k)-SUMMABILITY OF SERIES

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1. Introduction. Let $\gamma_k(x)$ denote the (C, k) mean of $\cos x$, so that

$$\gamma_0(x)=\cos x \ , \tag{1.1}$$

and

$$\begin{aligned}\gamma_k(x)&=\frac{k}{x^k}\int_0^x (x-u)^{k-1}\cos u\,du\ , \quad (k>0)\,,\\ &=k\int_0^1 (1-t)^{k-1}\cos xt\,dt\ ,\\ &=\Gamma(k+1)\frac{C_k(x)}{x^k}\ ,\end{aligned} \tag{1.2}$$

where $C_k(x)$, the kth fractional integral of $\cos x$, is commonly known as Young's function [6, p. 564].

We shall say that the infinite series $\sum_0^\infty a_n$ is summable (γ, k) if

(i) $\sum_0^\infty a_n\gamma_k(nt)$ converges for $0<t<A$

and

(ii) $\lim_{t\to 0}\sum_0^\infty a_n\gamma_k(nt)=S$, where S is finite.

We see that $(\gamma, 1)\equiv(R, 1)$ and $(\gamma, 2)\equiv(R, 2)$, where $(R, 1)$ and $(R, 2)$ are the well known Riemann summability methods. Hence the (γ, k)-summability methods constitute, in a sense, an extension of $(R, 1)$ and $(R, 2)$ summability methods to (R, k) methods where k may be non-integral. But this extension is not linked with the ideas which lie at the root of the Riemann summability methods, that is, taking generalised symmetric derivatives of repeatedly integrated Fourier series, so that the equivalence of (γ, k) and (R, k) for $k=1, 2$ may be considered to be somewhat accidental, and the extension artificial. However, the (γ, k) methods are also connected with certain aspects of the summability problems of Fourier series. For, let $\sum_0^\infty A_n(x)$ be the Fourier series of a periodic and Lebesgue integrable function $f(x)$ and let

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}\ .$$

Received June 3, 1957.

Then, by some well known theorems (see, for example, [1].) the problem of Cesàro summability of $\sum A_n(x)$ is connected with Cesàro continuity of $\phi(t)$ at $t=0$, $\phi(t)$ being said to be (C, k) continuous at $t=0$ if $k\int_0^1 (1-y)^{k-1}\phi(ty)\,dy$ exists for $0<t<A$ and tends to a finite limit as t tends to zero. On the other hand, under certain conditions (e.g., if $k\geqq 1$) we have

$$k\int_0^1 (1-y)^{k-1}\phi(ty)\ dy=\sum_0^\infty A_n(x)\gamma_k(nt)\ .$$

Thus the nature of the connexion between (γ, k) and Cesàro summability methods, when the series in question is a Fourier series, is immediately apparent.

Some known theorems which may be interpreted as results on (γ, k) methods are stated below. δ denotes any arbitrary positive number.

(A_1) *If a series is summable* $(\gamma, 1)$ *then it is summable* $(C, 1+\delta)$.

See Zygmund [11].

(A_2) *If a series is summable* $(\gamma, 2)$ *then it is summable* $(C, 2+\delta)$.

See Kuttner [7].

(A_3) *If a Fourier-Lebesgue series is summable* (γ, k), $k\geqq 1$, *then it is summable*, $(C, k+\delta)$.

See Bosanquet [1] and Paley [8].

Neither Bosanquet nor Paley actually states any such result, but if $\alpha\geqq 1$, then Bosanquet's Theorem 1 as well as Paley's Theorem 1 can be restated in the present form.

(B_1) *If a series is summable* $(C, -\delta)$ *then it is summable* $(\gamma, 1)$.

See Hardy and Littlewood [4].

(B_2) *If a series is summable* $(C, 1-\delta)$ *then it is summable* $(\gamma, 2)$.

See Bosanquet [1] and Verblunsky [10].

