ADDITIVE FUNCTIONALS OF A MARKOV PROCESS

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1. Introduction. We are concerned with functionals of the form $L=\int_0^t V[x(\tau)]d\tau$ where $x(t)$ is a temporally homogeneous Markov process in a locally compact Hausdorff space, X, and V is a non-negative measurable function on X. In studying the distribution of this functional various authors (e.g. [1], [3], and [7] have considered the following function

$$r(t, x, A)=E\{e^{-uL}\,|\,x(0)=x; x(t)\in A\}\,p(t, x, A) \tag{1.1}$$

where $p(t, x, A)$ is the transition probability function of $x(t)$. If one can determine r then one can in essence determine the distribution of L since $(u>0)$

$$r(t, x, A)=\int_0^\infty e^{-u\lambda}d_\lambda P[L\leqq\lambda|x(0)=x; x(t)\in A] \qquad p(t, x, A)\ .$$

Formally it is quite easy to see that if p satisfies an equation of diffusion type

$$\frac{\partial p}{\partial t}=\Omega p \tag{1.2}$$

that r should satisfy the equation

$$\frac{\partial r}{\partial t}=(\Omega-uV)r\ . \tag{1.3}$$

If $x(t)$ is the Wiener process in E^N and V satisfies a Lipschitz condition of order $\alpha>0$ Rosenblatt [12] has given a rigorous derivation of (1.3). In this paper we use the theory of semi-groups to give a meaning to (1.3) for a wide class of processes without assuming any smoothness conditions on V. Rosenblatt's result does not follow from ours since our results only imply that r is a "weak" solution of (1.3). However, for many applications (e.g. [10]) this is all that is really required.

Because of certain difficulties connected with the definition of the conditional expectation in (1.1) we define r directly and prove that if $p(t, x, A)>0$ then $\dfrac{r(t, x, A)}{p(t, x, A)}$ is the appropriate conditional expectation. Since we intend to apply analytic methods it is necessary to investigate the dependence of r on its various variables. This is done in § 2.

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Beginning in § 3 we assume that $p(t, x, A)$ has a density $f(t, x, y)$ with respect to a Radon measure m and we show (§§ 4 and 5) that if $U_t\varphi(x)=\int\varphi(y)p(t, x, dy)$ has infinitesimal generator Ω on $L_2(m)$ then $T_t\varphi(x) =\int\varphi(y)r(t, x, dy)$ has infinitesimal generator $\Omega-uV$ if V is bounded, subject to certain regularity conditions on f. If V is unbounded our results are less complete and are contained in Theorem 5.2. In the sequel we will suppress the parameter u.

We use throughout this paper the function space approach to stochastic processes. We also make use of certain elementary facts about integration in locally compact spaces. The reader is referred to [2], [4], and [5] for the basic facts required. In a future paper we plan to study the spectral properties of the operators defined here. In that paper X will be an open subset of an N dimensional Euclidean space.

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2. A class of integrals over a function space. Let X be a locally compact Hausdorff space and $\mathfrak{B}(X)$ the Borel sets of X; that is, the smallest σ-algebra of subsets of X containing the compact sets of X. Let $\mathfrak{X}$ be the set of all functions from $[0\leqq t<\infty]$ to X which are right continuous; that is, $x(t)\rightarrow x(t_0)$ if $t\downarrow t_0$. Let $p(t, x, A)$ be a transition probability function defined for $t>0$, $x\in X$, and $A\in\mathfrak{B}(X)$, such that given an arbitrary probability measure μ on $\mathfrak{B}(X)$ there exists a Markov process $x_\mu(t)$ with paths which are right continuous and which has μ as its initial distribution and $p(t, x, A)$ as its transition probability. In other words, if $\mathfrak{B}(\mathfrak{X})$ is the σ-algebra of subsets of $\mathfrak{X}$ generated by sets of the form

$$\Lambda=\{x(\cdot)|x(t_j)\in A_j; j=0, 1, \cdots, n; A_j\in\mathfrak{B}(X); 0=t_0<t_1<\cdots<t_n\}$$

then there exists a countably additive probability measure, P_μ, on $\mathfrak{B}(\mathfrak{X})$ such that

$$(2.1)\qquad P_\mu(\Lambda)=\int_{A_0}\int_{A_1}\cdots\int_{A_n}\mu(dx_0)p(t_1, x_0, dx_1)p(t_2-t_1, x_1, dx_2) \cdot p(t_n-t_{n-1}, x_{n-1}, dx_n) .$$

If μ assigns mass one to a single point, x, we write P_x for P_μ.

We assume that

(P_1) $p(\cdot, \cdot, A)$ is jointly measurable[1] in (t, x) for each $A\in\mathfrak{B}(X)$. We also pick a fixed μ, and $x(t)$ will always denote the processes having μ as

[1] Measurability conditions in t are understood to be with respect to the ordinary Borel sets of $[0\leqq t<\infty]$.

its initial distribution. Clearly (once we have established Theorem 2.1)

$$P[\Lambda \mid x(0)=x]=P_x(\Lambda)\ , \qquad (\Lambda \in \mathfrak{B}(\mathfrak{X}))\ . \tag{2.2}$$

If $A \in \mathfrak{B}(X)$ we define

$$A_t=\{x(\cdot) \mid x(\cdot) \in \mathfrak{X}\ ;\ x(t) \in A\} \in \mathfrak{B}(\mathfrak{X})\ . \tag{2.3}$$

