A NOTE ON A PAPER OF L. GUTTMAN

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In a recent paper L. Guttman [2] obtained, using a result of von Neumann on the theory of games, lower bounds for the largest characteristic root of the matrix AA' where A is a real matrix of order $m \times n$. As Guttman points out his bounds are non-trivial only if some row or column of A has only positive or only negative elements. I wish to show that Guttman's results, and even a better result, are an immediate corollary of a well known theorem on Hermitian matrices: that each diagonal element lies between the smallest and largest characteristic roots (see e.g. [1]). Moreover, if AA' be replaced by AA^* then A can be real or complex and a non-trivial result is always obtained.

THEOREM 1. *Let $A=(a_{ij})$ be an $m \times n$ matrix with real or complex elements. Let λ be the largest characteristic root of the $m \times m$ non-negative definite Hermitian matrix $B=AA^*=(b_{ij})$. Then*

$$\lambda \geqq \max_i \sum_{j=1}^{n} |a_{ij}|^2 \tag{1}$$

$$\lambda \geqq \max_j \sum_{i=1}^{m} |a_{ij}|^2 \tag{2}$$

Proof. Let b_{rr} be the largest diagonal element of B. Then

$$\lambda \geqq b_{rr} = \sum_{j=1}^{n} |a_{rj}|^2 = \max_i \sum_{j=1}^{n} |a_{ij}|^2 \,,$$

and (1) is proved. Now the non-zero characteristic roots of AA^* are the same as those of A^*A. Then (2) follows as above if we consider A^*A instead of AA^*.

The bounds in (1) and (2) can be replaced by the weaker bounds

$$\lambda \geqq n \cdot \max_i \left(\min_j |a_{ij}|^2 \right) \tag{3}$$

$$\lambda \geqq m \cdot \max_j \left(\min_i |a_{ij}|^2 \right) \tag{4}$$

respectively, and even these bounds are obviously better than Guttman's.

Theorem 1 can be improved further.

THEOREM 2. *Under the hypotheses of Theorem 1 we have*

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$$(5)\qquad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2+|a_{j\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2-|a_{j\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{i\nu}\bar{a}_{j\nu}\right|^2\right\}^{1/2}\right]$$

$$(6)\qquad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2+|a_{\nu j}|^2)+\left\{\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2-|a_{\nu j}|^2)\right]^2+4\left|\sum_{\nu=1}^{m}\bar{a}_{\nu i}a_{\nu j}\right|^2\right\}^{1/2}\right]$$

Proof. It was shown in [1] that the largest root of an Hermitian matrix is greater than or equal to the larger of the two roots of any principal minor of order two of the matrix. Suppose the principal minor or order two of B having the largest root lies in the r, s rows and columns of B. Then

$$\begin{aligned}2\lambda &\geqq b_{rr}+b_{ss}+[(b_{rr}-b_{ss})^2+4|b_{rs}|^2]^{1/2}\\ &=\sum_{\nu=1}^{n}(|a_{r\nu}|^2+|a_{s\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{r\nu}|^2-|a_{s\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{r\nu}\bar{a}_{s\nu}\right|^2\right\}^{1/2}\end{aligned}$$

and (3) follows. (4) is proved similarly by considering A^*A instead of B.

REFERENCES

1. Alfred Brauer and A. C. Mewborn, *Intervals for the characteristic roots of an Hermitian matrix*, Elisha Mitchell Scien. Soc., **73** (1957), 247-254.
2. Louis Guttman, *Some inequalities between latent roots and minimax (maximin) elements of real matrices*, Pacific J. Math., **7** (1957), 897-902.

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