REMARKS ON THE MAXIMUM PRINCIPLE FOR PARABOLIC EQUATIONS AND ITS APPLICATIONS

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Introduction. In [3] Nirenberg has proved maximum principles, both weak and strong, for parabolic equations. In § 1 of this paper we give a generalization of his strong maximum principle (Theorem 1). Hopf [2] and Olainik [4] have proved that if $Lu \geqq 0$ and L is a linear elliptic operator of the second order, if the coefficient of u in L is non-positive, and if u ($\not\equiv$const.) assumes its positive maximum at a point P^0 (which necessarily belongs to the boundary) then $\partial u/\partial \nu < 0$, where ν is the inwardly directed normal. In § 2 we extend this result to parabolic operators (Theorem 2). A further discussion of the assumptions made in Theorem 2 is given in § 3. Application of Theorem 2 to the Neumann problem is given in § 4. In § 5 we apply the weak maximum principle to prove a uniqueness theorem for certain nonlinear parabolic equations with nonlinear boundary conditions, and thus extend the special case considered by Ficken [1]. An even more special case arises in the theory of diffusion (for references, see [1]).

1. Consider the operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t)\frac{\partial u}{\partial x_i} + a(x, t)u - \frac{\partial u}{\partial t} \tag{1}$$

with $a(x, t) \leqq 0$. Here, $(x, t) = (x_1, \cdots, x_n, t)$ varies in the closure $\bar{D}$ of a given $(n+1)$-dimensional domain D. Assume that L is parabolic in $\bar{D}$, that is, for every real vector $\lambda \neq 0$ and for every $(x, t) \in \bar{D}$ we have

$$\sum a_{ij}(x, t)\lambda_i \lambda_j > 0 \ .$$

All the coefficients of L are assumed to be continuous in $\bar{D}$ and u is assumed to be continuous in $\bar{D}$ and to have a continuous t-derivative and continuous second x-derivatives in D. From [3; Th. 5] it follows that, under the above assumptions, *if* $Lu \geqq 0$ *and if* u *assumes its positive maximum at an interior point* P^0, *then* $u \equiv$ *const. in* $S(P^0)$. Here, $S(P^0)$ denotes the set of all points Q in D which can be connected to P^0 by a simple continuous curve in D along which the coordinate t is non-decreasing from Q to P^0. In the following theorem we consider the case

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in which P^0 is a boundary point of D. We may assume that P^0 is the origin. Let $t=\varphi(x)$ be the equation of the boundary of D near P^0. Assume that $t=0$ is the tangent hyperplane to the boundary of D at P^0. Therefore $\partial\varphi/\partial x_i|_{P^0}=0$. Let D be on the side $t<\varphi(x)$.

THEOREM 1. *If $Lu\geqq 0$ in D, if u assumes its positive maximum M at P^0, if*

$$\lim_{P\to P^0}\frac{\partial u(P)}{\partial x_i}=0,\ \lambda\equiv\lim_{P\to P^0}\sum a_{ij}(P)\frac{\partial^2 u(P)}{\partial x_i\partial x_j}\leqq 0 \qquad P\in D \tag{2}$$

and if

$$1+\sum a_{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\bigg|_{P^0}>0 \qquad \varphi\in C'' \tag{3}$$

then $u\equiv M$ in $S(P^0)$.

REMARK 1. Without making any use of (3) one can deduce the following:

Put $\mu\equiv\limsup_{P\to P^0}\dfrac{\partial u(P)}{\partial t}$ $(P\in D)$, then $\mu\geqq 0$ since $\mu<0$ will contradict $u(P^0)\geqq u(P)$. Letting $P\to P^0$ in $Lu(P)\geqq 0$ and using (2), we obtain $\lambda+a(P^0)M-\mu\geqq 0$, from which it follows that $\lambda\geqq 0$. Since, by (2), $\lambda\leqq 0$, we conclude that $\lambda=0$. Hence $a(P^0)M-\mu\geqq 0$, from which it follows that $\mu\leqq 0$ and, therefore, (since $\mu\geqq 0$) $\mu=0$. We also get $a(P^0)=0$.