(B_3) *If* $\sum_0^\infty a_n/n^2$ *is convergent and* $\sum_0^\infty a_n$ *is summable* (C, k), $k\geqq -1$, *then* $\sum_0^\infty a_n$ *is summable* $(\gamma, k+1+\delta)$.

See Bosanquet [1].

In view of the above results the question naturally arises whether we can remove the restrictions (a) that in (A_3) the series in question is a Fourier series and $k \geqq 1$ and (b) that in (B_3) $\sum a_n/n^2$ is convergent, and obtain the general results

(A'_3) *(γ, k) implies $(C, k+\delta)$, $k \geqq 0$,*

and

(B'_3) *(C, k) implies $(\gamma, k+1+\delta)$, $k \geqq -1$.*

But so far as (B_3) is concerned, we may state here that the convergence of $\sum a_n/n^2$ is essential for the truth of the conclusion, because we shall prove a result (Lemma 5) which implies that if $\sum a_n$ is summable (γ, k), $k \geqq 2$, then $\sum a_n/n^2$ is convergent.

In this paper we shall obtain the following results of the type of (A'_3):

(i) *If k is zero or a positive integer, then (γ, k) implies $(C, k+\delta)$.*

(ii) *If $[k] \geqq 4$, then (γ, k) implies $(C, k+\delta)$.*

The question of the truth of (A'_3) for fractional values of k less than 4 is still open.

2. Lemmas.

LEMMA 1. *If $k>0$, then for large positive values of x,*

$$\gamma_k(x) = \frac{A}{x^2} + O\left(\frac{1}{x^3}\right) + \frac{B \cos\left(x - \frac{k\pi}{2}\right)}{x^k} + O\left(\frac{1}{x^{k+1}}\right),$$

where A and B are non-zero constants; the asymptotic formulae for the derivatives of $\gamma_k(x)$ are obtained by formal differentiation of this formula.

This result is familiar. See, for example, Bosanquet [1].

LEMMA 2. *Let $f(x)$ be periodic with period 2π and Lebesque integrable, and let $\sum (a_n \cos nx + b_n \sin nx)$ be its Fourier series. Set*

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},$$

$$g(t)=\begin{cases}\phi(t)/t^r & \text{if } r \text{ is an even integer}\\ \psi(t)/t^r & \text{if } r \text{ is an odd integer.}\end{cases}$$

If $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0, \pi)$ and its Fourier series is summable (C, k) at $t=0$, then $\sum\left(\frac{d}{dx}\right)^r(a_n \cos nx+b_n \sin nx)$ is summable $(C, k+r)$.

For this result see Bosanquet [2, Theorem 2].

LEMMA 3. *Let $f(t)$ be an even periodic function with period 2π. If $f(t)\varepsilon C_\lambda L$ where $0<\lambda<1$ and $f(t)=o(1)(C, \lambda+1)$ as $t\rightarrow 0$, then the $(C_\lambda L)$-Fourier series of $f(t)$ at $t=0$ is summable (C, k) for every $k>\lambda+1$.*

For this result see Sargent [9, Theorem 4].

LEMMA 4. *If $\sum_0^\infty a_n\gamma_k(nt)$ is convergent for $\alpha<t<\beta$, then*

$$\text{(i)}\quad a_n=o(n^k) \quad \text{if} \quad 0<k\leqq 2 ,$$

$$\text{(ii)}\quad a_n=o(n^2) \quad \text{if} \quad k\geqq 2 .$$

Proof Case 1. $k=0$. The hypothesis implies that $\lim_{n\rightarrow\infty} a_n \cos nt=0$ for $\alpha<t<\beta$, which, by the Cantor-Lebesgue theorem, implies that $a_n=o(1)$.

