NUMERICAL SOLUTION OF VIBRATION PROBLEMS IN TWO SPACE VARIABLES

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1. Introduction. The classical theory of vibrating plates leads to the following non-dimensional fourth order partial differential equation in two space variables $W(x, y, t)$ for the transverse vibrations:

$$(1) \qquad \Delta\Delta W + W_{tt} = 0 ,$$

where $\Delta\Delta$ is the biharmonic operator

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} .$$

Solutions of this equation for two dimensional regions of arbitrary shape are of course not known, but even for those plate problems for which analytic solutions in series form for this equation are available, the series do not lend themselves easily to numerical calculations. Direct numerical solutions of this equation are therefore of considerable impotance. It is the purpose of this paper to present a new finite difference approximation to this equation which is stable for all values of the mesh ratios $\overline{\Delta t}/\overline{\Delta x}^2$ and $\overline{\Delta t}/\overline{\Delta y}^2$ and which involves an amount of work which is entirely feasible on large-scale digital computers. The method is a generalization of a method prepared by Douglas and Rachford [1] for solving the two dimensional diffusion equation.

2. The differential and difference equations. We consider first the specific problem of determining the transverse vibrations of a square homogeneous thin plate hinged at its boundaries and subjected to an arbitrary initial condition. The boundary value problem may be written

(2)

a) $\dfrac{\partial^4 W}{\partial x^4} + 2\dfrac{\partial^4 W}{\partial x^2 \partial y^2} + \dfrac{\partial^4 W}{\partial y^4} + \dfrac{\partial^2 W}{\partial t^2} = 0 , \qquad (x, y) \in R , \quad 0 \leqq t \leqq T ,$

b) $W(x, y, 0) = f(x, y) , \qquad (x, y) \in R ,$

c) $W_t(x, y, 0) = 0 , \qquad (x, y) \in R ,$

d) $W(x, y, t) = \dfrac{\partial^2 W}{\partial x^2}(x, y, t) = 0 , \qquad$ at $x=0, 1$ for $0<y<1, \ t>0 ,$

e) $W(x, y, t) = \dfrac{\partial^2 W}{\partial y^2}(x, y, t) = 0 , \qquad$ at $y=0, 1$ for $0<x<1, \ t>0 ,$

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where R is the open region $[0<x<1,\ 0<y<1]$. Letting $\Delta x=\Delta y=1/M$ we now lay a mesh over the region R and we introduce the following typical notation for difference operators

$$
\begin{aligned}
&w(i\Delta x, j\Delta y, n\Delta t)=w_{ijn}\ , \\
(3)\qquad &\Delta_t^2 w_{ijn}=(w_{i,j,n+1}-2w_{ijn}+w_{i,j,n+1})/\overline{\Delta t^2}\ , \\
&\Delta_x^4 w_{ijn}=(w_{i+2,j,n}-4w_{i+1,j,n}+6w_{ijn}-4w_{i-1,j,n}+w_{i-2,j,n})/\overline{\Delta x^4}\ .
\end{aligned}
$$

We now approximate (2) by the following finite difference system:

$$
\begin{aligned}
&\text{a)}\quad \frac{1}{2}\Delta_x^4[w^*_{i,j,n+1}+w_{i,j,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\Delta_y^4 w_{ijn} \\
&\qquad\qquad +\frac{w^*_{ijn+1}-2w_{ijn}+w_{i,j,n-1}}{\Delta t^2}=0\ , \\
&\text{b)}\quad \frac{1}{2}\Delta_y^4[w_{i,j,n+1}+w_{ij,n-1}]=\Delta_y^4 w_{ijn}-\frac{w_{i,j,n+1}-w^*_{i,j,n+1}}{\Delta t^2}\ , \\
(4)\qquad &\qquad\qquad (i\Delta x, j\Delta y)\in R'\ ,\quad 0\leqq n\Delta t\leqq T\ , \\
&\text{c)}\quad w_{i,j,0}=W_{ij0}=f_{ij}\ , \qquad (i,j=1,2,\cdots,M-1)\ , \\
&\text{d)}\quad w_{i,j,1}=W_{i,j,0}\ , \qquad (i,j=1,2,\cdots,M-1)\ , \\
&\text{e)}\quad \left.\begin{cases} w_{0,j,n}=w_{M,j,n}=0 \\ w_{i+1,j,n}=-w_{i-1,j,n}\ (i=0,M)\end{cases}\right\}\ (j=1,\cdots,M-1\ ;\ 0\leqq n\Delta t\leqq T)\ , \\
&\text{f)}\quad \left.\begin{cases} w_{i,0,n}=w_{i,M,n}=0 \\ w_{i,j+1,n}=-w_{i,j-1,u}\ (j=0,M)\end{cases}\right\}\ (i=1,\cdots,M-1\ ;\ 0\leqq n\Delta t\leqq T)\ ,
\end{aligned}
$$