REMARK 2. The assumptions (2) and (3) can be verified if we assume that $\varphi(x)=o(|x|^2)$ and that u belongs to C'' in the closure of the domain $V\cap\{t<0\}$, where V is some neighborhood of P^0. Indeed, by making an appropriate orthogonal transformation we can assume that $a_{ij}(P^0)=\delta_{ij}$. By the mean value theorem we have

$$u(x,t)-u(0,0)=\sum x_i\frac{\partial}{\partial x_i}u(\tilde{x},\tilde{t})+t\frac{\partial}{\partial t}u(\tilde{x},\tilde{t})\ .$$

Taking $(x,t)\in\bar{D}\cap V\cap\{t<0\}$ such that $|t|=o(|x|)$ and noting that $u(x,t)\leqq u(0,0)$, one can show that $\partial u(P^0)/\partial x_i=0$. Noting that $\varphi(x)=o(|x|^2)$ and expanding $[u(x,t)-u(0,0)]$ in terms of the first and second derivatives of u, one can show that $\partial^2 u(P^0)/\partial x_i^2\leqq 0$, and (2) is thereby proved. The proof of (3) is immediate.

PROOF OF THEOREM 1. For simplicity we shall prove the theorem only in case $n=1$; the proof of the general case is analogous. Lu takes the form

$$Lu\equiv A\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+cu-\frac{\partial u}{\partial t} \qquad c\leqq 0,\ A>0\ . \tag{4}$$

From the strong maximum principle [3; Th. 5] it follows that all we need to prove is that $u(P)\equiv M$ if $P\in V'\cap S(P^0)$ where V' is some neighborhood of P^0.

There are two possibilities: Either there exists a sequence $\{P^k\}$ such that $P^k\in S(P^0)$, $P^k\to P^0$, $u(P^k)=M$, or there exists a neighborhood $V=\{x^2+t^2<R^2\}$ of P^0 such that $u(P)<M$ for all $P\in V\cap S(P^0)$, $P\neq P^0$. In the first case we can use [3; Th. 5] to conclude that $u(P)\equiv M$ if $P\in V'\cap S(P^0)$ where V' is some neighborhood of P^0 (since $u(P)=M$ for all $P\in S(P^k)$).

It remains therefore to consider the case in which $u(P)<M$ for all $P\in V\cap S(P^0)$, $P\neq P^0$. We shall prove that this case cannot occur by deriving a contradiction. Writing

$$\varphi(x)=Kx^2+o(x^2)\ ,$$

we define a domain D_δ ($\delta>0$) as the intersection of $S(P^0)$ with the set of points (x, t) in V for which

$$t<\tilde{\varphi}(x)=(K-\delta)x^2\ .$$

If $K<0$ then, because of (3), we can choose δ sufficiently small such that

$$(5)\qquad 1+A\frac{\partial^2}{\partial x^2}\tilde{\varphi}(x)|_{x=0}>0\ .$$

If $K\geqq 0$, we can obviously take δ such that $K-\delta<0$ and such that (5) holds.

We now can take R sufficiently small such that $\tilde{\varphi}(x)<\min(0, \varphi(x))$ for all (x, t) in D_δ, $x\neq 0$. Consequently, $u(x, t)<M$ if $t=\tilde{\varphi}(x)$, $x\neq 0$. The function $h(x, t)=-t+\tilde{\varphi}(x)$ vanishes on $t=\tilde{\varphi}(x)$ and is positive in D_δ. Therefore, if $\varepsilon>0$ is sufficiently small, then $v=u+\varepsilon h$ is smaller than M at all points on the boundary of D_δ with the exception of P^0, where $v(P^0)=M$. Noting that $\tilde{\varphi}'(0)=0$ and using (5), we conclude that

$$Lh=1+A\tilde{\varphi}''(x)+a\tilde{\varphi}'(x)+ch>0$$

if R has been chosen sufficiently small. Hence, $Lv=Lu+\varepsilon Lh>0$. It follows that v cannot assume its positive maximum at interior points of D_δ and, therefore, it assumes its maximum M at P^0. We thus obtain $\partial v/\partial t\geqq 0$ at P^0 and, consequently,