If $\Lambda \in \mathfrak{B}(\mathfrak{X})$ and $A \in \mathfrak{B}(X)$ we define for $t>0$

$$P(\Lambda\, ;\, x\, ;\, t,\, A)=P_x[\Lambda \cap A_t]\ . \tag{2.4}$$

It is evident that $P(\cdot\, ;\, x\, ;\, t,\, A)$ is a finite measure on $\mathfrak{B}(\mathfrak{X})$ for fixed x, t, A and that $P(\Lambda\, ;\, x\, ;\, t,\, \cdot)$ is a finite measure on $\mathfrak{B}(X)$ for fixed Λ, x, t. It is easy to see that if t and A are such that $p(t, x, A)>0$ for all x, then (again assuming Theorem 2.1)

$$P[\Lambda \mid x(0)=x\, ;\, x(t) \in A]=\frac{P[\Lambda\, ;\, x\, ;\, t,\, A]}{p(t,\, x,\, A)}\ . \tag{2.5}$$

THEOREM 2.1. *$P[\Lambda\, ;\, \cdot\, ;\, \cdot,\, A]$ is a measurable function of (t, x) for fixed Λ, A.*

Proof. Let A be fixed and suppose

$$\Lambda=\{x(\cdot) \mid x(t_j) \in A_j\, ;\, j=1,\, \cdots,\, n\}$$

then $P[\Lambda\, ;\, x\, ;\, t,\, A]=P_x[\Lambda \cap A_t]$ which is measurable in (t, x) in view of (2.1) and (P_1). Hence $P[\Lambda\, ;\, x\, ;\, t,\, A]$ is measurable in (t, x) for Λ's which are finite disjoint unions of sets of the above form. But the measurability of $P[\Lambda\, ;\, x\, ;\, t,\, A]$ is preserved under monotone limits of Λ's and hence $P[\Lambda\, ;\, x\, ;\, t,\, A]$ is measurable for all $\Lambda \in \mathfrak{B}(\mathfrak{X})$. See [8].

The following lemmas will be of use in the sequel.

LEMMA 2.1. *Let $(Y, \mathfrak{S})$ and $(Z, \mathfrak{H})$ be measurable spaces and let $m(A, B)$ be defined for $A \in \mathfrak{S}$ and $B \in \mathfrak{H}$. Suppose that $m(\cdot, B)$ is a measure on $(Y, \mathfrak{S})$ for each fixed $B \in \mathfrak{H}$ and that $m(A, \cdot)$ is a measure on $(Z, \mathfrak{H})$ for each fixed $A \in \mathfrak{S}$. Let $f \geqq 0$ be a measurable function on $(Y, \mathfrak{S})$ then*

$$q(B)=\int f(y)m(dy, B) \tag{2.6}$$

is a measure on $(Z, \mathfrak{H})$.

Proof. The only thing that requires proof is that q is countably additive. Let $\{f_n\}$ be a sequence of simple functions such that $f_n \geqq 0$ and $f_n \uparrow f$. Clearly

$$q_n(B)=\int f_n(y)m(dy, B)$$

are measures and $q_n(B)\uparrow q(B)$ for each $B\in\mathfrak{H}$. Let $B=\bigcup_{j=1}^{\infty}B_j$ where the B_j's are disjoint. Put $B^{(k)}=\bigcup_{j=1}^{k}B_j$, then $\lim_n\lim_k q_n(B^{(k)})=q(B)$. Since $q_n(B^{(k)})$ is increasing in both n and k we can interchange the limits obtaining

$$q(B)=\lim_k\lim_n q_n(B^{(k)})=\lim_k q(B^{(k)})=\sum_{j=1}^{\infty}q(B_j)\ .$$

LEMMA 2.2. *Let $(Y,\mathfrak{G})$ be a measurable space and let $f(t,y)$ be an X valued function defined for $t\geqq 0$ and $y\in Y$. If $f(\cdot,y)$ is right continuous for each $y\in Y$ and $f(t,\cdot)$ is $\mathfrak{G}$-measurable for each t then $f(t,y)$ is jointly $\mathfrak{B}\times\mathfrak{G}$ measurable.* ($\mathfrak{B}$ is the σ-algebra of ordinary Borel sets.)

Proof. Define $g_n(t,y)=f((j+1)/n,y)$ if $j/n<t\leqq(j+1)/n$ for $j=0,1,2,\cdots$ and $n=1,2,\cdots$. Let $B\in\mathfrak{B}(X)$ and define $G_{jn}=f((j+1)/n,\cdot)^{-1}(B)$, then since $f(t,\cdot)$ is $\mathfrak{G}$-measurable $G_{jn}\in\mathfrak{G}$. Let $A_{jn}=\{t|j/n<t\leqq(j+1)/n\}\in\mathfrak{B}$, then

$$g_n^{-1}(B)=\bigcup_{j=0}^{\infty}A_{jn}\times G_{jn}$$

which is in $\mathfrak{B}\times\mathfrak{G}$. Hence g_n is jointly $\mathfrak{B}\times\mathfrak{G}$ measurable for each n, but $g_n(t,x)\to f(t,x)$ as $n\to\infty$ and thus f is $\mathfrak{B}\times\mathfrak{G}$ measurable.

If $\Phi\ [x(\cdot)]$ is a complex valued measurable[2] functional on $\mathfrak{X}$ we write $r[\Phi;t,x,A]$ for the integral of Φ over $\mathfrak{X}$ with respect to the measure $P[\cdot\,;x;t,A]$, provided the integral exists.

