A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS OF VARIATIONS

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1. In the classical problems involving a simple integral

$$I_1=\int L(t,\, q^i,\, \dot q^i)\, dt\ , \qquad i=1, \cdots, n\ , \tag{1}$$

one is led to the consideration of the Pfaffian form

$$\omega=L\, dt+\frac{\partial L}{\partial \dot q^i}\,\omega^i=\frac{\partial L}{\partial \dot q^i}\,dq^i-\Big(\dot q^i\frac{\partial L}{\partial \dot q^i}-L\Big)\,dt \tag{2}$$

where

$$\omega^i=dq^i-\dot q^i\, dt\ .$$

For example this form ω is the one which gives rise to the "relative integral invariant" of E. Cartan.

In a recent note [1] L. Auslander characterizes the form ω by a theorem equivalent to the following one.

THEOREM 1. *Among all semi-basic forms θ such that*

$$\theta\equiv L\, dt \bmod \omega^i \tag{3}$$

the form ω of (2) *is the only one satisfying the condition*

$$d\theta\equiv 0 \bmod \omega^i\ . \tag{4}$$

In this, a *semi-basic form* is a form for which the local expression contains only the differentials of t, q^i(not of $\dot q^i$). The integral I is defined over an arc $\bar c$ of a space $\mathscr{W}$ with local coordinates t, q^i, $\dot q^i$ satisfying the equations $\omega^i=0$: Therefore in (1) the form $L\,dt$ may be replaced by any θ satisfying (3).

Condition (4) is a special case of a congruence discovered by Lepage [5]. *The purpose of the present note is to give a natural reason for this congruence which goes beyond its nice algebraic expression.*

Let us observe that the space $\mathscr{W}$ is the manifold of 1-dimensional contact elements of a manifold $\mathscr{V}$ with local coordinates t, q^i. The map

$$(t,\, q^i,\, \dot q^i)\rightarrow(t,\, q^i)$$

is then the local expression of the natural projection π: $\mathscr{W}\rightarrow\mathscr{V}$. We

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remark that we do not integrate (1) on any arc $\bar{c}$ in $\mathscr{W}$ satisfying $\omega^i=0$ but on such an arc the projection c of which in $\mathscr{V}$ is regular.

2. Let U be the domain in $\mathscr{V}$ of the coordinates t, q^i; then the $t, q^i, \dot{q}^i$ are defined in an open subset $W\subset \mathscr{W}$ of projection $\pi(W)=U$. If we denote by L_i n real undeterminates, we have coordinates $t, q^i, \dot{q}^i, L_i$ in $W\times R^n$; we then define in this product the Pfaffian form

$$\Omega_W=L\,dt+L_i\omega^i\ . \tag{5}$$

Now, let us cover $\mathscr{W}$ with open sets $W, W', \cdots$; this way we get a family of products $W\times R^n$, $W'\times R^n$, $\cdots$ with forms $\Omega_W, \Omega_{W'}, \cdots$. Using fibre bundle techniques, one proves that over a non-empty intersection $W\cap W'$ the products $W\times R^n$ and $W'\times R^n$ can be glued together in such a way that the forms induced on $W\cap W'\times R^n$ coincide. This yields a fibre bundle $E(\mathscr{W}, R^n)$ over $\mathscr{W}$ as base, with fibre R^n. This bundle is covered by open subsets isomorphic with the products $W\times R^n$ and in which the $t, q^i, \dot{q}^i, L_i$ are local coordinates; there is also on E a global Pfaffian form Ω of local expression (5). Combining the projections $E\to\mathscr{W}$ and $\mathscr{W}\to\mathscr{V}$ we obtain a map $E\to\mathscr{V}$ locally defined by

$$(t, q^i, \dot{q}^i, L_i)\to(t, q^i)\ .$$

We want to characterize in E the extremal arcs c^* of $\int\Omega$ *which have a regular projection in* $\mathscr{V}$.