where R' is the set of lattice points $(i\Delta x, j\Delta y)$ in R and in condition e) and f) $w^*_{ijn}=w_{ijn}$.

Equation 4a) is implicit in x alone while equation 4b) is implicit in y alone. The numerical procedure consists of first solving equations 4a) to obtain $w^*_{ij,n+1}$. A system of $(M-1)$ equations in $(M-1)$ unknowns is obtained for the unknowns along a single line in the x-direction. The matrix of this system of equations has at most 5 non-zero elements in any one row (either on the main diagonal or on two adjacent diagonals). We shall call such matrices quidiagonal. These quidiagonal systems can be solved efficiently by an extension of an algorithm for solving tridiagonal matrices due to L. H. Thomas and involve about twice the amount of work as for tridiagonal matrices.

Use of equation 4a) above, however, is not sufficient to yield good values of w over a wide range in t because as will be shown the finite difference approximation is unstable. Equation 4b) then provides a corrective process which combined with 4a) does provide a stable, convergent process. Equation 4b) is implicit along lines parallel to the y-axis and again for rectangular regions yields $M-1$ systems of equations each involving $M-1$ unknowns. The matrices of these equations are again quidiagonal in form.

By eliminating $w^*_{ij,n+1}$ from equations 4a) and 4b) we obtain the following implicit finite difference equation

$$(5)\quad \frac{1}{2}\Delta_x^4[w_{ij,n+1}+w_{ij,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\frac{1}{2}\Delta_y^4[w_{ij,n+1}+w_{ij,n-1}] + \frac{w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}}{\Delta t^2}+\frac{1}{4}\overline{\Delta t}^2\Delta_x^4\Delta_y^4[w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}]=0\ ,$$

which lends itself more readily to a stability and convergence analysis.

3. Stability considerations. Let $v(x, y, t)$ be the error due to round-off. Then since equation (5) is linear, it follows that v_{ijn} will satisfy the system

$$\text{a)}\quad \frac{1}{2}\Delta_x^4[v_{ij,n+1}+v_{ij,n-1}]+2\Delta_x^2\Delta_y^2 v_{ijn}+\frac{1}{2}\Delta_y^4[v_{ij,n+1}+v_{ij,n-1}] + \frac{v_{ij,n+1}-2v_{ijn}+v_{ij,n-1}}{\Delta t^2}+\frac{1}{4}\overline{\Delta t}^2\Delta_x^4\Delta_y^4[v_{ij,n+1}-2v_{ijn}+v_{ij,n-1}]=0\ ,$$

$$(6)\quad \text{b)}\quad v_{i,j,0} \text{ and } v_{i,j,1} \text{ arbitrary} \qquad (i,j=1,\cdots,M-1)\ ,$$

$$\text{c)}\quad \begin{cases} v_{0,j,n}=v_{M,j,n}=0 \\ v_{i+1,j,n}=-v_{i-1,j,n}\ (i=0,M) \end{cases} \qquad (j=1,\cdots,M-1\ ;\ 0\leqq n\Delta t\leqq T)\ ,$$

$$\text{d)}\quad \begin{cases} v_{i,0,n}=v_{i,M,n}=0 \\ v_{i,j+1,n}=-v_{i,j-1,n}=0\ \ (j=0,M) \end{cases} \qquad (i=1,\cdots,M-1\ ;\ 0\leqq n\Delta t\leqq T)\ .$$