THEOREM 2.2. *If $\Phi\geqq 0$ is a measurable functional on $\mathfrak{X}$ then $r[\Phi;t,x,A]$ is a measure on $\mathfrak{B}(X)$ for fixed (t,x) and is measurable in (t,x) for fixed A.*

Proof. This is an immediate consequence of Lemma 2.1 and Theorem 2.1.

Let φ be a complex valued measurable function on X, then for each $t>0$ we define a measurable functional, φ_t, on $\mathfrak{X}$ as follows: $\varphi_t[x(\cdot)]=\varphi[x(t)]$. Also if Φ is a measurable functional on $\mathfrak{X}$ we denote its integral over $\mathfrak{X}$ with respect to the measure P_x by $E\{\Phi[x(\cdot)]|x(0)=x\}$.

THEOREM 2.3. *Let $\Phi\geqq 0$ be a measurable functional on $\mathfrak{X}$ and φ a complex valued measurable function on X; then*

$$\int\varphi(y)r[\Phi;t,x,dy]=E\{\Phi\cdot\varphi_t|x(0)=x\}\ , \tag{2.7}$$

provided either integral exists.

[2] Measurability of real or complex valued functions always means Borel measurability.

Proof. Suppose $\varphi = I_A$ where I_A denotes the characteristic function A, then the left hand side of (2.7) is $r[\Phi; t, x, A]$. Now if $\Phi = I_\Lambda$ then

$$r[I_\Lambda; t, x, A] = P[\Lambda; x; t, A] .$$

But

$$(I_A)_t[x(\cdot)] = I_A[x(t)] = I_{A_t}[x(\cdot)] ,$$

where $A_t = \{x(\cdot) | x(t) \in A]$. Thus

$$E\{I_\Lambda \cdot (I_A)_t | x(0) = x\} = P_x[\Lambda \cap A_t] = P[\Lambda; x; t, A] .$$

Let Φ_n be a sequence of simple functionals such that $\Phi_n \uparrow \Phi$, then $\Phi_n \cdot (I_A)_t$ is a sequence of simple functionals increasing to $\Phi \cdot (I_A)_t$. Therefore

$$E\{\Phi_n \cdot (I_A)_t | x(0) = x\} \uparrow E\{\Phi \cdot (I_A)_t | x(0) = x\} .$$

On the other hand $r(\Phi_n; t, x, A) \uparrow r(\Phi; t, x, A)$ by the monotone convergence theorem and since

$$E\{\Phi_n \cdot (I_A)_t | x(0) = x\} = r[\Phi_n; t, x, A]$$

it follows that if either of the integrals in (2.7) is finite the other is also and they are equal in the case $\varphi = I_A$.

If $\varphi \geqq 0$ let φ_n be a sequence of simple functions increasing to φ then if either of the integrals in (2.7) exists we have equality for each φ_n and by monotone convergence for φ. The result for a general φ now follows in the usual manner.

For each $t \geqq 0$ let $x_t(\tau) = x(t + \tau)$ for all $\tau \geqq 0$, then we define a map, S_t, from $\mathfrak{X}$ into $\mathfrak{X}$ by $S_t x(\cdot) = x_t(\cdot)$. Clearly S_t is a measurable transformation of $\mathfrak{X}$ into $\mathfrak{X}$. If Φ is a measurable functional we define $S_t\Phi[x(\cdot)] = \Phi[S_t x(\cdot)]$.

THEOREM 2.4. *Let Φ be a functional measurable with respect to*[3] $\mathfrak{B}_t$ *and Ψ be measurable with respect to $\mathfrak{B}_s$ such that $0 \leqq \Phi \leqq M$ and $0 \leqq \Psi \leqq M$, then*

$$\int r[\Phi; t, x, dy] r[\Psi; s, y, A] = r[\Phi \cdot S_t \Psi; t + s, x, A] . \tag{2.8}$$

Proof. Since Φ and Ψ are non-negative and bounded it is clear that the integral in question exists. If $\Phi = I_F$ and $\Psi = I_G$ with $F \in \mathfrak{B}_t$ and $G \in \mathfrak{B}_s$ then

$$S_t I_G[x(\cdot)] = I_G[S_t x(\cdot)] = I_{S_t^{-1} G} ,$$

thus to prove (2.8) for I_F and I_G we must show that

[3] $\mathfrak{B}[t_1, t_2]$ denotes the σ-algebra of subsets of $\mathfrak{X}$ generated by sets of the form $\{x(\cdot) | x(\tau_j) \in A_j;\ t_1 \leqq \tau_j \leqq t_2\}$, and $\mathfrak{B}_t = \mathfrak{B}[0, t]$.

$$(2.9)\qquad \int P[F; x; t, dy]P[G; y; s, A] = P[F \cap S_t^{-1}G; x; t+s, A] .$$

We first consider the case in which

$$F = \{x(\cdot) \mid x(t_j) \in A_j;\ j=1, \cdots, n;\ t_j < t\}$$
$$G = \{x(\cdot) \mid x(t'_k) \in B_k;\ k=1, \cdots, m;\ t'_k < s\} .$$

In this case

$$S_t^{-1}G = \{x(\cdot) \mid x(t+t'_k) \in B_k;\ k=1, 2, \cdots, m\} ,$$

thus

$$\begin{aligned}
&\int P[F; x; t, dy]P[G; y; s, A] \\
&= \iint_{A_1} \cdots \int_{A_n} p(t_1, x, dx_1)p(t_2-t_1, x_1, dx_2) \cdots p(t-t_n, x_n, dy) \\
&\qquad \cdot \int_{B_1} \cdots \int_{B_m} p(t'_1, y, dy_1) \cdots p(s-t'_m, y_m, A) \\
&= \int_{A_1} \cdots \int_{A_n}\int_{B_1} \cdots \int_{B_m} p(t_1, x, dx_1) \cdots p(t+t'_1-t_n, x_n, dy_1) \\
&\qquad \cdot p(t+t'_2-(t+t'_1), y_1, dy_2) \cdots p(t+s-(t+t'_m), y_m, A) \\
&= P[F \cap S_t^{-1}G; x; t+s, A] .
\end{aligned}$$

If $t=t_n$, or $s=t'_m$, or both, it is necessary to make only minor changes in the above argument.

