(C, ∞) AND (H, ∞) METHODS OF SUMMATION

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1. Introduction. Let

$$T: \qquad \beta_n = \sum_{m=1}^{\infty} \tau_{nm} \cdot \alpha_m \, , \qquad n = 1, 2, \cdots$$

be a linear transformation of the sequence $\alpha = \{\alpha_n\}$ into the sequence $\beta = \{\beta_n\}$; we write

$$\beta = T\alpha \, , \quad \beta_n = (\beta)_n = (T\alpha)_n \, .$$

α is said to be *summable T to the value a* if

$$(1) \qquad \lim_{n\to\infty} (T\alpha)_n = a \, ;$$

and U is said to contain T if every sequence summable T to the value a is also summable U to a. In particular T is called regular if it contains the identity transformation I.

We shall generalize the concept of regularity in several directions. A sequence of transformations $\{T_k\}$ $(k \geqq 0, T_0 = I)$ will be called *regular* if each T_k is included in T_{k+1}. As an example of a regular sequence we mention the iterates of a regular transformation T; they are defined

$$T^0\alpha = \alpha \, , \quad T^{k+1}\alpha = T(T^k\alpha) \, , \qquad k = 0, 1, 2, \cdots \, .$$

Given a regular sequence of transformations, $\{T_k\}$, we say that α is *summable T_∞ to a* if

$$\lim_{k\to\infty} (T_k\alpha)_n = a_n$$

exists for $n \geqq n_0$ and $\lim_{n\to\infty} a_n = a$. For T_∞ to be significant, it is plainly desirable that it shall contain each T_k. A regular sequence $\{T_k\}$ will be called *strongly regular* if, whenever for some k $\lim_{n\to\infty}(T_k\alpha)_n = a$, then α is summable T_∞ to a. With trivial modifications these definitions also apply to families of transformations T_λ which depend on a continuous real parameter $\lambda \geqq 0$.

Of particular interest are sequences of transformations of the type

$$(2) \qquad (T_k\alpha)_n = \frac{(T^k\alpha)_n}{(T^k\varepsilon)_n}$$

where ε is the unit sequence $\varepsilon_n = 1$, $n = 1, 2, \cdots$, and T is a transformation such that $T^k\varepsilon$ exists for $k \geqq 0$. By an extension of the concept of regularity we say that T is *strongly regular* if the particular sequence (2) is strongly regular, and summability T_∞ for this sequence will be denoted

Received Jaunary 27, 1958.

by summability (T, ∞).

To examine the usefulness of these concepts and in particular the possibility of 'infinite iteration', let us consider the Hölder process

$$(H\alpha)_n = \frac{1}{n}\sum_{m=1}^{n}\alpha_m .$$

It follows from

$$(H^{k+1}\alpha)_n = \frac{1}{n}\sum_{m=1}^{n}(H^k\alpha)_m = \frac{n-1}{n}(H^{k+1}\alpha)_{n-1} + \frac{1}{n}(H^k\alpha)_n ,$$

by induction on n, that

$$\lim_{k\to\infty}(H^k\alpha)_n = \alpha_1 ,$$

for every n. Thus we find the disconcerting result that every sequence is summable (H, ∞) to the first term of the sequence.

Similary if $(C\alpha)_n = \sum_{m=1}^{n}\alpha_m$, we have

$$(C^k\alpha)_n = \sum_{m=1}^{n}\binom{n-m+k-1}{n-m}\alpha_m , \quad (C^k\varepsilon)_n = \binom{n+k-1}{n-1} ,$$

and hence

$$\lim_{k\to\infty}\frac{(C^k\alpha)_n}{(C^k\varepsilon)_n} = \alpha_1$$

for every n. As in the case of (H, ∞), we find that the (C, ∞) limit always exists and is equal to the first term of the sequence. Thus neither the Hölder, nor the Cesàro process is strongly regular, and infinite iteration gives nothing useful.

