UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS OF METRIC SPACES

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In this paper we intend to find equivalent conditions under which continuous functions of a metric space are always uniformly continuous. Isiwata has attempted to prove a theorem in a recently published paper [3] by a method that has a close relation with ours. Unfortunately he does not accomplish his purpose, so we shall give a correct theorem (Theorem 3) in the last part of this paper and, for this purpose, give a condition for the existence of a uniformly continuous unbounded function in a metric space (Theorem 2).

In this paper the space S, unless otherwise specified, is the metric space with a distance function $d(x, y)$, and, for a positive number α, the α-sphere about a subset A $\{x ; d(A, x)<\alpha\}$ is denoted by $S(A, \alpha)$; the function is the real valued continuous mapping.

DEFINITION 1. Let us consider a family of neighborhoods U_n of x_n such that $\{x_n\}$ is a sequence of distinct points and $U_m \cap U_n = \phi$ (=empty) for $m \neq n$. Let $f_n(x)$ be a function such that $f_n(x_n)=n$ and $f_n(x)=0$ for $x \notin U_n$. Then a *mapping constructed from the family* is a mapping $f(x)$ defined by $f(x)=f_n(x)$ for x belonging to some U_n and $f(x)=0$ for the other x.

LEMMA. *Consider a family of neighborhoods U_n of x_n satisfying the following conditions*:

(1) $\{x_n\}$, *which consists of distinct points, has no accumulation point,*

(2) $\bar{U}_m \cap \bar{U}_n = \phi$, $m \neq n$ ($\bar{U}$ *a closure of* U), *and* $U_n \subset S(x_n, 1/n)$,

(3) *there is a sequence of points y_n such that distances of x_n and y_n converge to* 0 *and y_n does not belong to any U_m; then the mapping constructed from the family is continuous and not uniformly continuous. When $\{x_n\}$ is a sequence containing infinitely many distinct points and has no accumulation point, there is a family of neighborhoods of x_n satisfying* (2); *if $\{x_n\}$ further contains infinitely many distinct accumulation points, then the family besides satisfies* (3).

Proof. The continuity of the mapping constructed from the family follows from $\overline{\cup U_{n_i}} = \cup \bar{U}_{n_i}$ for any subsequence $\{n_i\}$ of indices; the mapping is not uniformly continuous by (3). Suppose $\{x_n\}$ consists of distinct accumulation points and has no accumulation point, then, by an inductive process, we have neighborhood V_n of x_n such that $V_n \subset S(x_n, 1/n)$ and $\bar{V}_m \cap \bar{V}_n = \phi$, and have y_n and a neighborhood U_n of x_n

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such that $U_n \not\ni y_n \in V_n$, $U_n \subset V_n$.

DEFINITION 2. Let x be isolated in a metric space, then we write $I(x)$ for a supremum of positive numbers α such that $S(x, \alpha)$ consists of x alone.

THEOREM 1. *The following conditions on a metric space S are equivalent*

(1) *If $\{x_n\}$ is a sequence of points without accumulation point, then all but finitely many members of x_n are isolated and* inf $I(x_n)$ *for the isolated points is positive.*

(2) *If a subset A of S has no accumulation point then all but finitely many points of A are isolated and* inf $I(x)$ *for all the isolated points of A is positive.*

(3) *The set A of all accumulation points in S is compact and* inf $I(x_n)$ *is positive for any sequence $\{x_n\}$ in S-A which has no accumulation point* (Isiwata [2], Theorem 2).

(4) $\bar{A} \cap \bar{B} = \phi$ *implies* $S(A, \alpha) \cap S(B, \alpha) = \phi$ *for some* α (Nagata [4], Lemma 1).

(5) $\bigcap_{n=1}^{\infty} \bar{A}_n = \phi$ *implies* $\bigcap_{n=1}^{\infty} S(A_n, \alpha) = \phi$ *for some* α.

(6) *For any function $f(x)$, there is a positive integer n such that every point of $A = \{x ; |f(x)| \geqq n\}$ is isolated and* $\inf_{x \in A} I(x)$ *is positive.*

(7) *All functions of S are uniformly continuous.*

(8) *All continuous mappings of S into an arbitrary uniform space S' are uniformly continuous.*

Proof. Since the equivalence of (1) and (3) is simple, we shall show (1)→(8)→(7)→(6)→(5)→(4)→(2)→(1).