This equality clearly extends to finite disjoint unions of such F's and G's and since S_t^{-1} is a σ-homomorphism it extends to monotone limits of such G's. Thus (2.9) holds for each F in the algebra of sets generated by sets of the given form and for each $G \in \mathfrak{B}_s$. For fixed $G \in \mathfrak{B}_s$ the left hand side of (2.9) is a measure in F by Lemma 2.1, hence (2.9) holds under monotone limits of such F's and thus finally (2.9) holds for all F and G in the appropriate σ-algebras.

Let Φ_n and Ψ_n be sequences of simple functionals increasing to Φ and Ψ, then by monotone convergence

$$\int r[\Phi_n; t, x, dy]r[\Psi; s, y, A] = r[\Phi_n \cdot S_t\Psi; t+s, x, A] .$$

Applying an argument similar to that used in the proof of Lemma 2.1 the equality (2.8) results. (This also follows from Theorem 2.3.)

We conclude this section with the following theorem which is easily proved using standard approximation techniques.

THEOREM 2.5. *Let Φ $(t, x(\cdot)) \geqq 0$ be jointly measurable in t and $x(\cdot)$ then $r[\Phi(t, x(\cdot)); t, x, A]$ is jointly measurable in (t, x).*

3. Additive functionals. For each pair (t_1, t_2) with $0 \leqq t_1 < t_2$ let $L[t_1, t_2; x(\cdot)]$ be a functional (L may be $+\infty$) on $\mathfrak{X}$ which is measurable with respect to $\mathfrak{B}[t_1, t_2]$ and which is jointly measurable in t_1, t_2, and $x(\cdot)$. We further assume that for $t_1 < t < t_2$ and each $x(\cdot) \in \mathfrak{X}$ we have

$$L[t_1, t_2; x(\cdot)] = L[t_1, t; x(\cdot)] + L[t, t_2; x(\cdot)]\ ; \tag{3.1}$$

and that

$$S_t L[t_1, t_2; x(\cdot)] = L[t_1 + t, t_2 + t; x(\cdot)]\ . \tag{3.2}$$

Such a functional will be called an additive functional on $\mathfrak{X}$ (See [1]).

THEOREM 3.1. *Let $V \geqq 0$ be a measurable function on X, then*

$$L[t_1, t_2; x(\cdot)] = \int_{t_1}^{t_2} V[x(\tau)] d\tau$$

is an additive functional on $\mathfrak{X}$.

Proof. Define $F(t, x(\cdot)) = x(t)$ then F is measurable in $x(\cdot)$ for fixed t and right continuous in t for fixed $x(\cdot)$. Thus by Lemma 2.2 F is jointly measurable in t and $x(\cdot)$. Since $V[x(t)] = V[F(t, x(\cdot))]$ is the composition of measurable transformations $V[x(\tau)]$ is jointly measurable in τ and $x(\cdot)$, and therefore (a simple argument using Lemma 2.2 shows that) $\int_{t_1}^{t_2} V[x(\tau)] d\tau$ is jointly measurable in t_1, t_2, and $x(\cdot)$. The other properties that L must satisfy are obvious.

We suppose that $L[t_1, t_2; x(\cdot)] \geqq -M$ where $M > 0$ is independent of t_1, t_2, and $x(\cdot)$. We define

$$r(t, x, A) = r[e^{-L[0, t; x(\cdot)]}; t, x, A]\ . \tag{3.3}$$

Theorems 2.2 and 2.5 imply that $r(t, x, A)$ is a measure on $\mathfrak{B}(X)$ for fixed (t, x) and is jointly measurable in (t, x) for fixed $A \in \mathfrak{B}(X)$. Moreover the fact that

$$0 \leqq r(t, x, A) \leqq e^M p(t, x, A) \tag{3.4}$$

is a simple consequence of our definitions.

THEOREM 3.2. $r(t+s, x, A) = \int r(t, x, dy) r(s, y, A)\ .$

Proof. This is a corollary of Theorem 2.4 once we observe that

$$S_t e^{-L[0, s; x(\cdot)]} = e^{-L[0, s; S_t x(\cdot)]} = e^{-S_t L[0, s; x(\cdot)]} = e^{-L[t, t+s; x(\cdot)]}\ ,$$

and therefore

$$e^{-L[0,t;x(\cdot)]}\cdot S_t e^{-L[0,s;x(\cdot)]}=e^{-L[0,t+s;x(\cdot)]}\ .$$

At this point we assume that there exists a Radon measure, m, on $\mathfrak{B}(X)$ such that $p(t, x, A)$ has a density $f(t, x, y)\geqq 0$ with respect to m for $t>0$; that is

$$p(t, x, A)=\int_A f(t, x, y)m(dy)\ , \qquad t>0\ . \tag{3.5}$$

We assume that f is jointly measurable in t, x, and y, but we do not assume that m is finite. We introduce the following conditions on $f(t, x, y)$:

(P_2) $\int f(t, x, y)m(dx)\leqq ke^{\alpha t}$ where k and α are positive constants independent of y and t.