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}-\varepsilon\frac{\partial h}{\partial t}\geqq\varepsilon>0$$

(Here

$$\frac{\partial g}{\partial t}=\liminf_{t\to 0}\frac{g(0, 0)-g(0, t)}{-t}\ .$$

On the other hand, letting in (4) $P \to P^0$ in an appropriate way and using (2) and the inequality $Lu(P) \geqq 0$, we get

$$0 \leqq \lim A(P)\frac{\partial^2 u(P)}{\partial x^2} + \lim a(P)\frac{\partial u(P)}{\partial x} + C(P^0)M - \lim\sup\frac{\partial u(P)}{\partial t} \leqq$$

$$-\lim\sup\frac{\partial u(P)}{\partial t} .$$

We have thus obtained

$$\limsup_{P\to P^0} \partial u(P)/\partial t \leqq 0 < \varepsilon \leqq \partial u/\partial t .$$

This is however a contradiction (since

$$\frac{\partial u}{\partial t} = \lim_{t_k\to 0} \frac{\partial u(0, t_k)}{\partial t} \leqq \limsup_{P\to P^0} \frac{\partial u(P)}{\partial t}$$

for an appropriate sequence $\{t_k\}$), and the proof is completed.

REMARK (a) Consider the following example: $n=1$, $P^0=(0, 0)$ and D defined by

$$x^2+t^2<R,\ t<\gamma_1 x,\ t<\gamma_2 x \qquad \gamma_1>0>\gamma_2 .$$

The function $u(x, t)=(t-\gamma_1 x)(\gamma_2 x-t)$ satisfies the following properties: $u<0$ in D, $u=0$ at P^0, and

$$Lu \equiv A\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=-2A\gamma_1\gamma_2+0(|x|+|t|)\geqq 0 ,$$

provided R is sufficiently small. Consequently, (3) and the second assumption in (2) are not satisfied and also the assertion of Theorem 1 is false.

REMARK (b). Consider now the case in which the tangent hyperplane at P^0 is not of the form $t=$const.. We shall prove that in this case Theorem 1 is false. Take $n=1$ and consider first the case in which D is defined by

$$x>0,\ x^2+t^2<R^2 .$$

If $Lu \equiv \partial^2 u/\partial x^2 - \partial u/\partial t$, then the function $u(x, t)=-x$ takes its maximum in $\bar{D}$ at $P^0=(0, 0)$, $Lu=0$, but $u\neq 0$ in $S(P^0)$.

Consider next the case in which $\bar{D}$ is defined by

$$x>\alpha t,\ x^2+t^2<R^2 .$$

and take $Lu=\partial^2 u/\partial x^2-\alpha\partial u/\partial x-\partial u/\partial t$.

The transformation $t'=t$, $x'=x-\alpha t$ carries the present case into the previous one.

Note that if the tangent hyperplane H at P^0 is not the plane $t=0$ and the axes are rotated so as to give H the equation $t'=0$ (in new x', t' coordinate), then Lu loses the form (1), for $u_{x't'}$ and $u_{t't'}$ will appear in it.

REMARK (c). If in Theorem 1 the domain D is on the side $t>\varphi(x)$, then the theorem is false. Indeed, as a counter-example take $u=-t$, and D bounded from below by $t=0$.