(1)→(8): If a continuous mapping $f(x)$ of S is not uniformly continuous, there is an "entourage" V (in the sense of Bourbaki) of S' such that $d(x_n, y_n) < 1/n$ and $(f(x_n), f(y_n)) \notin V$ for any positive integer n and for some x_n and y_n. $\{x_n\}$ contains infinitely many distinct points. If $\{x_n\}$ has an accumulation point x, there are subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x, and, since $f(x)$ is continuous, $(f(x), f(x_{n_i})) \in W$ and $(f(x), f(y_{n_i})) \in W$ for W satisfying $W \cdot W \subset V$ (we may assume $W^{-1} = W$) and for all sufficiently large i. Hence we have $(f(x_{n_i}), f(y_{n_i})) \in V$, which is excluded. Consequently $\{x_n\}$ hes no accmulation point and inf $I(x_n) = r > 0$ for all sufficiently large n, which contradicts the first inequality of f for n satisfying $r > 1/n$.

(8)→(7) is obvious.

(7)→(6): If, for some function $f(x)$ and every n, there is an accumulation point x_n such that $|f(x_n)| \geqq n$, $\{x_n\}$ contains infinitely many distinct elements and has no accumulation point, then, by the Lemma, we have a function which is not uniformly continuous. Suppose that every point of $A=\{x;\ |f(x)|\geqq n\}$ is isolated and inf $I(x)=0$. Then there is a sequence $\{x_n\}$ in A such that inf $I_n=0$, $I_n=I(x_n)$. $\{x_n\}$ has no accumulation point, and we may assume $I_n<1/n$. If distances of distinct points of $\{x_n\}$ are greater than a positive number e, then, for all n satisfying $e>4I_n$, x_n and y_n $(\neq x_n,\ \in S(x_n,\ 2I_n))$ satisfy the conditions of the Lemma. In the other case, there are arbitrarily large m and n satisfying $d(x_m,\ x_n)<e$ for any positive number e, and we have, by an inductive process, a subsequence $\{y_i\}$ of $\{x_n\}$ satisfying $d(y_{2i-1},\ y_{2i})<1/i$. Then y_{2i-1} and y_{2i} satisfy the conditions of the Lemma.

(6)→(5): Let $\bigcap_n S(A_n,\ 1/m)\neq\phi$ for every m in spite of $\bigcap_n \bar{A}_n=\phi$. We have a point x_1 contained in $\bigcap_n S(A_n,\ 1)$ and a point y_1 distinct from x_1 satisfying $d(x_1,\ y_1)<1$. Suppose $B_i=\{x_1,\cdots,\ x_i\}$ consists of distinct points such that $x_j\in\ \bigcap_n S(A_n,\ 1/j)$, x_j and y_j are distinct and $d(x_j,\ y_j)<1/j$, $j=1,\cdots,\ i$. Since, for any point x, $\bigcap_n S(A_n,\ 1/m)$ does not contain x for a sufficiently large m, $\bigcap_n S(A_n,\ 1/(i+1))$ contains a point x_{i+1} being not contained in B_i, and some A_n contains y_{i+1} distinct from x_{i+1} satisfying $d(x_{i+1},\ y_{i+1})<1/(i+1)$. Thus we have a sequence $\{x_n\}$ of distinct points and $\{y_n\}$ such that $x_m\ \in\ \bigcap_n S(A_n,\ 1/m)$, x_n and y_n are distinct, and $d(x_n,\ y_n)<1/n$. $\{x_n\}$ has no accumulation point because of $\bigcap \bar{A}_n=\phi$. The function obtained from the Lemma does not satisfy the condition (6) whether all but finitely many members of x_n are isolated or not.

(5)→(4) is obvious.

(4)→(2): Suppose A has infinitely many accumulation points x_n, $n=1,\ 2,\cdots$. Since $B=\{x_n\}$ has no accumulation point, there is a sequence $C=\{y_n\}$ having no accumulation point such that $d(x_n,\ y_n)<1/n$, $B\cap C=\phi$. $\bar{B}\cap\bar{C}=B\cap C=\phi$, and $S(B,\ \alpha)\cap S(C,\ \alpha)=\phi$ for no α. If every point of A is isolated and inf $I(x)=0$, we have a sequence $\{x_n\}$ such that lim $I(x_n)=0$, and have a sequence $\{y_n\}$ with the same properties as the above.