(P_3) Given $\varepsilon>0$ and a compact set $A\subset X$ there exists a compact set B such that

$$\int_{x\notin B} f(t, x, y)m(dx)<\varepsilon \text{ for } y\in A \text{ and } t\leqq 1\ .$$

We define operators on appropriate function spaces as follows:

$$(T_t\varphi)(x)=\int\varphi(y)r(t, x, dy) \tag{3.6}$$

$$(U_t\varphi)(x)=\int\varphi(y)p(t, x, dy)=\int\varphi(y)f(t, x, y)m(dy)\ . \tag{3.7}$$

THEOREM 3.3. *If $f(t,x,y)$ satisfies* (P_2) *then* $\{T_t; t>0\}$ *and* $\{U_t; t>0\}$ *are semi-groups of bounded operators on* $L_2(m)$.

Note. All Borel sets are m-measurable [4; 5].

Proof. From (3.4) we obtain

$$\begin{aligned}|T_t\varphi(x)|&\leqq\int|\varphi(y)|r(t, x, dy)\\&\leqq e^M\int|\varphi(y)|p(t, x, dy)=e^M U_t|\varphi|(x)\end{aligned}$$

and thus it will suffice to prove that U_t is a bounded operator on $L_2(m)$ for each $t>0$. But

$$\begin{aligned}|U_t\varphi(x)|^2&=\left|\int f(t, x, y)\varphi(y)m(dy)\right|^2\\&\leqq\int f(t, x, y)|\varphi(y)|^2 m(dy)\ ,\end{aligned}$$

and therefore

$$\int |U_t\varphi(x)|^2 m(dx) \leqq \int m(dx) \int f(t, x, y) |\varphi(y)|^2 m(dy)$$
$$\leqq k e^{\alpha t} \cdot \|\varphi\|^2 .$$

Thus $\|U_t\|^2 \leqq k e^{\alpha t}$ and $\|T_t\|^2 \leqq k e^{2M+\alpha t}$. The fact that $\{T_t; t>0\}$ and $\{U_t; t>0\}$ are semi-groups now follows from Theorem 3.2 and the fact that $p(t, x, A)$ satisfies the Chapman-Kolmogorov equation.

THEOREM 3.4. *If $f(t, x, y)$ satisfies* (P_2) *and* (P_3) *and* $\lim_{t\to 0} L[0, t; x(\cdot)]=0$ *for all* $x(\cdot) \in \mathfrak{X}$ *then the semi-groups* $\{U_t; t>0\}$ *and* $\{T_t; t>0\}$ *are strongly continuous*[4] *on* $L_2(m)$.

Proof. We prove the theorem for $\{T_t; t>0\}$ the results for $\{U_t; t>0\}$ being a special case (take $L\equiv 0$). We must show that $\|T_t\varphi-\varphi\| \to 0$ as $t\to 0$ for all $\varphi \in L_2(m)$. Since $\|T_t\|$ is uniformly bounded for $t \leqq 1$ it will be sufficient to show that $\|T_t\varphi-\varphi\|\to 0$ as $t\to 0$ for φ continuous with compact support, such functions being dense in $L_2(m)$ since m is a Radon measure, [2]. We first show that $T_t\varphi(x)\to\varphi(x)$ pointwise as $t\to 0$ if φ is continuous with compact support. According to Theorem 2.3

$$T_t\varphi(x) = \int \varphi(y) r(t, x, dy) = E\{e^{-L[0,t;x(\cdot)]} \cdot \varphi(x(t)) | x(0)=x\} .$$

Using the right continuity of $x(\cdot)$ and our assumption on L we see that

$$e^{-L[0,t;x(\cdot)]}\varphi[x(t)] \to \varphi[x(0)]$$

boundedly as $t \downarrow 0$ and hence by the bounded convergence theorem

$$T_t\varphi(x) \to E\{\varphi[x(0)] \,|\, x(0)=x\} = \varphi(x) \qquad \text{as} \quad t\downarrow 0 .$$

Let A be the support of φ, then if B is compact and $B\supset A$ we have

$$\|T_t\varphi-\varphi\|^2 = \int_B |T_t\varphi(x)-\varphi(x)|^2 m(dx) + \int_{x\notin B} |T_t\varphi(x)-\varphi(x)|^2 m(dx)$$
$$= I_1 + I_2 .$$

But

$$|T_t\varphi(x)| \leqq \int |\varphi(y)| r(t, x, dy) \leqq \sup_{x\in A} |\varphi(x)| \cdot e^M ,$$

hence $I_1 \to 0$ since B is compact. Now since $B\supset A$ we have

$$I_2 \leqq \int_{x\notin B} |T_t\varphi(x)|^2 m(dx) \leqq e^M \int_A |\varphi(y)|^2 \int_{x\notin B} f(t, x, y) m(dy) m(dx) ,$$

[4] By the strong continuity of a semi-group $\{T_t; t>0\}$ we will always mean strong continuity for $t \geqq 0$ where T_0 is the identity.

and so, if B is chosen properly, using (P_3), we see that I_2 is small. This completes the proof of the fact that $\{T_t; t>0\}$ is strongly continuous on $L_2(m)$.

4. The Darling-Siegert equations. In [3] Darling and Siegert showed that $r(t, x, A)$ has to satisfy two integral equations. We give a derivation of these equations based on the material of § 2. We assume that $p(t, x, A)$ satisfies (P_1) and that

$$L[t_1, t_2; x(\cdot)]=\int_{t_1}^{t_2} V[x(\tau)]d\tau$$

where V is a bounded, non-negative, measurable function on X. The formal outline of the derivation given below is exactly that of Darling and Siegert.

