A MEAN VALUE THEOREM FOR QUADRATIC FIELDS

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1. Introduction. Let K be an algebraic extension of the rationals of degree k, $F(n)$ denote the number of ideals whose norm is the rational integer n, $H(x)=\sum_{n\leqq x}F(n)$. Let $\zeta(s, K)$ denote the Dedekind zeta function for the field K, that is,

$$\zeta(s, K)=\sum_{\mathfrak{A}}\frac{1}{N(\mathfrak{A})s}=\sum_{n=1}^{\infty}\frac{F(n)}{n^s}$$

and α the residue of $\zeta(s, K)$ at its simple pole at $s=1$.

It has long been known [8] that

$$H(x)=\alpha x+\varDelta_k(x)$$

where

$$\varDelta_k(x) = 0(x^{1-1/k})$$

and Landau [3] proved that

$$\varDelta_k(x) = 0(x^{1-2/(k+1)})$$

The precise nature of the error term $\varDelta_k(x)$ seems rather intractable and seems to be intimately related to the behavior of the function $\zeta(s, K)$ in the critical strip. Of considerable interest is the particular case when K is the Gaussian field $R(i)$, for in that case $\varDelta_k(x)$ is the error term in the classical problem of the number of lattice points in a circle.

Using some results of class field theory, Suetuna [4] has obtained an improvement of Landau's result in the case when the field is normal and has abelian Galois group and $k\geqq 4$. For, when the field is abelian, the theorems of Weber-Takagi tell us that $\zeta(s, K)$ is the product of k Dirichlet L-functions belonging to primitive characters. Applying his approximate functional equation for the Dirichlet L-functions, and using refined estimates for these in the critical strip, Suetuna then obtains the desired result.

In the light of more recent techniques for dealing with the Riemann zeta function, further improvements are possible. The devices for handing the zeta function are used for the L-functions and the class

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field theorems again applied. We omit details.

It is our object here to study the problem of a mean value for $\Delta_k(x)$. We are able to obtain a precise result but only for quadratic fields. Some known results follow as corollaries when the quadratic field is specified: for example when the field is $R(i)$.

We use as our tools a result of Suetuna [4] and a technique devised by Titchmarsh [5], [6] for the corresponding results for the closely allied problem of $\sum_{n\leq x} d_k(n)$ where $d_k(n)$ is the number of solutions of $n=n_1 n_2\cdots n_k$. We follow closely Titchmarsh's method.

2. Notations and statement of Main theorem. Let $k=2$, $\Delta_2(x)=\Delta(x)$, $s=\sigma+it$. Following Hardy [6], we define the mean value of $\Delta(x)$ as the least number β such that

$$\frac{1}{x}\int_0^x \Delta^2(t)dt=0(x^{2\beta+\varepsilon})$$

It is our object to prove the following.

MAIN THEOREM. $\beta=\dfrac{1}{4}$.

We first relate β to the convergence of an integral.

THEOREM 2.1 *Let γ be the lower bound of positive numbers σ for which*

$$I=\int_{-\infty}^{\infty}\frac{|\zeta(\sigma+it, K)|^2}{|\sigma+it|^2}dt \tag{1}$$

converges. Then $\beta=\gamma$ and if $\sigma>\beta$, then

$$2\pi\int_0^\infty \Delta^2(x)x^{-2\sigma-1}dx=I \tag{2}$$

Proof. Using the classical formula for the sum of the coefficients of a Dirichlet series, we have,

$$H(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s, K)}{s}x^s ps \qquad (c>1)$$

$$=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\zeta(s, K)}{s}x^s ds$$

We move the line of integration to $\sigma=\delta$, where $0<\delta<1$. Using Cauchy's theorem and taking account of the residue at $s=1$, we get, if δ is chosen appropriately close to 1,

$$\Delta(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\delta-iT}^{\delta+iT} \frac{\zeta(s, K)}{s} x^s ds$$

The bound on δ follows from the implied calculation but we do not need it since we now prove the validity of (3) for the range $\gamma<\delta<1$. To do this, we note that by some general theorems of analysis [2], and taking account of (1), $\frac{\zeta(s, K)}{s}$ tends uniformly to 0 as $t\to\pm\infty$. With this established, we integrate around the rectangle defined by $\delta'-iT$, $\delta-iT$, $\delta+iT$, $\delta'+iT$ with $\gamma<\delta'<\delta<1$, let $T\to\infty$, and deduce the desired result.

