ON THE PERIODICITY OF THE SOLUTION OF A CERTAIN NONLINEAR INTEGRAL EQUATION

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In the following paper we will study the nonlinear integral equation

$$E(t)=F(t)-\int_0^t G(t-\tau)N\{E(\tau)\}\,d\tau \tag{1}$$

where $F(t)$ is a known periodic real function and $G(t)$ and $N(x)$ are known real functions. In particular we will investigate the behaviour of the solution $E(t)$ of the equation (1) for large values of t.

We assume that $G \in L[0, \infty]$ and that $N(x)$ is bounded almost everywhere and Borel-measurable in $[-\infty, \infty]$. Furthermore $N(x)$ is assumed expressible in the form

$$N(x) \sim N(0)+\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda x}-1}{i\lambda}d\lambda \tag{2}$$

with $\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda<\infty$ and with finite $N(0)$. This representation is to be valid almost everywhere in $[-\infty, \infty]$

Because $N(x)$ is Borel-measurable in $[-\infty, \infty]$ and $|N(0)|<\infty$, the measurability of x implies the measurability of $N(x)$. The following four classes of $N(x)$-functions are distinguished:

$$\begin{array}{llll} N\in K_{11} & \text{if} \quad x\in L[0,\ 1] & \text{implies } N(x)\in L[0,\ 1] \\ N\in K_{1\infty} & \text{if} \quad x\in L[0,\ 1] & \text{implies } N(x)\in L[0,\ \infty] \\ N\in K_{\infty 1} & \text{if} \quad x\in L[0,\ \infty] & \text{implies } N(x)\in L[0,\ 1] \\ N\in K_{\infty\infty} & \text{if} \quad x\in L[0,\ \infty] & \text{implies } N(x)\in L[0,\ \infty] \end{array} \tag{3}$$

The space of measurable and bounded functions defined on the finite interval $[0, A]$ will be denoted by $M[0, A]$. The norm of $x\in M[0, A]$ is defined, as usual, by

$$||x||=\inf_E\left\{\sup_{t\in[0,A]-E}|x|\right\}$$

where E ranges over the sets of measure zero in $[0, A]$, and the distance of $x\in M[0, A]$ and $y\in M[0, A]$ by $||x-y||$. The space $M[0, 1]$ is complete.

The proofs in this paper will be based on the following theorem by Tihonov (see for instance [1]) which is valid in $M[0, A]$: Let the operator B map $M[0, A]$ into itself and let $||B(x)-B(y)||\leqq\beta||x-y||$ for all x and

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y in $M[0, A]$, where $\beta<1$. Then the equation $y=B(y)$ has a unique solution $\bar{y}$ in $M[0, A]$. The function $\bar{y}$ may be obtained by iteration:

$$\bar{y}=\lim_{n\to\infty} y_n$$

where $y_n=B(y_{n-1})$ and where y_0 may be taken arbitrarily from $M[0, A]$.

We will prove the following theorem.

THEOREM. *Suppose that $F(t)$ is a periodic function in $[0, \infty]$ with period T, and that $F\in M[0, T]$. Furthermore suppose that $G\in L[0, \infty]$, $N\in K_{1\infty}$ and*

$$\left(\int_0^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right)<1\,.$$

If $E(t)$ is the solution of

$$(4)\qquad E(t)=F(t)-N(0)\int_0^t G(u)du-\int_0^t G(t-\tau)\int_{-\infty}^{+\infty}S(\lambda)\frac{e^{i\lambda E(\tau)}-1}{i\lambda}d\lambda d\tau$$

then $\lim E(nT+u)=v(u)$ *exists, as $n\to\infty$ through integer values. The convergence is uniform. Moreover, $v(u)$ has the period T, and satisfies*

$$(5)\qquad v(u)=F(u)-N(0)\int_0^{\infty}G(u)du-\int_0^{\infty}G(\tau)\int_{-\infty}^{+\infty}S(\lambda)\;\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau$$

This equation can be solved by iteration stating with any element of $M[0, T]$. The solution of (5) is unique.