We begin with the following identities which are easily verified (f measurable, non-negative, and bounded)

$$\exp\left[-\int_0^t f(\tau)d\tau\right]=1-\int_0^t f(s)\exp\left[-\int_s^t f(\tau)d\tau\right]ds \tag{4.1}$$

$$\exp\left[-\int_0^t f(\tau)d\tau\right]=1-\int_0^t f(s)\exp\left[-\int_0^s f(\tau)d\tau\right]ds . \tag{4.2}$$

Also using Theorem 2.4 we have

$$\begin{aligned} &r\left[V[x(s)]\exp\left(-\int_s^t V[x(\tau)]d\tau\right); t, x, A\right] \\ &=r\left[V[x(s)]\cdot S_s \exp\left(-\int_0^{t-s} V[x(\tau)]d\tau\right); (t-s)+s, x, A\right] \\ &=\int r[V[x(s)]; s, x, dy]r[\exp\left(-\int_0^{t-s} V[x(\tau)]d\tau\right); t-s, y, A] \\ &=\int V(y)p(s, x, dy)r(t-s, y, A) \end{aligned} \tag{4.3}$$

provided we show that

$$\int f(y)r[V[x(s)]; s, x, dy]=\int f(y)V(y)p(s,x, dy) \tag{4.4}$$

for measurable, bounded $f\geqq 0$. Suppose $f=I_A$ and $V=I_B$ then

$$\begin{aligned} &\int f(y)r[V[x(s)]; s, x, dy]=P[I_B[x(s)]; x; s, A] \\ &=P_x[B_s\cap A_s]=P(s, x, A\cap B)=\int f(y)V(y)p(s, x, dy) . \end{aligned}$$

The standard approximation technique now yields the desired result (4.4).

Putting $f(\tau)=V[x(\tau)]$ in (4.1) and applying (4.3) we obtain (the interchange in the order of integration is valid since

$$V[x(s)]\cdot\exp\left(-\int_s^t V[x(\tau)]\,d\tau\right)$$

is bounded and jointly measurable in s and $x(\cdot)$)

$$r(t, x, A)=p(t, x, A)-\int_0^t ds\int V(y)r(t-s, y, A)p(s, x, dy)\ . \tag{4.5}$$

In a similar manner using (4.2) we find

$$r(t, x, A)=p(t, x, A)-\int_0^t ds\int V(y)p(t-s, y, A)r(s, x, dy); \tag{4.6}$$

and these are the Darling-Siegert equations. In deriving (4.6) one needs the relation

$$r[V[x(0)]; t, y, A]=V(y)p(t, y, A) \tag{4.7}$$

which is obtained in much the same manner as (4.4).

Taking Laplace transforms in (4.5) and (4.6) yields (the necessary interchange of order of integration is again justified since the integrand is bounded and jointly measurable in its variables)

$$\hat{r}(\lambda, x, A)=p(\lambda, x, A)-\int V(y)\hat{r}(\lambda, y, A)\hat{p}(\lambda, x, dy) \tag{4.8}$$

$$\hat{r}(\lambda, x, A)=\hat{p}(\lambda, x, A)-\int V(y)\hat{p}(\lambda, y, A)\hat{r}(\lambda, x, dy) \tag{4.9}$$

where $\hat{r}$ and $\hat{p}$ are the Laplace transforms of r and p.

5. The infinitesimal generators. Let Ω and Ω' be the infinitesimal generators of $\{U_t; t>0)$ and $\{T_t; t>0\}$ respectively. We assume in this section that (P_1), (P_2), and (P_3) are satisfied. It then follows, since the semi-groups involved are strongly continuous on $L_2(m)$, that Ω and Ω' are closed densely defined operators on $L_2(m)$. See [6] and [9].

We assume that

$$L[t_1, t_2; x(\cdot)]=\int_{t_1}^{t_2} V[x(\tau)]d\tau \tag{5.1}$$

where V is a non-negative measurable function on X. Note that in this case $M=0$.

THEOREM 5.1. *If V is bounded then $\Omega'=\Omega-V$.*

Proof. Let J_λ be the resolvent of $\{T_t; t>0\}$ then for $\lambda>\alpha$ we have

$$J_\lambda\varphi(x)=\int_0^\infty e^{-\lambda t}T_t\varphi(x)dt\ ,$$

and thus

$$|J_\lambda\varphi(x)|^2\leqq\frac{1}{\lambda}\int_0^\infty e^{-\lambda t}|T_t\varphi(x)|^2dt\ .$$

Applying the Fubini theorem we see that $J_\lambda\varphi(x)$ exists for almost all $x(\lambda>\alpha)$ and is in $L_2(m)$, moreover for $\lambda>\alpha$ we have

$$\|J_\lambda\|^2\leqq\frac{k}{\lambda(\lambda-\alpha)}\ . \tag{5.2}$$

In view of the above facts we can write

$$J_\lambda\varphi(x)=\int\varphi(y)\hat{r}(\lambda,\,x,\,dy)\ . \tag{5.3}$$

From the general theory of semi-groups, [6] and [9], we know that for $\lambda>\alpha$ the range of J_λ is independent of λ and is, in fact, the domain of Ω', which we denote by $D_{\Omega'}$. In addition it is known that

$$(\lambda-\Omega')J_\lambda\varphi=\varphi \qquad \text{for all} \quad \varphi\in L_2(m); \tag{5.4}$$

$$J_\lambda(\lambda-\Omega')\varphi=\varphi \qquad \text{for all} \quad \varphi\in D_{\Omega'}\ . \tag{5.5}$$