The eigenfunctions of (6) are of the form

$$v_{ijn}=a_n \sin \pi p x_i \sin \pi q y_j\ , \qquad p,q=1,\cdots,M-1\ ,$$

where $x_i=i\Delta x$, $y_j=j\Delta y$. It is easily shown that, for example,

$$\overline{\Delta x}^4\Delta_x^4 v_{ijn+1}=16\sin^4 \pi p\frac{\Delta x}{2}v_{ij,n+1}\ ,$$

$$\overline{\Delta x}^2\overline{\Delta y}^2\Delta_x^2\Delta_y^2 v_{ijn}=16\sin^2 p\pi\frac{\Delta x}{2}\sin^2 q\pi\frac{\Delta y}{2}v_{ijn}\ .$$

Applying this to equation (6a) and rearranging we obtain the following recurrence relation in a_n:

$$(7)\quad a_{n+1}-2\alpha a_n+a_{n-1}=0\ ,$$

where

$$(8)\quad \alpha=\frac{1+\rho^2 s_p^4 s_q^4-2\rho s_p^2 s_q^2}{1+\rho^2 s_p^4 s_q^4+\rho(s_p^4+s_q^4)}=\frac{(1-\rho s_p^2 s_q^2)^2}{(1-\rho s_p^2 s_q^2)^2+\rho(s_p^2+s_q^2)^2}$$

and $s_p=\sin\frac{p\pi}{2M}$, $s_q=\sin\frac{q\pi}{2M}$, $\rho=8\frac{\Delta t^2}{\Delta x^4}=8\frac{\Delta t^2}{\Delta y^4}=8r^2$. The difference equation (6) will be stable provided that the roots of the characteristic equation

$$\xi^2-2\alpha\xi+1=0$$

corresponding to (7) are at most equal to one in absolute value. These roots are equal to one in absolute value if $|\alpha|\leqq 1$, a condition which follows at once from the definitions of s_p, s_q and ρ. Thus the finite difference system (4) is stable for all values of the mesh ratio ρ and for all values of p and q.

It should be pointed out that if (2) is replaced by an explicit finite difference approximation, a stability analysis leads to the requirement that

$$r=\frac{\Delta t}{\Delta x^2}=\frac{\Delta t}{\Delta y^2}\leqq 1/4\ .$$

This restriction on the time step ordinarily leads to an amount of computing time which is not feasible even with the most modern computers. On the other hand a straightforward implicit finite difference approximation to (2), while simpler than (4) and also stable for all values of the mesh ratio, leads to a system of $(M-1)^2$ equations in $(M-1)^2$ unknowns which must be solved at each time step. Even a 20×20 interior grid leads to a system of 400 equations in 400 unknowns again involving an unreasonable amount of computing time.

Finally if one attempts to use 4a) without the corrective equation 4b) the same stability analysis given above leads to the characteristic equation

$$\xi^2-2\beta\xi+1=0$$

where

$$\beta=\frac{1-\rho(s_q^4+2s_p^2s_q^2)}{1+\rho s_p^4}$$

It is easily verified that for some values of p and q, $|\beta|>1$ and hence equation 4a) is not stable for all values of ρ.

4. Treatment of other boundary conditions. The stability analysis of § 3 depends upon the existence of a set of eigenfunctions of the difference operator given in (6a) which satisfy the boundary conditions (6c) and (6d). If the boundary conditions (2d) and (2e) corresponding to the difference conditions (6c) and (6d) change, the eigenfunctions of the system (6) will also change. Let us consider then the error equation (6a) with the boundary conditions (6c), (6d) replaced by the general homogeneous conditions:

$$L_m(v_{ijn})=0\ , \qquad (i,j)\in S^1\ .\ (m=1,2,3,4)\ , \tag{9}$$

where S^1 is the set of boundary points affected by the conditions L_m.

Assume a set of eigenfunctions of (6a) of the form

$$v_{ijn}(p,q)=a_n\phi_{ij}(p,q)\ , \qquad p,q=1,\cdots,M-1\ .$$

Substituting into (6a) and rearranging, we obtain

$$a_{n+1}-2\alpha_{pq}a_n+a_{n-1}=0$$

where

$$\alpha_{pq}=\frac{-\overline{\Delta t}^2\Delta_x^2\Delta_y^2\phi_{ij}+\phi_{ij}+\frac{1}{4}\overline{\Delta t}^4\Delta_x^4\Delta_y^4\phi_{ij}}{\frac{1}{2}\overline{\Delta t}^2(\Delta_x^4\phi_{ij}+\Delta_y^4\phi_{ij})+1+\frac{1}{4}\overline{\Delta t}^4\Delta_x^4\Delta_y^4\phi_{ij}}=\frac{H[\phi_{ij}]}{K[\phi_{ij}]}$$