(2)→(1) is obvious.

Recently Isiwata has stated a theorem ([3], Theorem 4) which is related to our Theorem 1. However the first step in his proof is wrong. We shall give a correct form of the theorem in Theorem 3. Let us first give a counterexample for the statement "In a connected metric space which is not totally bounded, there exists a sequence $\{x_n\}$ and a uniformly continuous function f such that $f(x_n)=n$".

EXAMPLE. Denoting the points of the plane by polar-coordinate, we consider the following subsets of the plane:

$$A_m=\{(r,\ \theta)\ ;\ 0\leqq r\leqq 1,\ \theta=\pi/m\},$$
$$S=\bigcup_{m=1}^{\infty} A_m\ .$$

We define the distance of the points of S by

$$d((r,\ \theta),\ (r',\ \theta'))=|r-r'| \qquad \text{as } \theta=\theta' \text{ or } rr'=0\ ,$$
$$=r+r' \qquad \text{as } \theta\neq\theta'\ ,$$

then S is obviously a connected metric space which is not totally bounded. When $f(x)$, $x\in S$, is a uniformly continuous function of S, there is a positive integer n such that $d(x,\ y)<1/n$ implies $|f(x)-f(y)|<1$. If x is contained in A_m, there are points $y_0=0=$pole, $y_1, \cdots, y_r=x$, $r\leqq n+1$, of A_m such that $d(y_{i-1},\ y_i)<1/n$, $i=1, \cdots, r$.

$$|f(0)-f(x)|\leqq|f(0)-f(y_1)|+\cdots+|f(y_{r-1})-f(x)|\leqq n+1\ ;$$

namely $f(x)$ is bounded.

DEFINITION 3. Let e be a positive number, then the finite sequence of points $x_0, x_1, \cdots, x_m$ satisfying $d(x_{i-1},\ x_i)<e$, $i=1, \cdots, m$, is said to be an *e-chain with length m*. If, for any positive number e, there are finitely many points $p_1, \cdots, p_i$ and a positive integer m such that any point of the space can be bound with some p_j, $1\leqq j\leqq i$, by an e-chain with length m, then the space is said to be *finitely chainable.*

THEOREM 2. *A metric space S admits a uniformly continuous unbounded function if and only if S is not finitely chainable.*

Proof. Verification of "only if" part is analogous to that stated in the above example, hence is passed over. Let S be not finitely chainable, then there is a positive number e such that, for any finitely many points and a positive integer n, there is a point which cannot be bound with any one of points selected above by an e-chain with length n. We denote by A_0^n the set of all points which can be bound with a fixed x_0 by an e-chain with length n.

(1) When $A_0^n\neq A_0^{n+1}$ for every n, we put

$$f(x)=(n-1)e+d(x,\ A_0^{n-1})$$

for x belonging to A_0^n and not to A_0^{n-1}, and $f(x)=0$ for $x\notin A_0=\bigcup_n A_0^n$ ($f(x)=d(x_0,\ x)$ for $x\in A_0^1$). Since $S(A_0,\ e)=A_0$, $f(x)$ is uniformly continuous on S if it is so on A_0. Let $A_0^n\ni x\notin A_0^{n-1}$ and $d(x,\ y)<e'<e$, then $A_0^{n+1}\ni y\notin A_0^{n-2}$. (i) When y is in A_0^{n-1}, then

$$f(y)=(n-2)e+d(y,\ A_0^{n-2})$$

and $d(x,\ A_0^{n-1})<e'$, $d(y,\ A_0^{n-2})<e$, hence $f(y)\leqq f(x)$. If $d(y,\ A_0^{n-2})<e-e'$, then $d(y,\ y')<e-e'$ for some y' of A_0^{n-2} and $d(x,\ y')\leqq d(x,\ y)+d(y,\ y')<e$, so that x is in A_0^{n-1}, which is excluded. Therefore $d(y,\ A_0^{n-2})\geqq e-e'$ and

$$\begin{aligned}|f(x)-f(y)| &=f(x)-f(y)=e+d(x,\ A_0^{n-1})-d(y,\ A_0^{n-2})\\ &<e+e'-(e-e')=2e'\ .\end{aligned}$$