Let I_λ be the resolvent of $\{U_t;\,t>0\}$ and then in a similar manner we have

$$I_\lambda\varphi(x)=\int\varphi(y)\hat{p}(\lambda,\,x,\,dy)=\int\varphi(y)\hat{f}(\lambda,\,x,\,y)m(dy)\ . \tag{5.7}$$

From (4.8) we see that

$$\begin{aligned}J_\lambda\varphi(x)&=I_\lambda\varphi(x)-\int\varphi(z)\int V(y)\hat{r}(\lambda,\,y,\,dz)\hat{p}(\lambda,\,x,\,dy)\\&=I_\lambda\varphi(x)-I_\lambda[V\cdot J_\lambda\varphi](x)\\&=I_\lambda[\varphi-V\cdot J_\lambda\varphi](x)\ .\end{aligned}$$

The above steps are justified since $V\cdot J_\lambda\varphi\in L_2(m)$ under our assumption that V is bounded. Thus $D_{\Omega'}\subset D_\Omega$ and conversely using (4.9) $D_\Omega\subset D_{\Omega'}$, that is, $D_\Omega=D_{\Omega'}$. Now

$$\begin{aligned}(\lambda-\Omega)J_\lambda\varphi&=(\lambda-\Omega)I_\lambda[\varphi-VJ_\lambda\varphi]\\&=\phi-VJ_\lambda\varphi\ ,\end{aligned}$$

or equivalently,

$$[\lambda-(\Omega-V)]J_\lambda\varphi=\varphi \qquad \text{for all} \quad \varphi\in L_2\ .$$

Thus $\Omega - V$ is an extension of Ω', but since V is bounded the domain of $\Omega - V$ is $D_\Omega = D_{\Omega'}$. Hence $\Omega' = \Omega - V$.

COROLLARY. *If V is bounded and $f(t, x, y) = f(t, y, x)$ then Ω and Ω' are self-adjoint operators.*

Proof. Since $f(t, x, y) = f(t, y, x)$ each U_t is a bounded self-adjoint operator and hence Ω is also self-adjoint, although not necessarily bounded. The boundedness of V implies that V considered as an operator on $L_2(m)$ is bounded and self-adjoint, therefore $\Omega - V$ is self-adjoint, [**11**]. Thus $\Omega' = \Omega - V$ is a self-adjoint operator which in turn implies that each T_t is a bounded self-adjoint operator.

If V is not bounded our results are much less complete (V is no longer a bounded operator on $L_2(m)$ and one runs into the usual "domain problems"). It is natural to try to approximate V by bounded functions and then use a limiting procedure. Accordingly we define

$$V_N(x) = \begin{cases} V(x) & \text{if} \quad V(x) \leqq N\,, \\ N & \text{if} \quad V(x) \geqq N \end{cases} \tag{5.8}$$

and it is evident that each V_N is measurable and bounded. Let

$$D_V = \{\varphi \mid \varphi \in L_2(m)\,;\ V \cdot \varphi \in L_2(m)\}\,;$$

that is, D_V is the domain of V considered as an operator on $L_2(m)$. We are, of course, assuming that $f(t, x, y)$ satisfies (P_1), (P_2), and (P_3).

THEOREM 5.2. *If V is non-negative and measurable then $D_\Omega \cap D_V \subset D_{\Omega'}$ and if $\varphi \in D_\Omega \cap D_V$ then $\Omega' \varphi = (\Omega - V)\varphi$.*

Proof. We define

$$r_N(t, x, A) = r[e^{-\int_0^t V_N[x(\tau)]d\tau}\,;\ t, x, A]$$

and

$$T_t^{(N)} \varphi(x) = \int \varphi(y) r_N(t, x, dy)\ .$$

For each N we know that $\{T_t^{(N)}; t > 0\}$ is a strongly continuous semi-group of bounded operators on $L_2(m)$ whose infinitesimal generator is $\Omega - V_N$. Since $V_N \uparrow V$ we have by monotone convergence that

$$r_N(t, x, A) \downarrow r(t, x, A)\ . \tag{5.9}$$

We first show that for each $t > 0$ and all $\varphi \in L_2(m)$

$$\| T_t^{(N)} \varphi - T_t \varphi \| \to 0 \qquad \text{as } N \to \infty\ . \tag{5.10}$$

Since $\|T_t^{(N)}\| \leqq ke^{\alpha t}$ it will suffice to prove (5.10) for φ continuous with compact support. Let $\mu_N(A) = r_N(t, x, A) - r(t, x, A) \geqq 0$, then $\mu_N(A) \downarrow 0$ for each fixed A and is a measure on $\mathfrak{B}(X)$ for each fixed N. It is clear that

$$|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \leqq \int |\varphi(y)| \mu_N(dy) .$$

Let φ_j be a sequence of simple functions decreasing to $|\varphi|$, then since $\int \varphi_j(y)\mu_N(dy)$ is decreasing in both N and j we can interchange the limits obtaining $|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \to 0$ pointwise as $N \to \infty$ at least if φ is continuous with compact support. If the support of φ is A then (5.10) follows exactly as in the proof of Theorem 3.4 since

$$\int_{x \notin B} |T_t^{(N)}\varphi(x)|^2 m(dx) \leqq \int_A |\varphi(y)|^2 \int_{x \notin B} f(t, x, y) m(dx) m(dy)$$

for compact B. Thus (5.10) is established.