2. Consider the linear operator

$$L'u\equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i,j=1}^{m} b_{ij}(x,t)\frac{\partial^2 u}{\partial t_i \partial t_j}+\sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{m} b_i(x,t)\frac{\partial u}{\partial t_i}+a(x,t)u \qquad a(x,t)\leqq 0, \tag{6}$$

where $x=(x_1,\cdots,x_n)$ and $t=(t_1,\cdots,t_m)$ vary in the closure of a given $(n+m)$-dimensional domain D. We assume that L' is elliptic in the variables x and parabolic in the variables t, that is, for every real vector $\lambda\neq 0$,

$$\sum a_{ij}\lambda_i\lambda_j>0, \quad \sum b_{ij}\lambda_i\lambda_j\geqq 0 . \tag{7}$$

All the coefficients appearing in (6) are assumed to be continuous in $\bar{D}$ and u is assumed to be continuous in $\bar{D}$ and to have a continuous t-derivative and continuous second x-derivatives in D. Under these assumptions, Nirenberg [3 ; Th. 2] has proved a weak maximum principle from which it follows that, *if $L'u\geqq 0$ in D then u must assume its positive maximum on the boundary.*

Let $P^0=(x^0,t^0)$ be a point on the boundary of D such that $u(P^0)=M>0$ is the maximum of u in $\bar{D}$. Assume that there exists a neighborhood V: $|x-x^0|^2+|t-t^0|^2<R_0^2$ of P^0 such that $u(x,t)<M$ in $V\cap D$. We then can prove the following theorem.

THEOREM 2. *If there exists a sphere S: $|x-x'|^2+|t-t'|^2<R^2$ passing through P^0 and contained in $\bar{D}$, and if $x^0\neq x'$ then, under the assumptions made above (in particular, $L'u\geqq 0$, $u(x,t)<M$ in $V\cap D$), every non-tangential derivative $\partial u/\partial\tau$ at (x^0,t^0), understood as the limit inferior of $\Delta u/\Delta\tau$ along a non-tangential direction τ, is negative.*

By a non-tangential direction we mean a direction from P^0 into the interior of the sphere S.

REMARK (a). If $a(x, t) \equiv 0$ then the assumption $M>0$ is superflous.

REMARK (b). In § 3 we shall show that the assumption $x^0 \neq x'$ is essential. We shall also discuss the case in which $u(x, t)$ is not smaller than M at all the points of $V \cap D$.

Proof. For simplicity we give the proof in the case $m=n=1$, so that

$$L'u \equiv A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial t}+cu \quad A>0,\, B\geqq 0,\, c\leqq 0; \tag{8}$$

the proof of the general case is quite similar. Without loss of generality we can take $(x', t')=(0, 0)$ and $x^0>0$. Furthermore, we may assume that, with the exception of P^0, S lies in $V\cap D$, so that $u(x, t)<M$ in $S-P^0$. Denote by C the intersection of S with the plane $x>\delta$, where $0<\delta<x^0$. The function

$$h(x, t)=\exp(-\alpha(x^2+t^2))-\exp(-\alpha R^2)$$

satisfies the following properties: $h=0$ on the boundary of S, $h\geqq 0$ in C; if α is large enough, then

$$\begin{aligned} L'h=&\exp(-\alpha(x^2+t^2))[4\alpha^2(Ax^2+Bt^2)-2\alpha(A+B+ax+bt)+c] \\ &-c\exp(-\alpha R^2)>0. \end{aligned}$$

(Here we used $x>\delta>0$, $c\leqq 0$.)

If ε is sufficiently small, then the function $v=u+\varepsilon h$ is smaller than M at all points of the boundary of C with the exception of P^0, where $v(P^0)=M$. Since $L'v=L'u+\varepsilon L'h>0$, v cannot assume its positive maximum in $\overline{C}$ at the interior of C (since, otherwise, at such interior points $L'v$ would be non-positive). Hence, v assumes its maximum at P^0 and, consequently, $\partial v/\partial\tau=\liminf(\Delta v/\Delta\tau)\leqq 0$. Since along the normal ν (i.e., along the radius through P^0) $\partial h/\partial\nu>0$ and since along the tangential direction σ $\partial h/\partial\sigma=0$, it follows that $\partial h/\partial\tau>0$. Using the definition of v, we conclude that $\partial u/\partial\tau=\partial v/\partial\tau-\varepsilon\partial h/\partial\tau<0$, and the proof is completed.