In order to prove the theorem, we will first prove two lemmas. Put

$$H[\Delta(u+mT)]=\int_0^{t_0}G(\tau)d\tau\int_{-\infty}^{+\infty}S(\lambda)e^{i\lambda E(mT+u-\tau)}\frac{e^{i\lambda\Delta(mT+u-\tau)}-1}{i\lambda}d\lambda$$

where $\Delta(u+mT)=E(u+nT)-E(u+mT)$ and $0\leqq u\leqq T$. Here T is a finite positive real number, t_0 a positive real number which may be finite or infinite and m and n positive integers. $E(u+nT)\in M[0, T]$ and $E(u+mT)\in M[0, T]$ implies $\Delta(u+mT)\in M[0, T]$. The operator H will play an important role in the following considerations. For this reason we will first establish some of its properties. We will write more briefly $H(\Delta(mT+u))=H(\Delta)$.

LEMMA 1. *Suppose that $G\in L[0, \infty]$, and suppose that the function $N(x)$ belongs to one of the classes K_{11} and $K_{1\infty}$. Then $\Delta\in M[0, T]$ implies $H(\Delta)\in M[0, T]$ and*

$$||H(\Delta_1)-H(\Delta_2)||\leqq\beta||\Delta_1-\Delta_2||$$

where

$$\beta=\left(\int_0^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right).$$

Now

$$\int_0^t G(\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau$$
$$=\int_0^t G(\tau)I(t_0-\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau$$

where

$$N(E)=\int_{-\infty}^{+\infty}S(\lambda)\frac{e^{i\lambda E}-1}{i\lambda}d\lambda$$

and

$$I(t_0-t)=\begin{cases}1 \text{ if } t\leqq t_0\\ 0 \text{ if } t_0<t\end{cases}$$

$G\in L[0,\ \infty]$ implies $G(\tau)I(t_0-\tau)\in \mathrm{L}[0,\ \infty]$. Furthermore, from $x\in M[0,\ T]$ and the properties of $N(x)$ follows that $N(x)\in M[0,\ T]$. Consequently $N(x)\in L[0,T]$. From known properties of the convolution follows now that

$$\int_0^t G(\tau)I(t_0-\tau)[N\{E_1(t-\tau)\}-N\{E_2(t-\tau)\}]d\tau\in L[0,\ T].$$

Hence $H(\Delta)\in L[0,\ T]$. Now, as is easily seen,

$$||H(\Delta)||\leqq\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right)||\int_0^t|G(u)||\Delta|(t+mT-u)|du||\leqq\beta||\Delta||$$

which implies the boundedness of $H(\Delta)$. The function $H(\Delta)$ is thus measurable and bounded in $[0,\ T]$, $H(\Delta)\in M[0,\ T]$. Furthermore

$$||H(\Delta_2)-H(\Delta_1)||=||\int_0^{t_0}G(\tau)d\tau\int_{-\infty}^{+\infty}S(\lambda)e^{i\lambda E(u+mT-\tau)}\frac{e^{i\lambda\Delta_2(u+mT-\tau)}-e^{i\lambda\Delta_1(u+mT-\tau)}}{2}d\lambda||$$
$$\leqq\left(\int_{-\infty}^{+\infty}|S(\lambda)|\,d\lambda\right)||\int_0^{t_0}|G(\tau)||\Delta_2(u+mT-\tau)-\Delta_1(u+mT-\tau)|d\tau||$$
$$\leqq\beta||\Delta_2-\Delta_1||,$$

which completes the proof.

We will now consider the norm

$$(6)\qquad ||E(u+nT)-E(u+mT)+\int_0^{f(m)}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau||=Q$$

where m and n are positive integers, $f(m)$ an arbitrary function of m,

T a finite positive number and $E \in M[0, T]$. Furthermore it will be assumed that $G \in L[0, \infty]$ and $N \in K_{1\infty}$ and that they satisfy the condition

$$\left(\int_{-\infty}^{+\infty} |S(\lambda)|\, d\lambda\right)\left(\int_0^{\infty} |G(u)| du\right) < 1 \ .$$

The following lemma holds.

