A FIXED POINT THEOREM FOR MULTI-VALUED FUNCTIONS

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1. Introduction. If Y is a topological space then $S(Y)$ denotes the space of closed subsets of Y, endowed with the finite topology [2]. We say that a multi-valued function $F: X \rightarrow Y$ is continuous provided $F(x)$ is a closed set for each $x \in X$, and the induced (single-valued) function $f: X \rightarrow S(Y)$ is continuous in the usual sense. This definition of continuity for multi-valued functions is equivalent to that of Strother [4]. The space X is said to have the F.p.p. (= fixed point property for continuous multi-valued functions) if and only if for each such function $F: X \rightarrow X$, there is an $x \in X$ such that $x \in F(x)$. The space X has the f. p. p. if it has the fixed point property for continuous single-valued functions. It is hardly surprising that the T_1-spaces with the F. p. p. constitute a fairly small subclass of those with f. p. p. Indeed, Plunkett [3] has shown that a Peano continuum has the F. p. p. if and only if it is a dendrite. It is worth noting that Plunkett's argument employs the convex metric of a dendrite in much the same manner as the author [7] has used the order structure of certain acyclic continua to obtain fixed point theorems. A related argument has been used by Capel and Strother [1] to show that a tree has the fixed point property for continuous multi-valued functions for which the image of each point is connected. Their proof depends on being able to produce a continuous selection, in the sense of Michael, on the class of subcontinua of a tree.

In this paper an order-theoretic characterization of a wide class of acyclic spaces is given. This characterization is in the same spirit as the analogous results of [6] and [7] for trees and generalized trees. It is then shown, using their order properties, that such spaces have the F. p. p. To some extent the argument borrows from all of the proofs cited above.

2. Topologically chained continua. If X is a space and $\leqq$ is a partial order on X, we write $L(x) = \{a : a \leqq x\}$ and $M(x) = \{a : x \leqq a\}$. It is natural and convenient to define

$$[x, y] = M(x) \cap L(y)$$

and, if $A \subset X$, we write $M(A)$ for the union of all $M(x)$ for which

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$x \in A$. An *antichain* of X is a subset in which no two distinct elements are related under the partial order. A *zero* of a subset A of X is a member a_0 of A such that $A \subset M(a_0)$.

In what follows it will be convenient to use two theorems from [5]. These are stated below as Theorems A and B. The partial-order $\leqq$ is said to be *upper* (*lower*) *semicontinuous* provided $M(x)$ ($L(x)$) is closed, for each $x \in X$. It is *semicontinuous* if it is both upper and lower semicontinuous.

THEOREM A. *If the compact space X is endowed with an upper (lower) semicontinuous partial order then X admits a maximal (minimal) element.*

THEOREM B. *If the compact space X is endowed with an order-dense semicontinuous partial order and if the set of maximal elements or the set of minimal elements is connected, then X is connected.*

Recall that a continuum (= compact connected Hausdorff space) is *unicoherent* if it cannot be represented as the union of two subcontinua whose intersection is not connected. A continuum is *hereditarily unicoherent* if each of its subcontinua is unicoherent.

A continuum is *topologically chained* if each pair of points is contained in a subcontinuum which has exactly two non-cutpoints, and such a subcontinuum is called a topological chain. This concept is a natural generalization of the notion of an arc. Note that, if x and y are distinct elements of a continuum which is topologically chained and hereditarily unicoherent, there is a unique subcontinuum $C(x, y)$ which is irreducible about x and y. Moreover, $C(x, y)$ is a topological chain with x and y for endpoints.

Consider the following five properties enjoyed by some spaces X admitting a partial order, $\leqq$.