Added in proof. Theorem 2 was recently and independently proved also by R. Viborni, *On properties of solutions of some boundary value problems for equations of parabolic type*, Doklody Akad. Nauk SSSR, 117 (1957), 563–565.

3. **From now on we shall consider** only parabolic operators of the form (1). Suppose the assumption $u<M$ in $V\cap D$, made in Theorem 2, is replaced by $u\leqq M$. If there exists a sequence of points $\{P^k\}$ such

that $P^k \to P^0$, $P^k \in D$, $P^k=(x^k, t^k)$ and $t^k \geqq t^0$, $u(P^k)=M$, then, by [3 ; Th. 5], $u \equiv M$ in $S(P^k)$. Hence, if the boundary of D near P^0 is sufficiently smooth, $u \equiv M$ in some set $V' \cap D$ where V' is some neighborhood of P^0. Consequently $\partial u/\partial \tau=0$ for every τ.

If $u(P) \leqq M$ for all $P \in V \cap D$, if $u(P)$ is not strictly smaller than M for all $P \in V \cap D$, $P \neq P^0$, and if the previous situation does not arise, then one and only one of the following cases must occur:

(i) $u<M$ at all points (x, t) in $V \cap D$ with $t \geqq t^0$. Using [3; Th. 5] one can easily conclude that there exists a neighborhood V' of P such that $u<M$ in $V' \cap D$, and Theorem 2 remains true.

(ii) $u<M$ at all points (x, t) in $V \cap D$ with $t>t_0$ and $u \equiv M$ at all points (x, t) in $V \cap D$ with $t \geqq t_0$. We then consider only those directions τ along which $u<M$. We claim that *Theorem* 2 *is not true for the present situation.* To prove this, consider the following simple counter-example:

$$P^0=(0,0),\ M=0,\ Lu=\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\ ,\ u(x, t)=\begin{cases} -t^2 & \text{if } t>0 \\ 0 & \text{if } t<0\ . \end{cases}$$

u satisfies $Lu \geqq 0$ and assumes its maximum 0 for $t \leqq 0$. But, the derivative $\partial u/\partial \tau$ at $P^0=(0, 0)$, along any direction τ, is zero.

As another counter-example (with $Lu=0$) one can take a fundamental solution of the heat equation.

Note that the preceding counter-examples are valid without any assumptions on the behavior of the boundary of D near P^0.

We shall now consider the case $x^1=x^0$ which was excluded by the assumptions of Theorem 2. We shall assume that at $P^0=(0, 0)$ there passes a tangent hyperplane $t=0$. If D is above this hyperplane, then the preceding counter-examples show that Theorem 2 is not true. It remains to consider the case in which D is "essentially" below $t=0$, that is, if we denote by $t=\varphi(x)$ the equation of the boundary of D near P^0, then D is on the side $t<\varphi(x)$. In this case, however, Theorem 1 tells us that in general we cannot assume both $u(P^0)=\max\ u(P)>0$ $(P \in \overline{D})$ and $u<u(P^0)$ in $V \cap D$.

The example in § 1 Remark (a) can also serve as a counter-example to Theorem 2 in case P^0 is a vertex-point. Indeed, along the t-direction

$$\left.\frac{\partial u}{\partial t}\right|_{P^0}=\left.\frac{\partial}{\partial t}[(t-\gamma_1 x)(\gamma_2 x-t)]\right|_{x=0,\, t=0}=0\ .$$

By a small modification of this counter-example one can get a counter-example to the analogue of Theorem 2 for elliptic operators [2] [4] in case P^0 is a vertex. Indeed, define D by

$$x^2+y^2<R^2,\ y<\gamma_1 x,\ y>\gamma_2 x \qquad \gamma_1>0>\gamma_2,$$

and take $Lu=\partial^2u/\partial x^2+A\partial^2u/\partial y^2$, where $A>|\gamma_1\gamma_2|$. The function $u(x, y)=(y-\gamma_1x)(y-\gamma_2x)$ satisfies: $u<0$ in D, $u=0$ at the origin, $Lu=2\gamma_1\gamma_2+2A>0$. But along any direction τ within D, $\partial u/\partial\tau|_{x=0, y=0}=0$.