Case 2. $k>0$. The hypothesis implies that $\lim_{n\rightarrow\infty} a_n\gamma_k(nt)=0$ for $\alpha<t<\beta$, which, on account of Lemma 1, implies that

$$\lim_{n\rightarrow\infty} a_n\left\{\frac{A}{n^2t^2}+O\left(\frac{1}{n^3t^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{n^kt^k}+O\left(\frac{1}{n^{k+1}t^{k+1}}\right)\right\}=0 \quad \text{for } \alpha<t<\beta .$$

If $0<k\leqq 2$, we write this as

$$\lim_{n\rightarrow\infty}\frac{a_n}{n^k}\left\{\frac{A}{n^{2-k}t^2}+O\left(\frac{1}{n^{3-k}t^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{t^k}+O\left(\frac{1}{nt^{k+1}}\right)\right\}=0 ,$$

and if $k>2$, we write

$$\lim_{n\rightarrow\infty}\frac{a_n}{n^2}\left\{\frac{A}{t^2}+O\left(\frac{1}{nt^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{n^{k-2}t^k}+O\left(\frac{1}{n^{k-1}t^{k+1}}\right)\right\}=0 .$$

The result, for $0<k\leqq 2$, is now obtained by a slight modification of the

usual proof of the Cantor-Lebesgue theorem, whereas, for $k>2$ we get the result by noticing that the expression within brackets tends to a non-zero limit as n tends to infinity.

LEMMA 5. *If* $k \geqq 2$ *and* $\sum_0^\infty a_n \gamma_k(nt)$ *is convergent for* $\alpha<t<\beta$, *then* $\sum_1^\infty a_n/n^2$ is convergent.

Proof. Since Kuttner [7] has proved the result for $k=2$, we assume that $k>2$. We also assume without any loss of generality that $\alpha>0$. Now suppose that (α_0, β_0) is a subinterval of (α, β). Since

$$C_k(nt)=\frac{(nt)^{k-2}}{\Gamma(k-1)}-C_{k-2}(nt) , \tag{2.1}$$

therefore

$$\begin{aligned}
&\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\, dt \\
&\quad =\int_{\alpha_0}^{\beta_0} \frac{n^{k-2}t^{k-2}}{\Gamma(k-1)}(\beta_0-t)(t-\alpha_0)\, dt-\int_{\alpha_0}^{\beta_0} C_{k-2}(nt)(\beta_0-t)(t-\alpha_0)\, dt \\
&\quad =n^{k-2}\phi(\alpha_0, \beta_0)-\int_{\alpha_0}^{\beta_0} C_{k-2}(nt)(\beta_0-t)(t-\alpha_0)\, dt ,
\end{aligned}$$

where

$$\phi(\alpha_0, \beta_0)=\frac{1}{\Gamma(k-1)}\int_{\alpha_0}^{\beta_0} t^{k-2}(\beta_0-t)(t-\alpha_0)\, dt$$

is positive.

Hence, integrating by parts twice, we have

$$\begin{aligned}
&\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\, dt \\
&\quad =n^{k-2}\phi(\alpha_0, \beta_0)-\frac{1}{n^2}[(\beta_0-\alpha_0)\{C_k(n\beta_0)+C_k(n\alpha_0)\}]+\frac{2}{n^2}\int_{\alpha_0}^{\beta_0} C_k(nt)\, dt .
\end{aligned} \tag{2.2}$$

Now, since $k>2$,

$$C_k(nt)=\frac{n^k t^k}{\Gamma(k+1)}\gamma_k(nt)=O(n^{k-2}) ,$$

for any fixed t. Hence, from (2.2) we get

$$\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\, dt=n^{k-2}\phi(\alpha_0, \beta_0)+O(n^{k-4}) . \tag{2.3}$$