I. *$[x, y]$ is closed and simply ordered for each x and y in X.*

II. *$\leqq$ is order-dense.*

III. *There exists $e \in X$ such that $M(e) = X$.*

IV. *If x and y are points of the subcontinuum Y and $x \leqq y$, then $[x, y] \subset Y$.*

V. *If A is an antichain of X and P is a continuum contained in $M(A)$, then $P \subset M(x)$ for some $x \in A$.*

LEMMA 1. *If X is a compact Hausdorff space with a partial order satisfying* I *and* II, *and if $x < y$ in X, then $[x, y]$ is a topological chain,* *Proof.* For each $t \in [x, y]$ the sets

$$L(t) \cap [x, y] = [x, t],$$
$$M(t) \cap [x, y] = [t, y]$$

are closed so that $\leqq$ is semicontinuous on $[x, y]$. Since $[x, y]$ has only one minimal element it is connected by Theorem B. If $x < t < y$ then the continua $[x, t]$ and $[t, y]$ have only the point t in their intersection, so that x and y are the only non-cutpoints of $[x, y]$; that is, $[x, y]$ is a topological chain.

LEMMA 2. *If X is a compact Hausdorff space with a partial order satisfying* I, III, *and* V, *then each continuum contained in X has a zero.*

Proof. By I and III the set $L(x) = [e, x]$ is closed for each $x \in X$ and consequently, by Theorem A, each compact subset of X contains a minimal element. If Y is a subcontinuum of X then the set A of minimal elements of Y forms a nonempty antichain. It follows from V that A contains only one element, that is, Y has a zero.

THEOREM 1. *A necessary and sufficient condition that X be a topologically chained, hereditarily unicoherent continuum is that X be a compact Hausdorff space which admits a partial order satisfying* I-V.

Proof. Necessity. Let X be a topologically chained, hereditarily unicoherent continuum. Fix $e \in X$ and let $x \leqq y$ mean that $x \in C(e, y)$. Then $L(x) = C(e, x)$ and I, II and III are easily verified. If $x \leq y$ and x and y are elements of the subcontinuum Y then, by hereditary unicoherence, $[x, y] \cap Y$ is connected. Since x and y are elements of Y it follows that $[x, y] \subset Y$. This establishes IV. To see that V is satisfied let A be an antichain of X and suppose that P is a continuum contained in $M(A)$. If P meets $M(x)$ and $M(y)$ where x and y are distinct points of A, we may select $p \in M(x) \cap P$ and $q \in M(y) \cap P$. In view of I the points p and q are not comparable. Therefore, the continua $L(p) \cup P$ and $L(q) \cup P$ meet in the non-connected set $P \cup (L(x) \cap L(y))$, and this contradicts the hereditary unicoherence of X.

Sufficiency. Let X be a compact Hausdorff space which admits a partial order satisfying I-V. By Lemma 1 each set $L(x) = [e, x]$ is a topological chain and therefore X is a topologically chained continuum. Now suppose A and B are subcontinua of X and that x and y are distinct elements of $A \cap B$. In order to show that $A \cap B$ is connected and thus that X is hereditarily unicoherent it is sufficient to prove that

$$Z = [z, x] \cup [z, y] \subset A \cap B,$$

where z is the supremum of $L(x) \cap L(y)$. The existence of z is assured by Theorem A. By Lemma 2, A has a zero, a_0, so that a_0 is a predecessor of x and y and hence also of z. By IV,

$$[z, x] \subset [a_0, x] \subset A .$$

It is clear that, by a sequence of analogous arguments, $Z \subset A \cap B$.

3. **The fixed point theorem.** The proof of Theorem 2 which follows is patterned after that of Theorem 10 in [7] where it was assumed that the partial order had a closed graph. In view of our weaker hypotheses it has been necessary to revise that argument extensively. For the remainder of this paper the term *sequence* is used in its generalized sense, that is, a function defined on the predecessors of some ordinal number. If x is a sequence then a *subsequence* of x is the restriction of x to some cofinal subset of its domain.

A useful continuity property of a partial order satisfying I-V is given by the following lemma.