4. **Let *D* be a domain bounded** by the two hyperplanes $t=0$, $t=T>0$ and a surface B between them. Assume that the intersection $\{t=T\}\cap\bar{D}$ is the closure of an open set on $t=T$, and denote by A the boundary of D on $t=0$. The Neumann problem for the parabolic equation $Lu=0$ consists in finding a solution to the equation $Lu=0$ which satisfies the following initial and boundary conditions:

$$u=f \text{ on } A, \quad \frac{\partial u}{\partial\nu}=g \text{ on } B$$

(f, g are given functions).

From Theorem 2 and from the strong maximum principle [3; Th. 5] we conclude: *If for every point $P^0=(x^0, t^0)$ of B (i) there exists a sphere with center (x', t'), $x'\neq x^0$, passing through P^0 and contained in $\bar{D}$, and (ii) $\overline{S(P^0)}$ contains interior points of A, then the Neumann problem has at most one solution.* Clearly, this uniqueness property holds also for the more general problem where $\partial u/\partial\nu$ is replaced by $\partial u/\partial\tau$ and τ is a non-tangential direction which varies on B.

As another application to Theorem 2, one can deduce the positivity of $\partial G/\partial\nu$, where G is the Green's function of $Lu=0$.

5. **Let *D* be a domain bounded** by $t=0$, $t=T\ (0<T\leqq\infty)$ and surfaces Γ_k, $0\leqq k\leqq m$, Γ_0 being the outer boundary. Suppose further that the intersection of each Γ_k with $t=t_0\ (0\leqq t_0<T)$ is a simple closed curve $\gamma_k(t_0)$ which belongs to $C^{(3)}$ and does not reduce to a single point. Write $u_{x_i}=\partial u/\partial x_i$, $u_t=\partial u/\partial t$. We shall consider the following problem P:

$$\sum_{i,j=1}^{n} a_{ij}(x, t)u_{x_ix_j}-u_t=c(x, t, u, \nabla u) \tag{9}$$

(where ∇u denotes the vector $\partial u/\partial x_i$),

$$\frac{\partial u}{\partial\tau}\equiv\sum_{i=1}^{n}\alpha_i(x, t)u_{x_i}+\alpha(x, t)u_t=\varphi(x, t, u) \quad (x, t)\in\Gamma=\sum_{k=0}^{m}\Gamma_k \tag{10}$$

$$u(x, 0)=\psi(x) \text{ on } A \qquad A=\bar{D}\cap\{t=0\} \tag{11}$$

We make the following assumptions:

(a) $a_{ij}(x, t)$ is continuous in $\bar{D}$; $c(x, t, u, \nabla u)$ and it first derivatives with respect to u, ∇u are continuous for $(x, t)\in\bar{D}$ and for all values of u, ∇u.

(b) φ and $\partial\varphi/\partial u$ are continuous for all $(x, t) \in \Gamma$ and for all u.

(c) $\alpha_i(x, t), \alpha(x, t)$ are continuous for $(x, t) \in \Gamma$; $\psi(x)$ is continuous in A.

(d) (9) is parabolic in $\bar{D}$, that is, there exists a positive constant δ such that

$$\sum a_{ij}(x, t)\xi_i\xi_j \geqq \delta \sum \xi_i^2 \tag{12}$$

holds for all real ξ and for all $(x, t) \in \bar{D}$.

(e) On each surface Γ_k $(k=0, 1, \cdots, m)$ either all the directions $\tau=(\alpha_i, \alpha)$ are exterior or all are interior, and in the exterior case $\alpha \geqq 0$ and the directions $(\alpha_i, 0)$ are exterior while in the interior case $\alpha \leqq 0$ and the directions $(\alpha_i, 0)$ are interior.

