A METHOD OF APPROXIMATING THE COMPLEX ROOTS OF EQUATIONS

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1. The method described in this paper presents an algorithm by which at least two roots of an equation can be approximated starting with the same first approximation. This is achieved by introducing a parameter and choosing its numerical value appropriately. In particular, in case of real roots, two adjacent or the largest and the smallest roots are approximated by the use of two different values of the parameter. This is discussed in §3. In case of conjugate imaginary roots the real and imaginary parts of the approximations are easily separated. This is discussed in §4.

2. Let $f(z)$ be an analytic function within and upon a circle C, and let the roots of the equation $f(z)=0$ within and upon the circle be denoted by a_j, $j=1, 2, \cdots$, and their multiplicities by m_j respectively.

We consider the expansion into the partial fractions of $(u-z)^k f'(z)/f(z)$, where k is a positive integer and $u \neq a_j$ or z but otherwise arbitrary,

$$(u-z)^k f'(z)/f(z)=\sum_{j=1} m_j(u-a_j)^k/(z-a_j)+\psi_1(z) \ , \tag{1}$$

where $\psi_1(z)$ is analytic within and upon the circle, and the sum is taken over all the roots a_j starting with $j=1$.

By differentiating (1) $n-1$ times and dividing by $(-1)^{n-1}(n-1)!$ we derive

$$Q_{n,k}/[f(z)]^n=\sum_{j=1} m_j(u-a_j)^k/(z-a_j)^n+\psi_n(z) \ , \qquad n\geqq k, \tag{2}$$

where $\psi_n(z)$ is analytic within and upon C. The function $Q_{n,k}\equiv Q_{n,k}(z, u)$ can be evaluated by the formula

$$Q_{n,k}=\sum_{j=0}^{n-1}\binom{k}{j}(u-z)^{k-j}[f(z)]^j D_{n-j} \tag{3}$$

with D_n evaluated recursively

$$\begin{gathered} D_n=\sum_{j=0}^{n-2} f^{(j+1)}(z)[-f(z)]^j D_{n-j-1}/(j+1)!+f^{(n)}(z)[-f(z)]^{n-1}/(n-1)! \ , \\ D_0=1, \ D_1=f'(z) \end{gathered} \tag{4}$$

The function $Q_{n,k}$ can also be evaluated recursively, and both $Q_{n,k}$ and

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D_n can be expressed in the form of determinants [1, 2].

We rewrite (2) as follows

$$
(5)\qquad Q_{n,k}/[f(z)]^n = m_1(u-a_1)^k/(z-a_1)^n
$$
$$
\times\left\{1+(z-a_1)^n/m_1(u-a_1)^k\left[\sum_{j=2} m_j(u-a_j)^k/(z-a_j)^n+\psi_n(z)\right]\right\},
$$

where the summation starts with $j=2$. Now, if we assume that u and z are given such values that

$$
(6)\qquad |(u-a_1)/(z-a_1)|>|(u-a_j)/(z-a_j)|, \qquad j=2,\ 3,\ \cdots,
$$

and

$$
|(u-a_1)/(z-a_1)|>|(u-\zeta)/(z-\zeta)|,
$$

for any ζ on C, the following result follows:

$$
(7)\qquad (u-a_1)^{c-b}/(z-a_1)^a=\lim_{n\to\infty} Q_{n,n-b}/[f(z)]^a Q_{n-a,n-c},
$$

where a, b, and c are constants satisfying the conditions imposed on the subscripts of $Q_{n,k}$ in (2). An approximation to a_1 is obtained with a finite n.

Of particular practical value are the cases when the left hand side of the equation has only $u-a_1$, or $z-a_1$, or both of the first or second degree.

3. The reason for introducing the parameter u into the problem is that more than one root can be approximated with the same D_1, D_2, $\cdots D_n$ by using different appropriately chosen values of u. This will be illustrated when the left hand side of (7) is either $u-a_1$, $(u-a_1)/(z-a_1)$, or $1/(z-a_1)$, namely:

$$
(8)\qquad (z-a_1)/(u-a_1)=\lim_{n\to\infty} f(z)Q_{n-1,n-1}/Q_{n,n}
$$

$$
(9)\qquad z-a_1=\lim_{n\to\infty} f(z)Q_{n-1,n-1}/Q_{n,n-1}
$$

$$
(10)\qquad u-a_1=\lim_{n\to\infty} Q_{n,n}/Q_{n,n-1}
$$

$$
(11)\qquad (z-a_1)/(u-a_1)=\lim_{n\to\infty} f(z)Q_{n-1,n-2}/Q_{n,n-1}
$$

Let us assume that $z=x$ is real and that the two roots closest to x are also real, $a_1<x<a_2$, $x-a_1<a_2-x$. Then, as it can easily be verified, an approximation to a_1 can be obtained with any $u_1>x$, and $\infty<u_1<[(a_1+a_2)x-2a_1a_2]/(2x-a_1-a_2)$; an approximation to a_2 can be obtained with any $[(a_1+a_2)x-2a_1a_2]/(2x-a_1-a_2)<u_2<x$ (Diagram 1). The

above inequalities defining u_1 and u_2 should also be used when $a_1 < x$ is the largest real root of an equation and a_2 the smallest (Diagram 2).

