ASYMMETRY OF A PLANE CONVEX SET WITH RESPECT TO ITS CENTROID

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A. S. Besicovitch [1] proved that every bounded plane convex set K has a central subset of area at least $2m(K)/3$ where $m(K)$ denotes the area of K. His method is to construct a semi-regular hexagon of center N whose vertices belong to the boundary of K.

Ellen F. Buck and R. C. Buck [2] showed that for every K there exists at least one point X, called a six-partite point, such that there are three straight lines through X dividing K into six subsets each of area $m(K)/6$. H. G. Eggleston [3] showed that any six-partite point of K is the center of a semi-regular hexagon of area $2m(K)/3$ contained in K.

I. Fáry and L. Rédei [4] and S. Stein [5] defined for each point P the subset $S(P)$ of K determined by the intersection of K with its radial reflection in P and considered the function $f(P)=m(S(P))/m(K)$. By use of the Brunn-Minkowski theorem these authors showed that if a is a real number, then the set of points at which $f(P)\geqq a$ is convex; and the maximum f^* of $f(P)$ is attained at a single point. (Moreover, these results apply to an n-dimensional bounded convex set in n-dimensional Euclidean space.) Note that these conclusions may be false if the set K is not convex: for example, consider an L-shaped region formed by deleting one quarter of a square.

The results of Besicovitch and Eggleston imply $f(N)\geqq 2/3$ and $f(X)\geqq 2/3$, hence $f^*\geqq 2/3$.

We obtain the following theorem.

THEOREM. *If G is the centroid of K, then $f(G)\geqq 2/3$.*

To see that this result is not included in the theorems previously mentioned, consider the isosceles trapezoid with vertices $(-4, 0)$, $(4, 0)$, $(2, 2)$, $(-2, 2)$. For this example there is only one point N: $(0, 1)$ and only one point X: $(0, 4-4\sqrt{.6})$ and the closure of these points does not include G: $(0, 8/9)$.

Proof of the theorem. If K has central symmetry, then $f(G)=1$. In any case $S(G)$ has central symmetry about G; hence if K does not have central symmetry, the part M of K outside $S(G)$ has G at its centroid. Then as in Figure 1 let T be any maximal connected subset of M with

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A and B as terminal points of the boundary curve common to K and T. Let P' denote the reflection of a point P in G. Note that the congruent triangles AGB and $A'GB'$ are contained in $S(G)$.

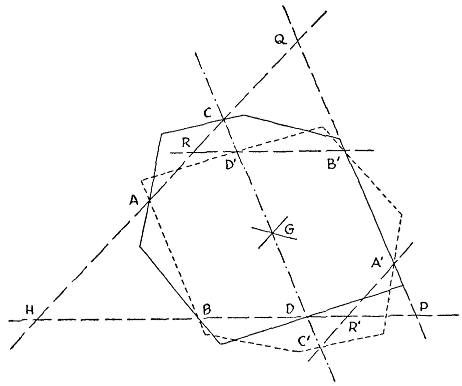

Fig. 1.

If for *every* T the area $m(T)$ is less than or equal to the area Δ of the corresponding triangle AGB, then $m(S(G)) \geqq 2m(M)$. Since

$$m(K) = m(M) + m(S(G)) \leqq 3m(S(G))/2\,,$$

it follows that $f(G) \geqq 2/3$.

In the contrary case, if we assume for *any* T that $m(T) > \Delta$, we can arrive at a contradiction of the fact that G is the centroid of M.

Let line L through G parallel to AB cut the boundary of K in points C and D. To fix ideas suppose in length $CG > GD$. Let lines BD and AC meet at H and intersect line $A'B'$ in P and Q, respectively. Let AC and $B'D'$ meet at R; then BD and $A'C'$ meet at R'; and R is on the side of L toward T.

Considerations of convexity imply that on the side of L away from T the *maximum* possible moment of M with respect to L is $u + w_2$ where u is the moment of triangle $R'A'P$ and w_2 is the moment of trapezoid $CQB'D'$. On the other side the *minimum* possible moment of M with respect to L is $w_1 + v$ where w_1 is the moment of triangle RCD' and v is the moment of a trapezoid of area $m(T)$ inscribed in triangle ABH and having AB as one base.

We will show that if $m(T) > \Delta$, then $w_1 + v > u + w_2$, in contradiction to G being the centroid of M. It will suffice to show $v > u + w$ where $w = w_2 - w_1$ is the moment of triangle RQB'.

Let $a = AB$, let d be the distance from G to AB and let h be the distance from H to AB. Let $a_1 = A'P$ and $a_2 = QB'$. From similar triangles $(a_1 + a_2 + a)/a = (2d + h)/h$, so that $a_1 + a_2 = 2ad/h$. The combined moments of triangles $R'A'P$ and RQB' are equivalent to those of a single triangle of base $a_1 + a_2$ and altitude d with centroid at a distance $2d/3$ from L, hence $u + w = 2ad^3/3h$.

Let c be the altitude of a trapezoid Z of area Δ inscribed in triangle ABH and having AB as one base. A direct computation shows the moment v' of Z with respect to L to be

$$v' = \frac{ad}{2}\left(d + \frac{c(3h - 2c)}{3(2h - c)}\right).$$

Since $m(T) > \Delta$ implies $v > v'$ the inequality $v > u + w$ will hold if

$v' > u+w$. Since $m(T) > \Delta$ also implies $h > d > c$, the inequality $v' > u+w$ reduces to

$$(6hd-3cd+3ch-2c^2)h > 4d^2(2h-c)\ .$$

Comparison of the areas of Z and triangle ABG shows $c^2 = 2ch - hd$. Then the previous inequality may be rearranged and factored to obtain the equivalent inequality

$$8hd(h-d) > c(h+4d)(h-d)$$

whose truth follows readily from $h > d > c$.

The case that length $CG = GD$ may be treated in the same manner (even if BD and AC are parallel). This completes the proof of the theorem.

We do not see how to extend the theorem about $f(G)$ to higher dimensions. Possibly the lower limit for $f(G)$ for the general bounded convex set is the same as $f(G)$ for a simplex of corresponding dimension. The value of the latter is given in [4] (but incorrectly given in Theorem 6 of [5], an error for which Professor Stein wishes this note to serve in lieu of a formal corrigendum).

Note that for as simple an example as a trapezoid $f^* > f(G)$. Some necessary conditions for determining P such that $f(P) = f^*$ have been given in [6].

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