Therefore, if p and q are two positive integers and $q>p$,

$$(2.4)\quad \int_{\alpha_0}^{\beta_0}\left\{\sum_{n=p}^{q}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\right\}dt=\sum_{n=p}^{q}\int_{\alpha_0}^{\beta_0}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\,dt$$

$$=\phi(\alpha_0,\beta_0)\sum_{n=p}^{q}\frac{a_n}{n^2}+\sum_{n=p}^{q}O(n^{-4})a_n \quad \text{by (2.3)}$$

$$=\phi(\alpha_0,\beta_0)\sum_{n=p}^{q}\frac{a_n}{n^2}+\sum_{n=p}^{q}o\left(\frac{1}{n^2}\right) \quad \text{by Lemma 4.}$$

If possible, let the lemma be false. Then we can find a positive number ε such that, for an infinity of pairs of integers (p_i, q_i), $q_i>p_i$, we have

$$(2.5)\qquad \left|\sum_{n=p_i}^{q_i}\frac{a_n}{n^2}\right|>2\varepsilon\ .$$

Again, if p_0 is sufficiently large, then

$$(2.6)\qquad \sum_{n=p_0}^{q_0}o\left(\frac{1}{n^2}\right)<\varepsilon\phi(\alpha_0,\beta_0)\ .$$

From (2.4), (2.5), (2.6) it follows that

$$(2.7)\qquad \left|\int_{\alpha_0}^{\beta_0}\left\{\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\right\}dt\right|>\varepsilon\phi(\alpha_0,\beta_0)\ .$$

But the quantity on the left hand side of (2.7)

$$(2.8)\qquad \leqq\left\{\underset{\alpha_0\leqq t\leqq\beta_0}{\text{l.u.b.}}\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_h(nt)\right|\right\}\int_{\alpha_0}^{\beta_0}(\beta_0-t)(t-\alpha_0)\,dt\ .$$

From (2.7) and (2.8).

$$\begin{aligned}\underset{\alpha_0\leqq t\leqq\beta_0}{\text{l.u.b.}}\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)\right| &>\varepsilon\phi(\alpha_0,\beta_0)\Big/\int_{\alpha_0}^{\beta_0}(\beta_0-t)(t-\alpha_0)\,dt\\ &>\varepsilon\alpha_0^{k-2}/\Gamma(k-1)\\ &\geqq\varepsilon\alpha^{k-2}/\Gamma(k-1)\\ &>M\varepsilon\ ,\end{aligned}$$

where M is a positive constant independent of the subinterval (α_0, β_0) Hence $\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)\right|<M\varepsilon$ at some point in (α_0, β_0) and therefore throughout a subinterval (α_1, β_1) of (α_0, β_0), since $\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)$ is a continuous function of t.

If, in the above argument, we now replace (α_0, β_0) by (α_1, β_1), (p_0, q_0) by (p_1, q_1) where $p_1>q_0$, we will reach the conclusion that

$$\left|\sum_{n=p_1}^{q_1} \frac{a_n}{n^k} C_k(nt)\right| > M\varepsilon$$

throughout a subinterval (α_2, β_2) of (α_1, β_1). We can thus determine a sequence of pairs of integers (p_i, q_i), tending to infinity with i, and a corresponding sequence of intervals (α_i, β_i) such that $\alpha_{i+1} \geqq \alpha_i$, $\beta_{i+1} \leqq \beta_i$ and

$$\left|\sum_{n=p_i}^{q_i} \frac{a_n}{n^k} C_k(nt)\right| > M\varepsilon$$

throughout (α_i, β_i). Therefore, there is at least one point t_0 common to all these intervals such that the infinite series $\sum \frac{a_n}{n^k} C_k(nt)$ diverges for $t=t_0$. This contradicts the hypothesis of the lemma.

3. Theorems.

THEOREM 1. *If $\sum_0^\infty a_n$ is summable (γ, k) where k is zero or a positive integer, then $\sum_0^\infty a_n$ is summable $(C, k+\delta)$, $\delta>0$.*

Proof. Case 1. $k>0$. As we have already noted in the introduction that the result is known to be true for $k=1$ and $k=2$ we take k to be an integer greater than 2 and assume that $\sum_0^\infty a_n$ is summable (γ, k) to S.