LEMMA 2. *For every $\varepsilon > 0$ there exists an integer m_0 such that $m \geqq m_0$ and $n \geqq m_0$ imply $Q < \varepsilon$, if and only if, with $v(u)$ from $M[0, T]$, $||E(u+pT)-v(u)|| \to 0$ as $p \to \infty$ through positive integral values.*

Suppose first that $||E(u+pT)-v(u)|| \to 0$, as $p \to \infty$, where E and v are in $M[0, T]$. Now

$$\begin{aligned}
&||\int_0^{f(m)} G(\tau)[N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\}]d\tau \\
&= ||\int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\,\frac{e^{i\lambda E(u+nT-\tau)} - e^{i\lambda E(u+mT-\tau)}}{i\lambda}\, d\lambda d\tau|| \\
&\leqq \left(\int_0^{\infty} |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)| d\lambda\right) ||E(u+nT-\tau) - E(u+mT-\tau)||
\end{aligned}$$

and consequently

$$\begin{aligned}
&||E(u+nT)-E(u+mT)+\int_0^{f(m)} G(\tau)[N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\}]d\tau|| \\
&\qquad \leqq \left[1+\left(\int_0^{\infty} |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)|\, d\lambda\right)\right] ||E(u+nT)-E(u+mT)||
\end{aligned}$$

where $\left(\int_0^{\infty} |G(u)| du\right)\left(\int_{-\infty}^{+\infty} |S(\lambda)| d\lambda\right) < 1$. Because $||E(u+pT)-v(u)|| \to 0$, as $p \to \infty$, there exists for every $\varepsilon > 0$ an integer m_1 such that $m_1 \leqq m < n$ implies

$$||E(u+nT)-E(u+mT)|| \leqq \frac{\varepsilon}{1+\beta}$$

from which the first part of the lemma follows.

Suppose now that (6) is valid for m and n greater than a given integer m_2. The inequality (6) may be written

$$\begin{aligned}
&||\varDelta(u+mT) \\
(7) \qquad &+\int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda E(u+mT-\tau)}\ \frac{e^{i\lambda \varDelta(u+mT-\tau)} - 1}{i\lambda}\, d\lambda d\tau|| \leqq \varepsilon
\end{aligned}$$

where $\varDelta(u+mT) = E(u+nT) - E(u+mT)$

Now let h be a function in $M[0, T] \cap S(\varepsilon, 0)$ where $S(\varepsilon, 0)$ is the sphere with radius ε and center at $h=0$. Put

$$\Delta(u+mT) + \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+mT-\tau)}\,\frac{e^{i\lambda\Delta(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau = h(u)\,. \tag{8}$$

The functions Δ obtained by solving (8) for all $h\in M[0, T]\cap S(\varepsilon, 0)$ are those which satisfy (7). $E(u+mT)$ is a known function.

The equation

$$\begin{aligned}\Delta(u+mT) &= h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+mT-\tau)}\,\frac{e^{i\lambda\Delta(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau \\ &= h(u) - H[\Delta(u+mT)]\end{aligned} \tag{9}$$

where H is the operator defined on page 3, may be solved by iteration.