Before applying any of (8)—(11), an approximation to a_1 can be obtained by using a more particular case of (7) [1, 2]:

$$z - a_1 = \lim_{n\to\infty} f(z)D_{n-1}/D_n = \lim_{n\to\infty} [f^{(n)}(z)D_n]^{1/n} \tag{12}$$

This gives an idea as to the location of the root closest to x.

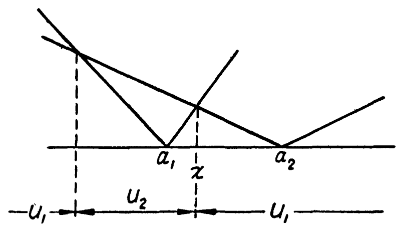

Diagram 1

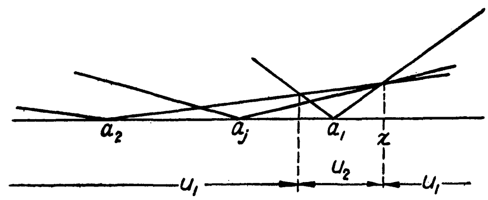

Diagram 2

4. Let now $z = x$ be real equidistant from two conjugate imaginary roots $a+bi$ and $a-bi$. Then u can be taken in the form $x+ti$ and the real and imaginary parts in the equations (8)—(11) can easily be separated. In this case, if x is closer to $a+bi$ and $a-bi$ than to any other root of the equation, and if the equation has no more imaginary roots, any positive t can be taken to approximate $a-bi$ (Diagram 3). If the equation has another imaginary root not much more distant from x than $a-bi$, and with real part closer to x than a, a large value of t would be required (Diagram 4). The imaginary root $a-bi$ can be approximated with some positive t (Diagram 5) even if there is a real root which is closer to x than to $a-bi$, but not very close, and if the real part, a, is close to x.

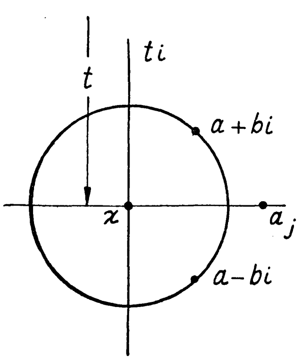

Diagram 3

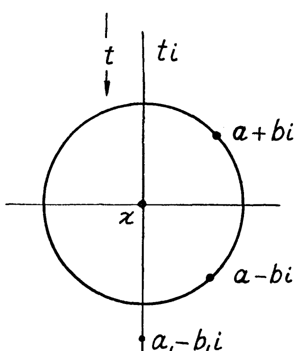

Diagram 4

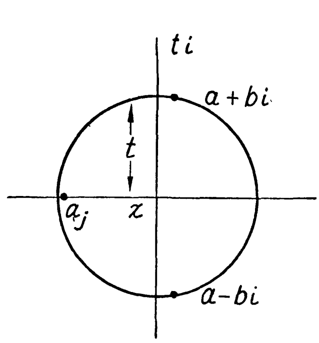

Diagram 5

We shall now give the explicit formulas for the real and imaginary parts of a root a_1 in the four cases given by (8)-(11).

We designate $z=x+ti$, as before, where x and t are real, $t>0$, $a_1=a-bi$, $Q_{n,n}=A_{n,n}+iB_{n,n}$, $Q_{n,n-1}=A_{n,n-1}+iB_{n,n-1}$, where

$$(13)\qquad \begin{aligned} A_{n,n}&=-t^2\sum_{j=1}(-1)^{j-1}\binom{n}{2j}t^{2j-2}[f(x)]^{n-2j}D_{2j}\ ,\\ B_{n,n}&=t\sum_{j=1}(-1)^{j-1}\binom{n}{2j-1}t^{2j-2}[f(x)]^{n-2j+1}D_{2j-1}\ ; \end{aligned}$$

$$(14)\qquad \begin{aligned} A_{n,n-1}&=\sum_{j=1}(-1)^{j-1}(2j^{n-1}-2)t^{2j-2}[f(x)]^{n-2j+1}D_{2j-1}\ ,\\ B_{n,n-1}&=t\sum_{j=1}(-1)^{j-1}(2j^{n-1}-1)t^{2j-2}[f(x)]^{n-2j}D_{2j}\ . \end{aligned}$$

The sums being taken over all j, for which the binomial coefficients do not vanish, starting with $j=1$.