Now let H and K have a common set of eigenfunctions subject to the condition (9), i. e.

$$H[\phi_{ij}]=\lambda_{pq}\phi_{ij}\ ,\quad L_m(\phi_{ij})=0\ ,$$

$$K[\phi_{ij}]=\beta_{pq}\phi_{ij}\ ,\quad L_m(\phi_{ij})=0\ .$$

We then have

$$\alpha_{pq}=\frac{\lambda_{pq}}{\beta_{pq}}$$

and the condition for stability is simply that for all p and q

$$|\alpha_{pq}|\leqq 1\ .$$

Thus the stability analysis of § 3 can be applied for any boundary conditions for which the operators H and K have common eigenfunctions.

5. A mean square convergence theorem for the square region. For the problem considered in § 2 assume that the function $f(x,y)$, is sufficiently regular in the closed region $\bar{R}$ to guarantee the existence and boundedness of

$$\frac{\partial^6 w}{\partial x^6}\ ,\ \frac{\partial^6 w}{\partial y^6}\ ,\ \frac{\partial^6 w}{\partial y^4\partial t^2}\ ,\ \frac{\partial^6 w}{\partial x^4\partial t^2}\ ,\ \frac{\partial^4 w}{\partial t^2}$$

in $\bar{R}$. Then it can be shown following the usual series expansions that

(10) $$\frac{1}{2}\Delta_x^4[w_{ij,n+1}+w_{ij,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\frac{1}{2}\Delta_y^4[w_{ij,n+1}+w_{ij,n-1}]$$
$$+\frac{w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}}{\overline{\Delta t}^2}=0(\overline{\Delta x}^2+\overline{\Delta t}^2)\ ,$$

and moreover that

(11) $$\overline{\Delta t}^2\Delta_x^4\Delta_y^4[w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}]=0(\overline{\Delta t}^2)\ .$$

Hence the difference operator (5) approximates the differential equation (2) to terms which are $0(\overline{\Delta x}^2+\overline{\Delta t}^2)$. In the notation of [2] the elementary truncation error h_{ijn} is

(12) $$h_{ijn}=0(\overline{\Delta x}^6+\overline{\Delta x}^4\overline{\Delta t}^2)$$

and by Theorem 1 of [2] we have

(13) $$\| W_{ijn}-w_{ijn}\|=0\left(\frac{\overline{\Delta x}^6}{\overline{\Delta t}^2}+\overline{\Delta t}^4\right)$$

uniformly in n, where

(14) $$\| W_{ijn}-w_{ijn}\|=\frac{1}{(M-1)}\left\{\sum_{i,j=1}^{M-1}| W_{ijn}-w_{ijn}|^2\right\}^{1/2}$$

It thus follows that if the boundary value problem (2) is sufficiently well defined in the sense that the derivatives mentioned above exist boundedly in the closed region $\overline{R}$, then the solution of (4) converges in the mean to the solution of (2) with errors given by (13) as Δx and Δt tend to zero.

The convergence proof given above holds for a rectangular region only. In practice one is usually interested in point-wise convergence rather than convergence in the mean square sense. Section 6 establishes point-wise convergence of the solution of the difference system to the solution of the differental system.

6. Point-wise convergence. A solution of the boundary value problem (2) can be given in series form

(15) $$W(x, y, t)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}A_{pq}\sin p\pi x\sin q\pi y\cos (p^2+q^2)\pi^2 t\ .$$

The initial condition (2b) will be satisfied provided that A_{pq} are taken to be the Fourier coefficients of $f(x, y)$, i.e.

(16) $$A_{pq}=4\int_0^1\int_0^1 f(x, y)\sin p\pi x\sin q\pi y\, dxdy\ .$$

The conditions on $f(x, y)$ are assumed to be such that the series (15) converges and is the unique solution of the boundary value problem (2). A solution $w(x, y, t)$ of the finite difference system consisting of (4a, b, e, f) can be obtained by separation of variables as follows:

$$w(x, y, t)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} B_{pq} \sin p\pi x \sin q\pi y \cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1(p, q)}{\lambda_2(p, q)}, \tag{17}$$

where

$$\lambda_1(p, q)=(1-\rho s_p^2 s_q^2)^2,$$
$$\lambda_2(p, q)=(1-\rho s_p^2 s_q^2)^2+\rho(s_p^2+s_q^2)^2,$$
$$\Delta x=\Delta y=1/M,$$

and B_{pq} are arbitrary constants. The series (17) satisfies the finite difference system (4) except for the initial condition 4c).