(ii) When y is in A_0^n and not in A_0^{n-1}, then

$$f(y)=(n-1)e+d(y,\ A_0^{n-1}),$$

and we have

$$|f(x)-f(y)|=|d(x,\ A_0^{n-1})-d(y,\ A_0^{n-1})|\leqq d(x,\ y)<e'$$

(cf. the proof of Prop. 3 of §2, [1]). (iii) The remaining case for y is similar to (i). Consequently $f(x)$ is uniformly continuous on A_0.

(2) When $A_0^n=A_0^{n+1}$ for some n, then $A_0^m=A_0^n$ for every $m\geqq n$, and, in the similar way to (1), $A_1=\cup A_1^n$ is obtained from a point of $S-A_0$. If we can make an unbounded function which is uniformly continuous on A_1, our proof will be complete.

(3) When we cannot, for every $m(0\leqq m\leqq n)$, construct a desired function on A_m obtained in the same way as (2), $A_0, \cdots, A_n$ cannot cover the space, because the space is not finitely chainable; namely we have a sequence of infinitely many subsets $A_0,\ A_1, \cdots$ when our proof is not complete in the similar way to (2). Then we put $f(x)=n$ for x of A_n and $f(x)=0$ for x which is not in any A_n. Then, since $S(A_m,\ e)\cap A_n=\phi$ for any $m\neq n$ and $S(\cup A_n,\ e)=\cup A_n$, $f(x)$ is uniformly continuous.

THEOREM 3. *If S is a connected metric space which is not finitely chainable, then the set of all uniformly continuous functions of S does not form a ring.*

Proof. The following verification is essentially due to Isiwata [3]. There is, by Theorem 2, a uniformly continuous unbounded function $f(x)$ of the space, and we have a sequence $A=\{x_n;\ n=1,\ 2, \cdots\}$ such that $f(x_n)=a_n$, $a_{n+1}-a_n\geqq 1$, $a_1\geqq 1$; A has no accumulation point. For some positive number α, $d(x,\ y)<\alpha$ imlies $|f(x)-f(y)|<1/3$, and so $S(x_m,\ \alpha)\cap S(x_n,\ \alpha)=\phi$ for $m\neq n$. We put

$$h(x)=1-d(A,\ x)/\alpha \qquad \text{and} \qquad G=\bigcup_n S(x_n,\ \alpha)$$

and

$$f'(x) = \begin{cases} h(x) & \text{for } x \in G , \\ 0 & \text{for } x \notin G . \end{cases}$$

$h(x)$ is uniformly continuous on the space, because $d(A, x)$ is so (cf. Prop. 3 of §2, [1]). $h(x)>0$ and $h(y)\leqq 0$ for x of G and y of $S-G$ respectively, so we have

$$|h(x)-h(y)| = h(x)-h(y) \geqq h(x) = |f'(x)-f'(y)| .$$

Hence $f'(x)$ is uniformly continuous on the space. $g(x)=f(x)f'(x)$ is not uniformly continuous. In fact, if it is uniformly continuous, $d(x, y)<\beta$ implies

$$(*) \qquad |g(x)-g(y)|<1 \quad \text{and} \quad |f(x)-f(y)|<1$$

for some β ($\leqq \alpha$). We select a positive integer n such that a_n is greater than $1+4\alpha/\beta$, and take a point y such that $\beta/2 \leqq d(x_n, y)<\beta$ (it is possible to take such a point because of the connectedness of the space). Then, by (*), we have $|a_n-f(y)|<1$, $f(y)>a_n-1\geqq 0$, and

$$\begin{aligned} |g(x_n)-g(y)| &= |a_n-(1-d(A, y)/\alpha)f(y)| = |a_n-f(y)+d(x_n, y)f(y)/\alpha| \\ &\geqq |d(x_n, y)f(y)/\alpha| - |a_n-f(y)| > d(x_n, y)f(y)/\alpha-1 \\ &> \beta(a_n-1)/2\alpha-1 > \beta(1+4\alpha/\beta-1)/2\alpha-1=1 , \end{aligned}$$

which contradicts (*).

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