Indeed, by Lemma 1 the operator H is defined in $M[0, T]$, $\Delta\in M[0, T]$ implies $H(\Delta)\in M[0, T]$ and

$$\|h(u)-H(\Delta_1)-(h(u)-H(\Delta_2))\| = \|H(\Delta_2)-H(\Delta_1)\| \leqq \beta\|\Delta_2-\Delta_1\|$$

where $\beta=\left(\int_0^\infty |G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)<1.$

The conditions of the Tihonov's theorem are thus satisfied. We begin the iteration process with an h from $M[0, T]\cap S(\varepsilon, 0)$:

$$\Delta_1(u+nT) = h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+nT-\tau)}\,\frac{e^{i\lambda h(u-\tau)}-1}{i\lambda}\,d\lambda d\tau$$

and generally

$$\Delta_{p+1}(u+nT) = h(u) - \int_0^{f(m)} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda E(u+nT-\tau)}\,\frac{e^{i\lambda\Delta_p(u+mT-\tau)}-1}{i\lambda}\,d\lambda d\tau$$

The unique solution of (9) is then $\lim_{k\to\infty}\Delta_k(u+nT)=\Delta(u+nT)$ where $\Delta(u+mT)$ is in $M[0, T]$.

Now

$$\begin{aligned}\|\Delta_{p+1}\| &\leqq \|h\| + \left\|\int_0^{f(m)} |G(\tau)|\int_{-\infty}^{+\infty}|S(\lambda)|\,|e^{i\lambda E(u+mT-\tau)}|\,\frac{e^{i\lambda\Delta_p}-1}{i\lambda}\,|d\lambda d\tau|\right\| \\ &\leqq \varepsilon + \left(\int_0^\infty |G(\lambda)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)\|\Delta_p\| \leqq \varepsilon + \beta\|\Delta_p\|\end{aligned}$$

From this inequality one obtains now, remembering that $\|\Delta_0\|=\|h\|\leqq\varepsilon$ and that $\beta<1$,

$$\|\Delta_{p+1}\| \leqq (1+\beta+\beta^2+\cdots+\beta^{p+1})\varepsilon \leqq \frac{\varepsilon}{1-\beta}$$

This inequality holds true for all p. Consequently

$$||\Delta(u+nT)||\leqq\frac{\varepsilon}{1-\beta}$$

or, in view of the definition of $\Delta(u+nT)$,

$$||E(u+nT)-E(u+mT)||\leqq\frac{\varepsilon}{1-\beta}$$

for m and n greater than m_2. But such m_2 exists for every $\varepsilon>0$. From this and from the completeness of the space $M[0,\ T]$ follows that there exists a $v_1\in M[0,\ T]$ such that

$$||E(u+pT)-v_1(u)||\longrightarrow 0$$

as $p\to\infty$ through integral values.

We now proceed to prove the Theorem.

Because of the periodicity of $F(t)$ one obtains from (1)

$$\begin{aligned}E(u+nT)&+\int_0^{u+nT}G(\tau)N\{E(u+nT-\tau)\}d\tau\\&=E(u+mT)+\int_0^{u+mT}G(\tau)N\{E(u+mT-\tau)\}d\tau\end{aligned}$$

where $0\leqq u\leqq T$ and where m and n are positive integers.

Suppose that $m<n$ and $t_0\leqq mT$. Then

$$\begin{aligned}E(u+nT)-E(u+mT)&+\int_0^{t_0}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau\\&=\int_{t_0}^{u+mT}G(\tau)N\{E(u+mT-\tau)\}d\tau-\int_{t_0}^{u+nT}G(\tau)N\{E(u+nT-\tau)\}d\tau\end{aligned}$$

and

$$\begin{aligned}&||E(u+nT)-E(u+mT)+\int_0^{t_0}G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau||\\&\leqq||\int_{t_0}^{u+nT}|G(\tau)||N\{E(u+nT-\tau)\}|d\tau+\int_{t_0}^{u+mT}|G(\tau)||N\{E(u+mT-\tau)\}|d\tau||\\&\leqq\left(||\int_{t_0}^{u+nT}|G(\tau)|d\tau||+||\int_{t_0}^{u+mT}|G(\tau)|d\tau||\right)||N||\leqq2||N||\int_{t_0}^{\infty}|G(\tau)|d\tau\end{aligned}$$