An extremal arc c^* of $\int\Omega$ has to satisfy the local equations

$$\frac{\partial(d\Omega)}{\partial(dt)}=\frac{\partial(d\Omega)}{\partial(\omega^i)}=\frac{\partial(d\Omega)}{\partial(d\dot{q}^i)}=\frac{\partial(d\Omega)}{\partial(dL_i)}=0\ .$$

We have

$$d\Omega=\frac{\partial L}{\partial\dot{q}^i}\omega^i\wedge dt+\left(\frac{\partial L}{\partial\dot{q}^i}-L_i\right)d\dot{q}^i\wedge dt+dL_i\wedge\omega^i\ .$$

These equations are therefore

$$\omega^i=0\ ,\qquad\left(\frac{\partial L}{\partial\dot{q}^i}-L_i\right)dt=0\ ,\qquad\frac{\partial L}{\partial\dot{q}^i}dt-dL_i=0\ .$$

Since an arc c^* of regular projection in $\mathscr{V}$ cannot satisfy simultaneously $\omega^i=0$ and $dt=0$ it has to lie in the submanifold F of E locally characterized by

$$\frac{\partial L}{\partial\dot{q}^i}=L_i$$

or equivalently by condition (4).

THEOREM 2. *Every arc* c^* *in* E *for which* $\int \Omega$ *is stationary and the projection of which in* $\mathscr{V}$ *is regular necessarily lies in the submanifold* F *of* E *locally defined by the congruence* (4). *Furthermore the projection* c *of* c^* *in* $\mathscr{V}$ *extremizes in the classical sense the integral* (1). *Finally if* c *is a regular extremal are of* (1) *in* $\mathscr{V}$ *let* c^* *be the arc of* F *the projection* $\bar{c}$ *of which in* $\mathscr{W}$ *is the arc of tangent directions to* c; *then* c^* *extremizes* $\int \Omega$.

3. **The submanifold** F can be identified with $\mathscr{W}$ in an obvious way so that $\mathscr{W}$ can be considered as a submanifold of E. Then clearly Ω induces ω on $\mathscr{W}$.

THEOREM 3. *If the integral* (1) *is regular there exists a* (*one-to-one*) *correspondence between the regular extremal arcs* c *in* $\mathscr{V}$ *of* (1) *and the extremal arcs* $\bar{c}$ *of* $\int \omega$ *in* $\mathscr{W}$ *which have a regular projection in* $\mathscr{V}$. *Starting from an extremal* c, *the corresponding* $\bar{c}$ *is the arc the points of which are the tangent directions to* c; *starting from* $\bar{c}$ *the corresponding* c *is its projection in* $\mathscr{V}$.

In this statement, regularity of (1) means that the matrix $(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j)$ is everywhere non singular.

Theorem 2 and 3 give a complete justification of condition (4). Theorem 3 was actually proved by E. Cartan [2]. These theorems are special cases of similar theorems involving multiple integrals and even those in which the function L depends on higher order contact elements. Theorem 2 was first proved by the author [3], as well as the alluded generalizations.

Combining Theorems 2 and 3 yields the following.

THEOREM 4. *In the regular case, every arc* $\bar{c}$ *in* $\mathscr{W}$ *of regular projection in* $\mathscr{V}$ *which extremizes* $\int \omega$ *with respect to variations confined to* $\mathscr{W}$ *does also extremize* $\int \Omega$ *with respect to variations in the larger space* E.

4. **There is a last question** to be answered: why in Theorem 1 restrict oneself to semi-basic forms?

We can only add to $L.dt$ a linear combination of Pfaffian forms vanishing with ω^i; every such form is a linear combination of the ω^i

and is therefore semi-basic. Hence the restriction to semi-basic forms in Theorem 1 was actually redundant.