We will now show that it is possible to choose the coefficients B_{pq} so that the solution $w(x, y, t)$ of the difference system will converge to the solution $W(x, y, t)$ of the differential system as $M\to\infty$. We first define an integer $k(M)$ such that $k(M)<M^{1/5}$ and $\lim_{M\to\infty} k(M)=\infty$. We then choose the B_{pq} so that $B_{pq}=0$ for $p>k(M)$, $q>k(M)$ and the remaining B_{pq} so that for any $\varepsilon>0$ there exists an $M_1(\varepsilon)$ such that for $M>M_1$,

$$|B_{pq}-A_{pq}|<\varepsilon M^{-2/5} \text{ uniformly for } p, q=1, \cdots, k(M). \tag{18}$$

One way of satisfying (18) for instance is to choose $B_{pq}=A_{pq}$ for $p, q=1, \cdots, k(M)$. An exact solution of the difference equation then is

$$w_M(x, y, t)=\sum_{p=1}^{k(M)}\sum_{q=1}^{k(M)} B_{pq} \sin p\pi x \sin q\pi y \cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1}{\lambda_2}. \tag{19}$$

This solution satisfies the initial condition

$$w_M(x, y, 0)=\sum_{p=1}^{k}\sum_{q=1}^{k} B_{pq} \sin p\pi x \sin q\pi y \tag{20}$$

and of course does not satisfy the exact initial condition $w(x, y, 0)=f(x, y)$. However, it will satisfy this initial condition in the limit as $M\to\infty$.

LEMMA 1. *For any* $\rho>0$, $0\leqq z_1\leqq\frac{\pi}{2}M^{-4/5}$, $0\leqq z_2\leqq\frac{\pi}{2}M^{-4/5}$, *there exists an* $M_2(\rho)$ *such that for* $M>M_2$ *and for any* $\varepsilon>0$

$$\left|4r(z_1^2+z_2^2)-\operatorname{arc\,cos}\frac{\lambda_1}{\lambda_2}\right|\leqq\frac{r\varepsilon}{M^3}, \tag{21}$$

where

$$\lambda_1=(1-\rho \sin^2 z_1 \sin^2 z_2)^2 ,$$

$$\lambda_2=(1-\rho \sin^2 z_1 \sin^2 z_2)^2+\rho(\sin^2 z_1+\sin^2 z_2)^2 .$$

Proof. We first choose $M_3(\rho)$ such that $M>M_3$ and for all admissible z_1, z_2

$$1-\rho \sin^2 z_1 \sin^2 z_2>0 . \tag{22}$$

Let

$$F(z_1, z_2)=4r(z_1^2+z_2^2)-\text{arc cos}\frac{\lambda_1}{\lambda_2} .$$

It is obvious that $F(0, 0)=0$ and it can be shown by direct calculation that the partial derivatives of $F(z_1, z_2)$ up to and including those of order 3 all vanish at $z_1=0$, $z_2=0$. Thus in the Taylor series expansion of $F(z_1, z_2)$ the remainder term is

$$R(z_1, z_2)=a_{41}z_1^4+a_{42}z_1^3z_2+a_{43}z_1^2z_1^2+a_{44}z_1z_2^3+a_{45}z_2^4 ,$$

where the coefficients $a_{4i}(\bar{z}_1, \bar{z}_2)$, $i=(1, \cdots, 5)$, are related to the fourth derivatives of $F(z_1, z_2)$ and $0<\bar{z}_1<\frac{\pi}{2}M^{-4/5}$, $0\leqq\bar{z}_2<\frac{\pi}{2}M^{-4/5}$. Using the inequality (22) it is possible to show that the a_{4i} are bounded functions of ρ. Thus using the extreme limits of z_1, z_2 we have

$$|F(z_1, z_2)|\leqq|R(z_1, z_2)|\leqq A(\rho) . \frac{\pi^4}{2^4}\cdot M^{-16/5} ,$$

and hence it follows that there exists an $M_2(\rho)$ such that for $M>M_2$,

$$|F(z_1, z_2)|<\frac{\varepsilon r}{M^3} ,$$

as the lemma asserts.