LEMMA 3. *Let X be a compact Hausdorff space which contains no indecomposable continuum and which admits a partial order satisfying* I-V. *If x is an increasing sequence in X then* $\lim x$ *exists and* $x_\alpha \leqq \lim x$ *for each α in the domain of x.*

Proof. Let x_0 be a limit point of x and suppose $x_0 \in X - M(x_\beta)$ for some β. Without loss of generality we may take $\beta = 1$. Let C be the union of the topological chains $[x_\alpha, x_{\alpha'}]$, $\alpha' > \alpha$. If σ is the zero of $\overline{C}$ then $x_0 \in \overline{C}$ implies $\sigma < x_1$. If K is a subcontinuum of $\overline{C}$ which contains the values of some subsequence of x, then by IV, $x_\alpha \in K$ implies $[x_\alpha, x_\alpha'] \subset K$ for each $\alpha' > \alpha$ and hence K contains x_0 and σ. Again by IV, $K = \overline{C}$. It follows that $\overline{C}$ is the union of no two proper subcontinua and this contradicts our hypothesis that X contains no indecomposable continuum. Hence $x_\alpha \leqq x_0$ for each α in the domain of x. If $y \in X$ is not a predecessor of x_0 then y is a member of the open set $U = X - L(x_0)$. Since $x_\alpha \in X - U$ for each α, y is not a limit point of x. Therefore x_0 is the unique limit point of x, that is, $x_0 = \lim x$.

LEMMA 4. *Let K be a connected topological space and let F be a continuous multi-valued function defined on K. Suppose that $F(x)$ is a compact set for each $x \in K$. If Q is a quasi-component of $F(K)$ then $F(x) \cap Q$ is nonempty for each $x \in K$.*

Proof. By definition $Q = \cap \{V_\alpha\}$ where $\{V_\alpha\}$ is the family of open and closed sets containing Q. If $F(x)$ meets each V_α then, by the compactness of $F(x)$, it meets Q and therefore $F^{-1}(Q) = \cap \{F^{-1}(V_\alpha)\}$. By an elementary continuity argument it can be verified that each set $F^{-1}(V_\alpha)$ is both open and closed and hence is equal to K. Therefore, $K = F^{-1}(Q)$ and the lemma follows.

For the remainder of this section, X denotes a topologically chained, hereditarily unicoherent continuum which contains no indecomposable subcontinuum, and $F: X \rightarrow X$ is a continuous multi-valued function. By Theorem 1 the space X admits a partial order satisfying I-V. Let J be the set of all elements x of X such that (i) there exists a minimal element t_x in the set $F(x) \cap M(x)$, and (ii) if $x < p \leqq t_x$ then $F(p)$ and $[p, t_x]$ are disjoint.

LEMMA 5. *If F is fixed point free then J is not empty. Indeed if $a \in X$ such that $F(a) \cap M(a)$ is not empty then $M(a) \cap J$ is not empty.*

Proof. By Lemma 2, X has a zero and hence there exists $a \in X$ such that $F(a) \cap M(a)$ is not empty. If $b \in F(a) \cap M(a)$ then by Theorem A there is a minimal element t_a in the compact set $F(a) \cap [a, b]$. Clearly t_a is minimal in $F(a) \cap M(a)$. Let K denote the set of all $p \in [a, t_a]$ such that $F(p) \cap [p, t_a]$ is not empty. Then $a \in K$ and $t_a \in X - K$. If $x_0 = \sup K$ it is easy to construct an increasing sequence y in K such that $\lim y = x_0$. For each α in the domain of y let $z_\alpha \in F(y_\alpha) \cap [y_\alpha, t_a]$. Then the sequence z has a limit point z_0 and by continuity $z_0 \in F(x_0)$. Further, since $y_\alpha < z_\alpha \leqq t_a$ for each α, it follows that $z_0 \in [x_0, t_a]$ and hence $x_0 \in K$. Now let t_0 be minimal in $F(x_0) \cap [x_0, z_0]$. If $x_0 < p \leqq t_0$ and $F(p) \cap [p, t_0]$ is not empty then $t_0 \leqq t_a$ implies that $F(p) \cap [p, t_a]$ is not empty. But this is a contradiction since x_0 is the supremum of K. Therefore $F(p)$ and $[p, t_0]$ are disjoint, that is, $x_0 \in J$ and the lemma is proved.