We prove next that $D_\Omega \cap D_V \subset D_{\Omega'}$. Let $J_\lambda^{(N)}$ and J_λ be the resolvents of $\{T_t^{(N)}; t>0\}$ and $\{T_t; t>0\}$ respectively. Since $\|T_t^{(N)}\| \leqq ke^{\alpha t}$ and $T_t^{(N)}\varphi \to T_t\varphi$ it follows that $J_\lambda^{(N)}\varphi \to J_\lambda\varphi$ for each $\varphi \in L_2(m)$ and $\lambda > \alpha$. Choose a $\lambda > \alpha$ and let it be fixed for the remainder of the present proof. If $\varphi \in D_\Omega \cap D_V$ then $\varphi \in D_{\Omega - V_N}$ for each N, hence there exist $\psi_N \in L_2(m)$ such that $\varphi = J_\lambda^{(N)}\psi_N$. Moreover $[\lambda - (\Omega - V_N)]\varphi = \psi_N$ or $\psi_N = \lambda\varphi - \Omega\varphi + V_N\varphi$. Clearly $V_N\varphi \to V\varphi$ pointwise and since $|V_N\varphi| \leqq |V\varphi|$ it follows that $\|V_N\varphi - V\varphi\| \to 0$. Thus $\psi_N \to \lambda\varphi - \Omega\varphi + V\varphi = \psi$ as $N \to \infty$ in $L_2(m)$. But

$$\|J_\lambda^{(N)}\psi_N - J_\lambda\psi\| \leqq \|J_\lambda^{(N)}\psi_N - J_\lambda^{(N)}\psi\| + \|J_\lambda^{(N)}\psi - J_\lambda\psi\|$$

and therefore $J_\lambda^{(N)}\psi_N \to J_\lambda\psi$ as $N \to \infty$ since $\|J_\lambda^{(N)}\|$ is uniformly bounded in N. However, $\varphi = J_\lambda^{(N)}\psi_N$ for all N and hence $\varphi = J_\lambda\psi$ which implies that $\varphi \in D_{\Omega'}$.

Since $\varphi = J_\lambda\psi$ where $\psi = \lambda\varphi - \Omega\varphi + V\varphi$ we see that $(\lambda - \Omega')\varphi = \psi = \lambda\varphi - \Omega\varphi + V\varphi$ or equivalently that $\Omega'\varphi = (\Omega - V)\varphi$ for $\varphi \in D_\Omega \cap D_V$. This completes the proof of Theorem 5.2.

COROLLARY. *If Ω is self-adjoint (that is, $f(t, x, y) = f(t, y, x)$) then Ω' is self-adjoint. Let $E_N(\lambda)$ denote the spectral resolution of $\Omega - V_N$ and $E(\lambda)$ the spectral resolution of Ω', then $E_N(\lambda)\varphi \to E(\lambda)\varphi$ for all $\varphi \in L_2(m)$ provided that λ is a continuity point of $E(\lambda)$.*

Proof. We use the same notation as in the proof of Theorem 5.2. From the corollary to Theorem 5.1 it follows that each $T_t^{(N)}$ is self-adjoint and T_t being the strong limit of self-adjoint operators is self-adjoint for each $t>0$. Hence the infinitesimal generator, Ω', of $\{T_t; t>0\}$ is self-adjoint. The strong continuity of $\{T_t; t>0\}$ implies that $T_t\varphi=0$ if and only if $\varphi=0$. A similar statement holds for $T_t^{(N)}$. Under these circumstances $E_N^{(\lambda)}=F_N(e^\lambda)$ and $E(\lambda)=F(e^\lambda)$ where F_N and F are the spectral resolutions of $T_1^{(N)}$ and T_1 respectively. See [11]. Thus if we show that $F_N(\lambda)\varphi\to F(\lambda)\varphi$ at all continuity points of F we will have proved the corollary. Since $T_1^{(N)}\varphi\to T_1\varphi$ this follows from a theorem of Rellich (See [11], p. 366).

References

1. A. Blanc-Lapierre and R. Fortet, *Théorie des functions aléatiores.* Paris, 1953.
2. N. Bourbaki, *Éléments de mathematique, Livre, VI, Integration.* Paris, 1952.
3. D. A. Darling and A. J. F. Siegert, *On the distribution of certain functionals of Markov chains and processes*, Proc. Nat. Acad. Sci. U. S. A., **42** (1956), 525–529.
4. Edwin Hewitt, *Integration on locally compact spaces, I*, University of Washington Publ. in Math., **3** (1952), 71–75.
5. E. Hewitt and H. S. Zuckerman, *Integration on locally compact spaces, II*, Nagoya Math. J., **3** (1951), 7–22.
6. E. Hille, *Functional analysis and semi-group.* Amer. Math. Soc. Colloquium Publications, **31**, 1948.
7. M. Kac, *On some connections between probability theory and differential and integral equations*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. Berkeley, 1951.
8. M. Loeve, *Probabilily theory*, New York, 1955.
9. R. S. Phillips, *Perturbation theory for semi-gvoups of linear operators*, Trans. Amer. Math. Soc. **74** (1953), 199–221.
10. D. Ray, *On spectra of second order differential operators*, Trans. Amer. Math. Soc. **77** (1954), 299–321.
11. F. Riesz and B. Sz. Nagy, *Leçons d'analyse fonctionnelle*, Budapest, 1953.
12. M. Rosenblatt, *On a class of Markov processes*, Trans. Amer. Math. Soc. **71** (1951), 120–135.

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