Now multiplying (21) by $\frac{M^2t}{r}$ and putting $z_1=\frac{p\pi}{2M}$, $z_2=\frac{q\pi}{2M}$ we have

$$\left|\frac{M^2t}{r}\text{ arc cos }\frac{\lambda_1(p, q)}{\lambda_2(p, q)}-\pi^2(p^2+q^2)t\right|<\frac{\varepsilon t}{M} ,$$

and therefore

$$\left|\cos\frac{M^2t}{r}\text{ arc cos }\frac{\lambda_1(p, q)}{\lambda_2(p, q)}-\cos \pi^2(p^2+q^2)t\right|<\frac{\varepsilon t}{M} . \tag{23}$$

THEOREM 1 (THE CONVERGENCE THEOREM). *Under the assumptions*

a) $t>0\,,\quad 0<x<1\,,\quad 0<y<1\,;\quad \rho>0\,;\quad \rho, t, x, y$ fixed ;

b) $|A_{pq}|\leqq P\,,$ P constant, for all $p, q=1, 2, \cdots, \infty$;

c) $k(M)<M^{1/5}\,,\quad \lim_{M\to\infty} k(M)=\infty$;

d) $|B_{pq}-A_{pq}|<\varepsilon M^{-2/5}\,,$ $1\leqq p, q\leqq k(M)$,

we have

$$\lim_{M\to\infty} w_M(x, y, t)=W(x, y, t)$$

or

$$\lim_{M\to\infty}\sum_{p=1}^{k(M)}\sum_{q=1}^{k(M)} B_{pq}(M)\sin p\pi x\sin q\pi y\cos\frac{M^2t}{r}\operatorname{arc}\cos\frac{\lambda_1(p, q)}{\lambda_2(p, q)}$$
$$=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} A_{pq}\sin p\pi x\sin q\pi y\cos(p^2+q^2)\pi^2 t\ .$$

Proof.

$$w_M(x, y, t)-W(x, y, t)=\sum_{p=1}^{k}\sum_{q=1}^{k}(B_{pq}-A_{pq})\sin p\pi x\sin q\pi y\cos(p^2+q^2)\pi^2 t$$
$$+\sum_{p=1}^{k}\sum_{q=1}^{k}(B_{pq}-A_{pq})\sin p\pi x\sin q\pi y\left[\cos\frac{M^2t}{r}\operatorname{arc}\cos\frac{\lambda_1}{\lambda_2}-\cos(p^2+q^2)\pi^2 t\right]$$
$$+\sum_{1}^{k}\sum_{1}^{k} A_{pq}\sin p\pi x\sin q\pi y\left[\cos\frac{M^2t}{r}\operatorname{arc}\cos\frac{\lambda_1}{\lambda_2}-\cos(p^2+q^2)\pi^2 t\right]$$
$$+\sum_{k+1}^{\infty}\sum_{k+1}^{\infty} A_{pq}\sin p\pi x\sin q\pi y\cos(p^2+q^2)\pi^2 t\ ,$$
$$=I_1+I_2+I_3+I_4\ .$$

By conditions c) and d) above and Lemma 1,

$$|I_1|\leqq k^2(M)\cdot\varepsilon M^{-2/5}\leqq\varepsilon\ ,$$

$$|I_2|\leqq k^2(M)\cdot\varepsilon M^{-2/5}\frac{\varepsilon t}{M}\leqq\frac{\varepsilon^2}{M}<\varepsilon^2 t\ .$$

By condition b) and c) and Lemma 1,

$$|I_3|\leqq P\cdot k^2(M)\cdot\frac{\varepsilon t}{M}\leqq P\varepsilon t\ ,$$

and because the series for $w(x, y, t)$ converges there exists an M_4 such that for $M>M_4$

$$|I_4|<\varepsilon\ .$$

Thus for $M > \max(M_1, M_2, M_4)$,

$$|w_M(x, y, t) - W(x, y, t)| \leqq \varepsilon(2 + \varepsilon t + Pt) .$$

This establishes the convergence theorem.

References

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