In the next section we shall reconsider the problem of (H, ∞) and (C, ∞) from the point of view of generalized limits of functions; this will lead quite naturally to strongly regular transformations. Here we mention an interesting example of a strongly regular family of transformations, known as the 'circle methods' of Hardy and Littlewood. For $\lambda > 0$ define

$$(3) \qquad (T_\lambda\alpha)_n = \rho^{n+1}\sum_{m=n}^{\infty}\binom{m}{n}(1-\rho)^{m-n}\alpha_m , \qquad n \geqq 0 ,$$

where $\rho = e^{-\lambda}$; (3) certainly exists if

$$(4) \qquad \overline{\lim_{n\to\infty}}\,|\alpha_n|^{1/n} \leqq 1 .$$

If α satisfies this condition and $\mu > \lambda \geqq 0$, then T_μ contains T_λ; this follows from the regularity of T_λ and the formula

$$(5) \qquad T_\mu\alpha = T_{\mu-\lambda}(T_\lambda\alpha)$$

which is valid for sequences satisfing (4) (See [4, p. 218]). It is seen directly from the definition of T_λ that if $\lim_{n\to\infty} \alpha_n = a$ then for every fixed $n \geqq 0$,

$$\lim_{\lambda\to\infty} (T_\lambda \alpha)_n = a ;$$

it follows, by (5), that $\lim_{n\to\infty}(T_\lambda\alpha)_n = a$ for some $\lambda > 0$ implies $\lim_{\mu\to\infty}(T_\mu\alpha)_n = a$. Hence T_∞ contains T_λ, and the family T_λ is strongly regular, at least for sequences which satisfy the condition (4).

2. (C, ∞) and (H, ∞) limits. Let us consider the problem of infinite iteration from the point of view of generalized limits of functions. For the sake of definitness we consider limits at $x = \infty$. We say $f(x)$ has a (generalized) limit a at $x = \infty$ by the process

$$T: \quad Tf(x) = \int_0^\infty \tau(x, t) f(t) dt \tag{6}$$

if $\lim_{x\to\infty} Tf(x)/Te(x) = a$, where $e(x)$ is the unit function $e(x) = 1$ for all $x > 0$; we assume that

$$T^k e(x) = \int_0^\infty \tau(x, t_1)dt_1 \int_0^\infty \tau(t_1, t_2)dt_2 \cdots \int_0^\infty (t_{k-1}, t_k)dt_k$$

exists for every $k > 0$.[1] Integrals of the form $\int_0^\infty \varphi(t)dt$ are understood to be improper Lebesgue integrals in the following sense: $\varphi(t)$ is assumed to be L-integrable in every interval $0 < t_1 \leqq t \leqq t_2 < \infty$ and

$$\int_0^\infty \varphi(t)dt = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \varphi(t)dt + \lim_{x \uparrow \infty} \int_1^x \varphi(t)dt .$$

The *domain* of T is the class of functions $f(x)$ for which the integral (6) exists; but since we are interested in the limit of $f(x)$ when $x \to \infty$, it is convenient also to consider the subclass of these functions in the domain of T for which $f(x) = 0$ for $x < x_0$ (where x_0 is not necessarily the same number for every $f(x)$). This subclass will be called the *essential domain* of T.

The definitions given in § 1 apply equally well to transformations of the form (6); in particular, T is called strongly regular if the sequence

$$T_k f(x) = \frac{T^k f(x)}{T^k e(x)}, \qquad k \geqq 0, \tag{7}$$

is regular and $\lim_{x\to\infty} T_k f(x) = a$ for some $k \geqq 0$ implies

$$\lim_{x\to\infty} \lim_{k\to\infty} T_k f(x) = a .$$

The natural analogues of the Cesàro and Hölder limits at $x = \infty$ are obtained by the transformations

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[1] Unless the contrary is stated, letters $k, m, n, \cdots$ denote non-negative integers.

$$Cf(x) = \int_0^x f(t)dt$$

and

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(xt)dt ;$$

their domain is the class of functions L-integrable over every interval $0 < t_1 \leqq t \leqq t_2 < \infty$ for which $\int_0^1 f(t)dt$ (as an improper integral) exists. Denote this class by $\Phi_{1,0}$. Clearly $f(x) \in \Phi_{1,0}$ implies the existence and continuity of $Cf(x)$ for $x > 0$ and $\lim_{x\downarrow 0} Cf(x) = 0$; therefore $C^k f(x)$ exists for every $k > 0$. On the other hand $H^k f(x)$ does not always exist when $f(x) \in \Phi_{1,0}$; for example $f(x) = (x \log^2 x)^{-1}$ is in $\Phi_{1,0}$, but not $f(x) = (x \log x)^{-1}$, so that $H^2 f(x)$ does not exist ; We denote by $\Phi_{k,0}$ the class of functions for which $H^k f(x)$ exists ; $\Phi_{\infty,0}$ denotes the intersection of all classes $\Phi_{k,0}$; $\Phi_{0,0}$ denotes the class of functions L-integrable over every interval $0 < t_1 \leqq t \leqq t_2 < \infty$. For later use we also define : $\Phi_{0,m}$, the class of functions $s(x)$ such that $f(x) \equiv s(1/x)$ is in $\Phi_{m,0}$; $\Phi_{k,m} = \Phi_{k,0} \cap \Phi_{0,m}$. If $s(x) \in \Phi_{k,m}$ then $f(x) \equiv s(1/x) \in \Phi_{m,k}$. Finally Φ_I shall denote the class of functions for which $\int_0^\infty f(t)dt$ exists, and Φ_B is the subclass of bounded functions of $\Phi_{0,0}$; clearly Φ_I is a subclass of $\Phi_{1,1}$ and Φ_B is a subclass of $\Phi_{\infty,\infty}$.