Consider the set S of all sequences x such that J contains the range of x and

$$x_1 < x_2 < \cdots < x_\alpha < \cdots,$$

$$t_\alpha \leqq x_{\alpha+1} \text{ for each } \alpha + 1 \text{ in the domain of } x,$$

where $t_\alpha = t_{x_\alpha}$. If J is not empty then S contains at least one nonempty sequence. We partially order the elements of S in the following manner: x precedes y provided x is an initial segment of y. Clearly, if N is a simply ordered subset of S then $\cup N$ is a member of S. By Zorn's lemma S contains a maximal element.

LEMMA 6. *If F is fixed point free and x is an element of S then the domain of x is not a limit ordinal.*

Proof. If the domain of x is a limit ordinal, let $x_0 = \lim x$, $t_0 = \lim t$. By Lemma 3, $x_\alpha \leqq x_0$ and $t_\alpha \leqq t_0$ for each α. From the definition of J we have $x_\alpha < t_\alpha$ for each α and hence $x_0 \leqq t_0$. But by the definition of S, $t_{\alpha+1} \leqq x_\alpha$ for each α and hence $t_0 \leqq x_0$. This implies that $x_0 = t_0$ is a fixed point of F, which is a contradiction.

LEMMA 7. *If F is fixed point free and x is an element of S then the domain of x is finite.*

Proof. Suppose the domain of x is not finite. We denote by ω the first infinite ordinal. Then by Lemma 6, ω is a member of the domain of x. From the definition of S it is clear that x, restricted to ω, is an element of S, contrary to Lemma 6.

THEOREM 2. *Each topologically chained, hereditarily unicoherent continuum which contains no indecomposable continuum has the* F. p. p.

Proof. If $F: X \rightarrow X$ is fixed point free then, by Lemma 5, J is not empty and hence S contains a maximal element, x. By Lemma 7 the domain of x is a set $\{1, 2, \cdots, N\}$ of integers. We write $t_N = t_{x_N}$. By Lemma 5 and the maximality of x it follows that $M(t_N) \cap F(t_N)$ is empty. Now from Corollary 9.6 of [2] the set $F([x_N, t_N])$ is compact and hence its components and quasi-components are identical. Let D be that component of $F([x_N, t_N])$ such that $t_N \in D$. By Lemma 4, $F(t_N) \cap D$ is not empty and hence $D - M(t_N)$ is not empty. Now D is a continuum and hence has a zero z which precedes t_N. By IV, $[z, t_N] \subset D$ but since F is fixed point free there exists $q \in X$ with $x_N < q < t_N$ such that $[q, t_N] \subset D - F(t_N)$. Let b be an infinite increasing sequence in $[q, t_N]$ such that $\lim b = t_N$. For each b_α there exists a_α, $x_N < a_x < t_N$, such that $b_\alpha \in F(a_\alpha)$. Since $x_N \in J$ we infer that $b_\alpha < a_\alpha$. Clearly $\lim a = t_N$ from which it follows that $t_N \in F(t_N)$, a contradiction. Therefore F cannot be fixed point free.

4. Remarks.

The following question is of some interest. Can the assumption of hereditary decomposability be omitted from the hypotheses of Theorem 2? This assumption is essential to Lemma 3 so that an affirmative answer to our question would require a substantially different proof. Even more difficult is the problem of characterizing the F. p. p. among the topologically chained continua. In view of Plunkett's characterization [3] of the F. p. p. for Peano continua, there would seem to be some hope of discovering a succinct characterization.

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