Denote by $\sum$ the class of functions $u(x, t)$ defined and continuous in $\bar{D}$ and satisfying the following conditions:

(α) $\partial u/\partial t, \partial u/\partial x_i, \partial^2 u/\partial x_i \partial x_j$ are continuous in D ;

(β) For every $R>0$, $\partial u/\partial x_i$ is bounded in $D \cap \{|x|^2+t^2<R^2\}$.

THEOREM 3. *Under the assumptions* (a)−(e) *the problem P cannot have two different solutions in the class $\sum$.*

We shall need the following lemma.

LEMMA. *There exists a function $\zeta(x)$ defined in A and having the following properties*: (i) *ζ has continuous first derivatives in A and continuous second derivatives in the interior of A*; (ii) *$\partial\zeta/\partial\nu=-1$ and $\partial\zeta/\partial\mu=0$ on $\gamma_0(0), \cdots, \gamma_m(0)$, where $\partial/\partial\nu$ and $\partial/\partial\mu$ denote the derivatives with respect to the interior normal and to any tangential direction, respectively.*

PROOF OF THE LEMMA. It will be enough to construct a function $\chi_0(x)$ which is C'' in A, which vanishes in a neighborhood of $\gamma_i(0)$ $(i=1, \cdots, m)$ and for which $\partial\chi_0/\partial\nu=-1$, $\partial\chi_0/\partial\mu=0$ along $\gamma_0(0)$; constructing $\gamma_1(x)$ in a similar manher, we can then take $\zeta(x)=\sum \chi_1(x)$. Since $\gamma_0(0)$ belongs to $C^{(3)}$, the normals issuing from $\gamma_0(0)$ and inwardly directed cover in a one-to-one manner a small inner neighborhood of $\gamma_0(0)$, call it A_0. To each point x in A_0 there corresponds a unique point x^0 on the boundary of $\gamma_0(0)$, such that x lies on the normal through x^0. Denote by $\sigma(x)$ the distance $|x-x^0|$. It is elementary to show that $\sigma(x)$ has continuous second derivatives in A_0. Denote by A_1 the domain $0 \leqq \sigma \leqq \varepsilon_0$, where $\varepsilon_0>0$ is small enough so that $A_1 \subset A_0$. The function

$$\chi_0(x)=\begin{cases} \dfrac{1}{3\varepsilon_0^2}(\varepsilon_0-\sigma(x))^3 & \text{if } x \in \bar{A}_1 \\ 0 & \text{if } x \in A-A_1 \end{cases}$$

belongs to C'' in A and satisfies: $\partial\chi_0/\partial\nu=\partial\chi_0/\partial\sigma=-1$ and $\partial\chi_0/\partial\nu=0$ on $\gamma_0(0)$, and χ_0 vanishes near $\gamma_k(0)$, $(1 \leqq k \leqq m)$; the proof is completed.

Proof of Theorem 3. We first consider the case $n>1$. We may suppose that the vectors (α_i, α) are exterior directions on $\Gamma_0, \cdots, \Gamma_q$ and that (α_i, α) are interior directions on $\Gamma_{q+1}, \cdots, \Gamma_m$. Suppose now that u and v are two solutions in $\sum$ of the problem P, and define $w=v-u$. Writing

$$C(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u} c(x, t, u+\lambda w, \nabla u+\lambda \nabla w)d\lambda$$

$$C_i(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u_{x_i}} c(x, t, u+\lambda w, \nabla u+\lambda \nabla w)d\lambda$$

$$\Phi(x, t, u, v)=\int_0^1 \frac{\partial}{\partial u} \varphi(x, t, u+\lambda w)d\lambda$$

and using (9), (10) and (11), we obtain for w the following system:

$$\sum a_{ij} w_{x_i x_j} - w_t = Cw + \sum C_i w_{x_i} \tag{13}$$

$$\frac{\partial w}{\partial \tau} = \sum \alpha_i w_{x_i} + \alpha w_t = \Phi w \tag{14}$$