However, as mentioned above and as I have proved in various papers (e.g. [3, 4]), the above properties generalize to a multiple integral

$$I_p=\int L(t^\alpha, q^i, q^i_\alpha)\, dt\ , \tag{6}$$

$$dt=dt^1\wedge\cdots\wedge dt^p, \qquad \alpha=1, 2, \cdots, p; \qquad i=1, 2, \cdots, n\ ,$$

to be integrated over a p-surface c defined by $q^i=q^i(t^\alpha)$ and where q^i_α stands for $\partial q^i/\partial t^\alpha$. Then $\mathscr{V}$ is of dimension $n+p$ and $\mathscr{W}$ (which is geometrically the manifold of p-dimensional contact elements of $\mathscr{V}$) is of dimension $n+p+np$. We can consider that we integrate (6) in $\mathscr{W}$ over a p-surface $\bar{c}$ of regular projection in $\mathscr{V}$ and solution of the Pfaffian equations

$$\omega^i=dq^i-\sum q^i_\alpha\, dt^\alpha=0\ .$$

Such a p-surface $\bar{c}$ is formed of the contact elements of dimension p to a regular p-surface in $\mathscr{V}$ and will be called a *p-multiplicity.*

Now in (6) we can add to $L.dt$ any p-form vanishing on all p-multiplicities and all such forms are no longer semi-basic if $p>1$: for example $d\omega^i\wedge dt^3\wedge\cdots\wedge dt^p$ is such one. Nevertheless, the semi-basic forms satisfying the Lepage congrences [5]:

$$\theta\equiv L\, dt \qquad \mathrm{mod}\ \omega^i\ , \tag{7}$$

$$d\theta\equiv 0 \qquad \mathrm{mod}\ \omega^i\ . \tag{8}$$

play an important role for a deeper reason which is actually a *transversality condition.* We briefly discuss this below referring the reader to my memoir [4] for further details.

5. Let $\mathscr{K}$ be a p-dimensional manifold and K a domain of $\mathscr{K}$ with regular boundary K. A map

$$c:\ K\to\mathscr{V}$$

is a domain of integration of (6); it gives rise canonically to a map

$$\bar{c}:\ K\to\mathscr{W}$$

such that for $k\in K$, $\bar{c}(k)$ is the contact element of dimension p to c at k. *A variation* (or *homotopy*) of c is a family of maps

$$c_t:\ K\to\mathscr{V}, \qquad t\in R\ , \qquad c_0=c\ ;$$

this yields a variation of $\bar{c}$:

$$\bar{c}_t: K \to \mathscr{W}.$$

We also define $C: K \times R \to \mathscr{V}$, $\overline{C}: K \times R \to \mathscr{W}$ by

$$C(k, t) = c_t(k) , \qquad \overline{C}(k, t) = \bar{c}_t(k) .$$

The corresponding variation of $\int \theta$ is then

$$\Delta = \int_{\bar{c}_t} \theta - \int_{\bar{c}_0} \theta$$

which may be expressed as a sum of two terms:

$$\Delta = \int_{\overline{C}_{0t}} d\theta + \int_{\lambda_{0t}\overline{C}} \theta . \tag{9}$$

The domains of integration $\overline{C}_{0t}$ and $\lambda_{0t}\overline{C}$ are the restrictions of $\overline{C}$ to $K \times I_{0t}$ and $\dot{K} \times I_{0t}$ respectively (where $I_{0t} = [0, t] \subset R$). We say that the variation $\overline{C}$ is *transversal* to θ if this form vanishes on $\lambda\overline{C}$ (restriction of $\overline{C}$ to $\dot{K} \times R$). This being the case, the last integral (or boundary term) in (9) is zero.

Now the variations usually considered are those for which the restriction of C to $\dot{K}$ is constant (fixed boundary variations): for those, $\lambda\overline{C}$ has an everywhere non-regular projection in $\mathscr{V}$, so that every semi-basic form vanishes on $\lambda\overline{C}$. Therefore if we replace in (6) $L.dt$ by a semi-basic p-form θ satisfying (7), all variations with fixed boundary are transversal to it. This would of course not be the case, should we add to $L.dt$ a non-semi-basic p-form vanishing on all p-multiplicities.

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