Suppose k is an even integer. Then by (1.2) and repeated application of (2.1), we find that, if $n \geqq 1$ and $t \neq 0$,

$$\gamma_k(nt) = \left(\frac{A_1}{n^2t^2} + \frac{A_2}{n^4t^4} + \cdots + \frac{A_{k/2}}{n^kt^k}\right) + \frac{R\cos nt}{n^kt^k},$$

where R, A_1, A_2, etc. are some constants. Therefore

$$\sum_{n=0}^\infty a_n\gamma_k(nt) = a_0 + \sum_{n=1}^\infty a_n \left\{\left(\frac{A_1}{n^2t^2} + \frac{A_2}{n^4t^4} + \cdots + \frac{A_{k/2}}{n^kt^k}\right) + \frac{R\cos nt}{n^kt^k}\right\}, \tag{3.1}$$

for $t \neq 0$. Since, by Lemmas 4 and 5, $\sum_1^\infty a_n/n^2$, $\sum_1^\infty a_n/n^4$, etc. converge respectively to S_1, S_2, etc. say, it follows from (3.1) that $\sum_1^\infty \frac{a_n}{n^k}\cos nt$ is covergent for $0<t<A$. It is also convergent for $t=0$.

From (3.1)

$$a_0 + \frac{R}{t^k}\sum_1^\infty \frac{a_n}{n^k}\cos nt = \sum_0^\infty a_n\gamma_k(nt) - \left(\frac{A_1S_1}{t^2} + \frac{A_2S_2}{t^4} + \cdots + \frac{A_{k/2}S_{k/2}}{t^k}\right). \tag{3.2}$$

By suitably altering $k/2+1$ terms of the series $\sum a_n$ and working with the resulting series, say $\sum a'_n$, we can simultaneously have $S=a_0=a'_0$, $S_1=S_2=\cdots=S_{k/2}=0$, so that by (3.2),

$$\sum_1^\infty \frac{a'_n}{n^k}\cos nt=o(t^k) \text{ as } t\to 0 . \tag{3.3}$$

Again, $\frac{a'_n}{n^k}=o(n^{2-k})=o\,(n^{-1})$, so that $\sum_1^\infty \frac{a'_n}{n^k}\cos nt$ is a Fourier series converging to a function $f(t)$, say, in a neighbourhood of the origin. Since $f(t)=o(t^k)$ for small t, it follows that the kth symmetric generalised derivative of $f(t)$ exists at $t=0$, and is equal to zero there. Hence, by virtue of well known results in the theory of Fourier series, we can immediately conclude that $(C, k+\delta)\sum_1^\infty a'_n=0$, so that $(C, k+\delta)\sum_0^\infty a_n=S$. The proof, when k is an odd integer, is similar.

Case 2. $k=0$. We are given that $\sum_0^\infty a_n \cos nt$ converges to a function $f(t)$ for $0<t<A$ and $\lim_{t\to 0} f(t)$ is a finite number S. Therefore $f(t)$ is bounded in some interval $0<t<\eta$.

Let $\sum_0^\infty b_n \cos nt$ be the Fourier series of an even periodic function $\lambda(t)$ defined as follows.

$$\lambda(t)=\begin{cases} 1 & \text{for} \quad 0\leqq t\leqq \eta'<\eta \\ 0 & \text{for} \quad \eta\leqq t\leqq \pi . \end{cases}$$

Moreover, let $\lambda(t)$ change smoothly as t increases from η' to η, so that $\lambda'''(t)$ exists and is continuous. Hence $b_n=o(1/n^3)$. (See [5 Theorem 40]).