$$w(x, 0)=0 . \tag{15}$$

Substituting $w(x, t)=z(x, t)\exp(Kt+M\zeta(x))$, where $\zeta(x)$ is the function constructed in the lemma and K, M are constant to be determined later, we get for z the following system:

$$\begin{aligned} \sum a_{ij} z_{x_i x_j} - z_t = & -M\sum a_{ij}\zeta_{x_i x_j} z - M^2 \sum a_{ij}\zeta_{x_i}\zeta_{x_j} z \\ & -2M\sum a_{ij}\zeta_{x_i} z_{x_j} + Kz + Cz + M\sum C_i \zeta_{x_i} z + \sum C_i z_{x_i} \end{aligned} \tag{13'}$$

$$\frac{\partial z}{\partial \tau} \equiv \sum \alpha_i z_{x_i} + \alpha z_t = -M\sum \alpha_i \zeta_{x_i} z - \alpha K z + \Phi z \tag{14'}$$

$$z(x, 0)=0 . \tag{15'}$$

If $0\leqq k\leqq q$, $\alpha \geqq 0$ and $\sum \alpha_i(x, 0)\zeta_{x_i}(x)>0$ on $\gamma_k(0)$, since the angle between the vectors (α_i) and grad ζ is $<\pi/2$. By continuity we get $\sum \alpha_i(x, t)\zeta_{x_i}(x) \geqq \eta > 0$ on $\gamma_k(t)$, provided $0\leqq t \leqq T'$ and T' is sufficiently small. Hence, we can choose M sufficiently large such that

$$-M\sum \alpha_i \zeta_{x_i} - \alpha K + \Phi < 0 \tag{16}$$

holds on $\gamma_k(t)$, provided $K\geqq 0$ and $0\leqq t\leqq T'$.

If $q+1\leqq k \leqq m$, $\alpha\leqq 0$ and $\sum \alpha_i(x, 0)\zeta_{x_i}(x)<0$, since the angle between (α_i) and $-$grad ζ is $<\pi/2$. Again, if $K\geqq 0$ and M is sufficiently large, then

$$-M\sum \alpha_i \zeta_{x_i} - \alpha K + \Phi > 0 \tag{17}$$

on $\gamma_k(t)$, $0 \leqq t \leqq T'$.

Having fixed M, we now choose K sufficiently large so that the coefficient of z on the right side of (13′) becomes positive in the domain $D_{T'} = D \cup \{0 < t < T'\}$. We claim that $z \equiv 0$ in $D_{T'}$. Indeed, if this is not the case then, using (15′) and the weak maximum principle [3 ; Th. 2] we conclude that z assumes either its positive maximum or its negative minimum on the boundary $\sum_{k=0}^{m} \gamma_k(t)$, $0 \leqq t \leqq T'$, of $D_{T'}$. It will be enough to consider the case in which z assumes its positive maximum at a point P^0 on $\gamma_k(t)$. If $0 \leqq k \leqq q$, then $\partial z / \partial \tau \geqq 0$ since τ is outwardly directed. On the other hand, using (14′) and (16) we get $\partial z / \partial \tau < 0$, which is a contradiction. If $q+1 \leqq k \leqq m$, then $\partial z / \partial \tau \leqq 0$ since τ is inwardly directed. On the other hand, using (14′) and (17) we get $\partial z / \partial \tau > 0$ which is a contradiction. We have thus proved that $z \equiv w \equiv 0$ in $D_{T'}$. We can now apply a classical procedure of continuation and thus complete the proof of the theorem for the case $n > 1$.

In the case $n=1$, $\Gamma = \Gamma_0$ is composed of two curves Γ_{01} aud Γ_{02}. Suppose Γ_{0k} intersects $t=0$ at a_k, $a_1 < a_2$. The function

$$\zeta(x) = \frac{(x-a_1)(x-a_2)}{a_2 - a_1}$$

can be used in the preceding proof. Note that it is not necessary to make any assumptions on the smoothness of the curves Γ_{0k}.

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