The examination of the infinite iteration of the C and H methods for functions leads to a result which is analogous in some respects to the corresponding result for sequences. It turns out that the limit by (C, ∞) or (H, ∞), if it exists at all, depends on the behaviour of the function in the neighbourhood of zero rather than infinity. If in particular $\lim_{x\downarrow 0} f(x) = a$ exists then

$$\lim_{k\to\infty} H_k f(x) = \lim_{k\to\infty} C_k f(x) = a$$

for all $x > 0$. More generally we shall show :

THEOREM 1. *Suppose that* $f(x) \in \Phi_{1,0}$ *and*

$$\lim_{x\downarrow 0} C_k f(x) = a \tag{8}$$

for some $k \geqq 0$. *Then*

$$\lim_{n\to\infty} C_n f(\xi) = a \tag{9}$$

for every fixed $\xi > 0$.

THEOREM 1*. *Suppose that* $f(x) \in \Phi_{k,0}$ *for some* $k \geqq 0$ *and*

$$\lim_{x\downarrow 0} H_k f(x) = a , \tag{8*}$$

Then $f(x) \in \Phi_{\infty,0}$ *and*

$$\lim_{n\to\infty} H_n f(\xi) = a \tag{9*}$$

for every fixed $\xi > 0$.

Theorem 1 and Theorem 1* show that although C and H are not strongly regular with respect to $x \uparrow \infty$, they are strongly regular with respect to limits of the form

$$\lim_{x\downarrow 0} C_k f(x) \quad \text{and} \quad \lim_{x\downarrow 0} H_k f(x) . \tag{10}$$

Generalized limits of the type (10) were first considered by Hardy and Littlewood in connection with the summability problem of Fourier series [5]; in the present context they appear as natural extensions of the Cesàro and Hölder processes distinguished by the property that they admit infinite iteration.

Proof of Theorem 1. For $k > 0$ we find by repeated integration by parts

$$C^k f(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)dt , \tag{11}$$

$$C^k e(x) = \frac{1}{k!} x^k , \qquad x > 0 \tag{11*}$$

and

$$C_k f(x) = k\int_0^1 (1-t)^{k-1} f(xt)dt , \qquad k > 0 . \tag{12}$$

The relations (11) and (12) define C^k and C_k also for non-integral $k > 1$. The existence of $C^\sigma f(x)$ for $\sigma > 1$ can be seen from

$$\int_\varepsilon^x (x-t)^{\sigma-1} f(t)dt = [(x-t)^{\sigma-1} Cf(t)]_{t=\varepsilon}^{t=x} - (\sigma-1)\int_\varepsilon^x (x-t)^{\sigma-2} Cf(t)dt ;$$

the expressions on the right have a limit when $\varepsilon \downarrow 0$, since $\lim_{x\downarrow 0} Cf(x) = 0$. Clearly $C^\sigma f(x)$ is continuous for $\sigma \geqq 1$, $x > 0$, and $\lim_{x\downarrow 0} C^\sigma f(x) = 0$. By partial integration we obtain, for fixed $\xi > 0$ and $\sigma > k$ [2]

$$C_\sigma f(\xi) = \frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}\int_0^1 (1-t)^{\sigma-k-1} t^k C_k f(\xi t)dt . \tag{13}$$

This can be regarded (for fixed ξ and k) as a transformation from $C_k f(\xi t)$ to $C_\sigma f(\xi)$; and Theorem 1 is proved if we can show that this transformation is regular. Now regularity follows immediately from a remark to § 3.5(3) in [4, p. 61], since the following three conditions are satisfied :

[2] This is a well-known identity; see for example [2, p. 3].