Because $G\in L[0,\ \infty]$, there exists a positive integer m_3 for every $\varepsilon>0$ such that for $t_0=m_3T$

$$\int_{m_3T}^{\infty}|G(\lambda)|du\leqq\frac{\varepsilon}{2||N||}$$

But $m_3\leqq m<n$. Consequently, for every $\varepsilon>0$ there exists a positive integer m_3 such that $m_3\leqq m<n$ implies

$$\|E(u+nT)-E(u+mT) \\ +\int_0^{m_2T} G(\tau)[N\{E(u+nT-\tau)\}-N\{E(u+mT-\tau)\}]d\tau\| \leqq \varepsilon$$

By Lemma 2 it follows now that there exists a $v \in M[0, T]$ such that $\|E(u+pT)-v(u)\| \to 0$ as $p \to \infty$ through positive integral values. Consequently $E(u+pT)$ converges uniformly to $v(u)$ in $[0, T]$. That $v(u)$ is periodic with period T is immediate.

We substitute now

$$E(u+nT)=v(u)+H_n(u)$$

where $H_n \in M[0, T]$ and $\|H_n\| \to 0$, as $n \to \infty$ and where $0 \leqq u \leqq T$, into (1) and obtain

$$v(u)+H_n(u)=F(u)-N(0)\int_0^{u+nT} G(\tau)d\tau \\ -\int_0^{u+nT+} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau$$

As is seen at once, this may be rewritten as follows:

$$v(u)-F(u)+\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau+N(0)\int_0^{\infty} G(\tau)d\tau \\ +\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)e^{i\lambda v(u-\tau)}\frac{e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau+H_n(u)+ \\ -\int_{nT+u}^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}e^{i\lambda H_n(u-\tau)}-1}{i\lambda}d\lambda d\tau-N(0)\int_{nT+u}^{\infty} G(\tau)d\tau=0$$

which yields the inequality

$$\|v(u)-F(u)+\int_0^{\infty} G(\tau)\int_{-\infty}^{+\infty} S(\lambda)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}d\lambda d\tau\| \\ \leqq \|H_n(u)\|+\left(\int_0^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)\|H_n(u)\| \\ +\left(\int_{nT+u}^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)\|v(u)+H_n(u)\|+N(0)\int_{nT+u}^{\infty}|G(u)|du \\ =(1+\beta)\|H_n(u)\|+\left(\int_{nT}^{\infty}|G(u)|du\right)\left(\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda\right)(\|v(u)\|+\|H_n(u)\|) \\ +N(0)\int_{nT}^{\infty}|G(u)|d\lambda\,.$$

But β, $\int_{-\infty}^{+\infty}|S(\lambda)|d\lambda$, $\|v(u)\|$ and $N(0)$ are finite, $\|H_n\| \to 0$ as $n \to \infty$ and $\int_{nT}^{\infty}|G(u)|du \to 0$ as $n \to \infty$. Consequently

$$||v(u)-\left(F(u)-N(0)\int_0^\infty G(u)du-\int_0^\infty G(\tau)\int_{-\infty}^{+\infty} S(\omega)\frac{e^{i\lambda v(u-\tau)}-1}{i\lambda}\,d\lambda d\tau\right)||\longrightarrow 0\ ,$$

as $n\rightarrow\infty$ through integral values, from which the equation (5) follows for $v(u)$.

The right side of (5) satisfies the conditions of Tihonov's theorem. This follows by Lemma 1 where we substitute $t_0=\infty$, $E(mT+u-\tau)=0$ and $\varDelta(mT+u-\tau)=v(u-\tau)$. If the right side of (5) is denoted by $c(v)$, then, by Lemma 1, $v\in M[0,\ T]$ implies $c(v)\in M[0,\ T]$ and $||c(v_1)-c(v_2)||\leqq\beta||v_1-v_2||$ for v_1 and v_2 from $M[0,\ T]$. By Tihonov's theorem it follows then that the equation (5) has a unique solution $v\in M[0,\ T]$ which may be obtained by iteration, beginning with an arbitrary function from $M[0],T$.

Reference

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