Now by using (8)-(11) we get respectively

$$(15)\qquad \begin{aligned} x-a=\lim_{n\to\infty} t\,f(x)[B_{n-1,n-1}(f(x)A_{n-1,n-1}-A_{n,n})\\ -A_{n-1,n-1}(f(x)B_{n-1,n-1}-B_{n,n})]/\Delta\ , \end{aligned}$$

$$(15_1)\qquad \begin{aligned} b=-\lim_{n\to\infty} t\,f(x)[A_{n-1,n-1}(f(x)A_{n-1,n-1}-A_{n,n})\\ -B_{n-1,n-1}(f(x)B_{n-1,n-1}-B_{n,n})]/\Delta\ , \end{aligned}$$

where

$$\Delta=[f(x)A_{n-1,n-1}-A_{n,n}]^2+[f(x)B_{n-1,n-1}-\mathrm{B}_{n,n}]^2\ ;$$

$$(16)\qquad x-a=\lim_{n\to\infty} f(x)(A_{n,n-1}A_{n-1,n-1}+B_{n,n-1}B_{n-1,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ,$$

$$(16_1)\qquad b=\lim_{n\to\infty} f(x)(A_{n,n-1}B_{n-1,n-1}-B_{n,n-1}A_{n-1,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ;$$

$$(17)\qquad x-a=\lim_{n\to\infty}(A_{n,n}A_{n,n-1}+B_{n,n}B_{n,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ,$$

$$(17_1)\qquad t-b=\lim_{n\to\infty}(A_{n,n}B_{n,n-1}-B_{n,n}A_{n,n-1})/(A^2_{n,n-1}+B^2_{n,n-1})\ ;$$

$$(18)\qquad \begin{aligned} x-a=\lim_{n\to\infty} t\,f(x)[B_{n-1,n-2}(f(x)A_{n-1,n-2}-A_{n,n-1})\\ -A_{n-1,n-2}(f(x)B_{n-1,n-2}-B_{n,n-1})]/\Delta_1\ , \end{aligned}$$

$$(18_1)\qquad \begin{aligned} b=\lim_{n\to\infty} t\,f(x)[A_{n-1,n-2}(f(x)A_{n-1,n-2}-A_{n,n-1})\\ -B_{n-1,n-2}(f(x)B_{n-1,n-2}-B_{n,n-1})/\Delta_1\ , \end{aligned}$$

where

$$\Delta_1=[f(x)A_{n-1,n-2}-A_{n,n-1}]^2+[f(x)B_{n-1,n-2}-B_{n,n-1}]^2\ .$$

5. Results analogous to those presented above can be obtained by considering other expansions similar to those given by (2). We mention

here one such result assuming that $f(z)$ has only simple zeros. We consider then $(u-z)^k/f(z)$ instead of $(u-z)^k f'(z)/f(z)$ and derive the equation

$$Q^1_{n,k}/[f(z)]^n = \sum_{j=1} A_j(u-a_j)^k/(z-a_j)^n + \psi^1_n(z) , \tag{19}$$

where A_j are constants.

$$Q^1_{n,k} = \sum_{j=0}^{n-1} \binom{k}{j} (u-z)^{k-j}[f(z)]^j P_{n-j-1} \tag{20}$$

$$P_n = \sum_{j=0}^{n-1} f(z)^{(j+1)}[-f(z)]^j P_{n-j-1} , \qquad P_0 = 1. \tag{21}$$

It would suffice now to replace $Q_{n,k}$ by $Q^1_{n\ k}$ in all the previous formulas.

6. If $f(z)$ is a polynomial of degree N and $k=n$, then the last member on the right hand side of (2) equals $-N$. If taken to the left hand side, it would contribute the term $N[f(z)]^n$ to $Q_{n,n}$, consequently, $N[f(z)]^n$ and $N[f(z)]^{n-1}$ will be contributed to $A_{n,n}$ and $A_{n-1,n-1}$ respectively in (8), (9), and (10). In case of equation (19), however, the last member would be reduced to $-A_1 - A_2 - \cdots - A_N$.

REFERENCES

1. S. Kulik, *A new method for evaluating the zeros of analytic functions*, Interim Technical Report No. 1, OOR., U.S, Army, 1956.

2. ———, *A method for approximating the zeros of analytic functions*, Duke Math. J. **24**, (1957), 137–142.

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