(1) $$\frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}(1-t)^{\sigma-k-1}t^k \geqq 0 \qquad \text{for } \sigma > k,\ 0 \leqq t \leqq 1 .$$

(2) $$\frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}\int_0^1 (1-t)^{\sigma-k-1}t^k dt = 1 .$$

(3) For a fixed x, $0 < x < 1$, and a fixed k,

$$\lim_{\sigma\uparrow\infty} \frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}\int_x^1 (1-t)^{\sigma-k-1}t^k dt = 0$$

since

$$\begin{aligned} 0 &\leqq \frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}\int_x^1 (1-t)^{\sigma-k-1}t^k dt \\ &\leqq \frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\sigma-k)}\int_x^1 (1-t)^{\sigma-k-1} dt \\ &= \frac{\Gamma(\sigma+1)}{\Gamma(k+1)\Gamma(\rho-k+1)}(1-x)^{\sigma-k} \to 0 \qquad \text{as } \sigma\uparrow\infty . \end{aligned}$$

Proof of Theorem 1*. We note first that $H^k e(x) = 1$ for every $k \geqq 0$, $x > 0$, and conditon (8*) implies that $H^k f(x)$ is bounded for $0<x\leqq M+\infty$. Therefore $H^m f(x)$ exists for $m \geqq k$ and hence $f(x) \in \Phi_{\infty,0}$. Condition (8*) implies

(14) $$H^k f(x) = a + \varphi(x)$$

where $\lim_{x\downarrow 0}\varphi(x) = 0$; we have to show that $\lim_{n\to\infty} H^n\varphi(x) = 0$. For bounded functions repeated partial integration gives

(15) $$H^{k+1}f(x) = \frac{1}{k!}\int_0^1 \left(\log\frac{1}{t}\right)^k f(xt)dt , \qquad k \geqq 0 .$$

Thus the statement to be proved is, that for any fixed $\xi > 0$,

(16) $$\lim_{n\uparrow\infty}\int_0^1 \frac{1}{n!}\left(\log\frac{1}{t}\right)^n \varphi(\xi t)dt = 0 .$$

Choose $\delta > 0$ so that $|\varphi(\xi t)| < \varepsilon/2$ for $0 < t < \delta$ and let $|\varphi(\xi t)| < K$ for $0 < t \leqq 1$; then

$$\left|\frac{1}{n!}\int_0^\delta \left(\log\frac{1}{t}\right)^n \varphi(\xi t)dt\right| < \frac{1}{2}\varepsilon \frac{1}{n!}\int_0^1 \left(\log\frac{1}{t}\right)^n dt = \frac{1}{2}\varepsilon ,$$

$$\left|\frac{1}{n!}\int_\delta^1 \left(\log\frac{1}{t}\right)^n \varphi(\xi t)dt\right| < K\frac{1}{n!}\left(\log\frac{1}{\delta}\right)^n < \frac{1}{2}\varepsilon , \qquad \text{for } n \geqq n_0 ,$$

which proves (16) and our theorem.

The proof of Theorem 1 suggests that $\lim_{n\to\infty} C^n f(\xi) = a$ implies $\lim_{\sigma\uparrow\infty} C_\sigma f(\xi) = a$. The following lemma shows that this is in fact so.

LEMMA 1. *Suppose that* $g(x) \in \Phi_{1,0}$ *and*

$$\lim_{n\to\infty} n \cdot \int_0^1 (1-t)^{n-1} g(t)dt = a \ . \tag{17}$$

Then

$$\lim_{\sigma\uparrow\infty} \sigma \cdot \int_0^1 (1-t)^{\sigma-1} g(t)dt = a \ . \tag{18}$$

Proof. The proof follows easily on integration by parts and by noticing that the function $G(x)$, defined by $G(x) = \int_0^x g(t)dt$, is bounded for $0 < x < 1$ and that $\lim_{x\downarrow 0} G(x) = 0$.

In the formulation of Theorem 1 and Theorem 1* we have made a distinction between the limits (8) and (8*). This is not really necessary: the two limits are equivalent.

THEOREM 2. *$f(x) \in \Phi_{k,0}$ and*

$$\lim_{x\uparrow\infty} C_k f(x) = a \tag{19}$$

for some $k > 0$ imply

$$\lim_{x\uparrow\infty} H_k f(x) = a \ . \tag{20}$$

Conversely, $f(x) \in \Phi_{k,0}$ and (20) imply (19).