With Titchmarsh, we now use the theory of Mellin transforms. The Parseval theorem for the Mellin integral gives [7],

$$(4) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\delta+it), k|^2}{|\delta+it|^2} = \int_0^\infty \Delta^2\left(\frac{1}{x}\right) x^{2\delta-1} dx$$

$$= \int_0^\infty \Delta^2(x) x^{-2\delta-1} dx$$

as long as $\gamma<\delta<1$. This implies that $\beta\leqq\gamma$: by (4)

$$\int_{\frac{x}{2}}^{x} \Delta^2(x) x^{-2\delta-1} dx < C(\delta) = C_1$$

that is,

$$\int_{\frac{x}{2}}^{x} \Delta^2(x) dx < C_2 x^{2\delta+1}$$

Replacing x by $x/2$, $x/4$, $x/8,\cdots$ and adding, we deduce

$$\int_1^x \Delta^2(x) dx < C_3 x^{2\delta+1}$$

whence $\beta\leqq\delta$, that is, $\beta\leqq\delta$.

To prove the reverse inequality, we have by Plancherel's form of the inverse Mellin transform [7],

$$(5) \qquad \frac{\zeta(s, K)}{s} = \int_0^\infty \Delta(x) x^{-s-1} dx$$

where the right hand integral exists in the mean square sense for $\gamma<\delta<1$. Actually the right hand side is uniformly convergent for the range $\beta'<\sigma<\beta''$ where $\beta<\beta'<\beta''<1$. For, using the Schwartz inequality,

$$\int_{\frac{x}{2}}^{x}|\Delta(x)|x^{-\sigma-1}dx \leqq \left(\int_{\frac{x}{2}}^{x}\Delta^2(x)dx\right)^{1/2}\left(\int_{\frac{x}{2}}^{x}x^{-2\sigma-2}\right)^{1/2} = O(x^{2\beta+1+\varepsilon})^{1/2}(x^{-2\sigma-1})^{1/2} = O(x^{\beta-\sigma+\varepsilon})$$

putting $x=2, 4, 8, \cdots$ and adding, we get

$$\int_1^{\infty}|\Delta(x)|x^{-\sigma-1}dx < \infty$$

By a similar argument,

$$\int_0^{\infty}\Delta^2(x)x^{-2\delta-1}$$

converges for $\beta<\delta<1$. Now

$$\int_0^{\infty}\Delta(x)x^{-s-1}dx$$

is an analytic function for $\beta<\sigma<1$, and hence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\delta+it, K)|^2}{|\delta+it|^2}dt<\infty$$

for $\beta<\delta<1$, whence $\gamma\leqq\delta$ that is, $\gamma\leqq\beta$ and the theorem is proved.

So far we have not made use of the condition $k=2$. Indeed Titchmarsh's method applies in quite a general setting. We require the condition however in the proof of the main theorem.

LEMMA (Suetuna). *If* $\sigma>\frac{1}{2}$, *then*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(s, K)|^2dt=\sum_{n=1}^{\infty}\frac{F^2(n)}{n^{2\sigma}}$$

Proof of the Main Theorem. We first prove that $\beta\geqq\frac{1}{4}$. By the above lemma, we have for $\frac{1}{2}<\sigma<1$,

$$\int_{\frac{T}{2}}^{T}|\zeta(s, K)|^2<CT$$

Therefore for $0<\sigma<\frac{1}{2}$, $T>1$ and using Hecke's functional equation for $\zeta(s, K)$, (see for example. Landau [3]), we get

$$\int_{-\infty}^{\infty}\frac{|\zeta(\sigma+it, K)|^2}{|\sigma+it|^2}dt>\int_{\frac{T}{2}}^{T}\frac{|\zeta(\sigma+it, K)|^2}{|\sigma+it|^2}dt>\frac{C_1}{T^2}\int_{\frac{T}{2}}^{T}|\zeta(\sigma+it, K)|^2dt$$

$$>C_2T^{-4\sigma}\int_{\frac{T}{2}}^{T}|\zeta(1-\sigma-it,\ K)|^2dt>C_3T^{1-4\sigma}$$

The right hand side tends to infinity if $\sigma<\frac{1}{4}$, whence $\beta\geqq\frac{1}{4}$.

Again by the above lemma, for $\frac{1}{2}<\sigma<1$

$$\int_1^T|\zeta(\sigma+it,\ K)|^2dt=O(T)$$

Using the functional equation, we get for $0<\sigma<\frac{1}{2}$

$$\int_1^T|\zeta(\sigma+it,\ K)|^2dt=O(T^{2-4\sigma})\int_1^T|\zeta(1-\sigma-it,\ K)|^2dt$$
$$=O(T^{3-4\sigma})$$

Hence

$$\int_{\frac{T}{2}}^{T}\frac{|\zeta(\sigma+it,\ K)|^2}{|\sigma+it|^2}dt=O(T^{-\eta}) \qquad \eta>0$$

provided that $\sigma>\frac{1}{4}+\varepsilon$. It then follows by a simple argument that

$$\int_T^\infty\frac{|\zeta(\sigma+it,\ K)|^2}{|\sigma+it|^2}dt\longrightarrow 0$$

for $\sigma>\frac{1}{4}+\varepsilon$, and therefore that $\gamma\leqq\frac{1}{4}$ that is, that $\beta\leqq\frac{1}{4}$.

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