If $\sum_0^\infty c_n \cos nt$ is the formal product of $\sum_0^\infty a_n \cos nt$ and $\sum_0^\infty b_n \cos nt$, then it follows from Rajchman's theory of formal multiplication [12, section 11.42] that $\sum_0^\infty c_n \cos nt$ converges to $f(t)$ in $0<t\leqq\eta'$, to $\lambda(t)f(t)$ in $\eta'\leqq t\leqq \eta$, and to zero in $\eta\leqq t\leqq\pi$. Hence it follows that $\sum_0^\infty c_n \cos nt$ is a Fourier series [12, Theorem 11.33], and therefore $\sum_0^\infty c_n$ is summable (c, δ) for $\delta>0$, because $\lim_{t\to 0} f(t)=S$. Consequently, $\sum_0^\infty a_n$ is also summable (c, δ) (See [12, section 11.42].

THEOREM 2. *Let* $\sum_0^\infty a_n$ *be summable* (γ, k) *where* $k\geqq 1$ *and let* $\sum_1^\infty \frac{a_n}{n^{[k]-1}}\cos nx$ (*when* $[k]$ *is odd*) *or* $\sum_1^\infty \frac{a_n}{n^{[k]-1}}\sin nx$ (*when* $[k]$ *is even*) *be* *a Fourier series. Then* $\sum_0^\infty a_n$ *is summable* $(c, k+\delta)$, $\delta>0$.

Proof. Since, in Theorem 1, we have already proved the result for integral k under more general conditions, we assume k to be non-integral. We also take $[k]$ to be an odd integer. The proof, when $[k]$ is even, is similar.

By making $\frac{[k]-1}{2}$ applications of (2.1), using Lemma 5, and arguing as in the deduction of (3.3), we get

$$\sum_1^\infty \frac{a_n'}{n^k} C_{k-[k]+1}(nt)=o(t^k) , \tag{3.4}$$

where $\sum_0^\infty a_n'$ differs from $\sum_0^\infty a_n$ in a finite number of terms only, $\sum_1^\infty a_n'/n^2=0$, $\sum_1^\infty a_n'/n^4=0$, etc., and $\sum_0^\infty a_n'$ is summable (γ, k) to $a_0'=a_0$. Let $\sum_1^\infty \frac{a_n'}{n^{[k]-1}} \cos nx$ be the Fourier series of an even function $\phi(x)\varepsilon L$. Then it can be easily shown that

$$\begin{aligned}\phi_{k-[k]+1}(t) &= \sum_1^\infty \frac{a_n'}{n^k} C_{k-[k]+1}(nt) \\ &= o(t^k) \quad \text{by (3.4)} .\end{aligned} \tag{3.5}$$

Again, $\phi(t) \in L$ obviously implies that $\phi(t)$ is Cesàro-Lebesgue integrable $C_\lambda L$ for any $\lambda \geqq 0$ so that

$$\phi(t) \in C_{k-[k]}L \tag{3.6}$$

From (3.5) and (3.6), we have (See [3, Theorem 2])

$$\frac{\phi(t)}{t^{[k]-1}} \in C_{k-[k]}L$$

and

$$\frac{\phi(t)}{t^{[k]-1}} = o(1)(C, k-[k]+1) \quad \text{as} \quad t\to 0 .$$

Hence, in view of Lemma 3, we conclude that the (Cesàro-Lebesgue) Fourier series of $\frac{\phi(t)}{t^{[k]-1}}$ is summable $(C, k-[k]+1+\delta)$ at $t=0$ for any $\delta>0$. Now it follows from Lemma 2, where we take $r=[k]-1$, that $\sum_1^\infty a_n'$ is summable $(C, k+\delta)$. Hence $\sum_0^\infty a_n$ is also summable $(C, k+\delta)$.

COROLLARY. *If* $\sum_0^\infty a_n$ *is summable* (γ, k), $k \geqq 4$, *then* $\sum_0^\infty a_n$ *is summable* $(C, k+\delta)$, $\delta > 0$.

The corollary follows immediately from the theorem because $a_n = o(n^2)$ by Lemma 4.

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