THEOREM 2*. *$f(x) \in \Phi_{1,0}$ and*

$$\lim_{x\downarrow 0} C_k f(x) = a \tag{19*}$$

for some $k > 0$ imply that $f(x) \in \Phi_{k,0}$ and

$$\lim_{x\downarrow 0} H_k f(x) = a \ . \tag{20*}$$

Conversely, $f(x) \in \Phi_{k,0}$ and (20*) *imply* (19*).

Theorem 2 and Theorem 2* are the continuous analogues of the well-known Knopp-Schnee equivalence theorem for sequences. Note that in the first statement of Theorem 2 it is necessary to assume $f(x) \in \Phi_{k,0}$; otherwise it may happen that $H^k f(x)$ does not exist, for example $f(x) = (x^2 \log x)^{-1}$. This is not a serious restriction, though, since the assumption only affects the behaviour of $f(x)$ in the neighbourhood of 0 and is obviously satisfied in the essential domain of (C, k). There is no restriction of this kind in Theorem 2* where the assumption $f(x) \in \Phi_{1,0}$ and the existence of the limit (19*) automatically ensures that $f(x) \in \Phi_{k,0}$.

Theorem 2 is due to Landau [7]; Theorem 2 is stated (without proof and without specifiying the precise conditions of $f(x)$) by Hardy and Littlewood [5, p. 96]. It can be proved by an argument similar to the one used in [4, p. 112].

Once we have established the strong regularity of the limits (10), there is no difficulty in constructing strongly regular methods for $x \uparrow \infty$ and hence for sequences. We first note that limits at 0 by any method T can be converted into limits at ∞ by a 'reciprocal' method T^* which is defined as follows.

Suppose that T is given by

$$Tf(x) = \int_0^\infty \tau(x, t)f(t)dt .$$

To indicate clearly the function that is transformed and the point where the transform is taken, we shall use the notation

$$Tf(x) = T[f(t)](x) .$$

By an obvious change of variable

$$\begin{aligned} T[f(t)]\left(\frac{1}{x}\right) &= \int_0^\infty \tau\left(\frac{1}{x}, t\right)f(t)dt \\ &= \int_0^\infty \tau^*(x, t)f\left(\frac{1}{t}\right)dt \\ &= T^*\left[f\left(\frac{1}{t}\right)\right](x) \end{aligned}$$

where

$$\tau^*(x, t) = t^{-2}\tau\left(\frac{1}{x}, \frac{1}{t}\right) .$$

Clearly

$$T^k[f(t)]\left(\frac{1}{x}\right) = T^{*k}\left[f\left(\frac{1}{t}\right)\right](x) , \qquad k = 1, 2, \cdots .$$

Therefore

$$\lim_{x \downarrow 0} T_k[f(t)](x) = a$$

implies

$$\lim_{x \uparrow \infty} T_k^*\left[f\left(\frac{1}{t}\right)\right](x) = a ,$$

and conversely. Also

$$\lim_{k \to \infty} T_k[f(t)]\left(\frac{1}{x}\right) = \lim_{k \to \infty} T_k^*\left[f\left(\frac{1}{x}\right)\right](x) ,$$

in the sense that if one of the expressions exists then so does the other and the two are equal. It follows therefore that if T is strongly regular with respect to $x \downarrow 0$ then T^* is strongly regular with respect to $x \uparrow \infty$. In particular

$$C^*s(x) = \int_x^\infty t^{-2}s(t)dt , \tag{25}$$

and

$$H^*s(x) = x\int_x^\infty t^{-2}s(t)dt \equiv \int_1^\infty t^{-2}s(xt)dt \tag{26}$$

are strongly regular for $x \uparrow \infty$. Note that the domain of C^* and H^* is $\Phi_{0,1}$.

The processes (25) and (26) can easily be converted, if we wish, into strongly regular methods for sequences; for instance

$$(H^*\alpha)_n = n \sum_{m=n}^{\infty} \frac{\alpha_m}{m(m+1)} \tag{27}$$

is such a method. Its strong regularity is proved if it is shown that $\beta_n \to 0$ implies $\lim_{k\to\infty}(H^{*k}\beta)_n = 0$ for every fixed n. This can be shown for instance by comparing the sequence $(H^{*k}\beta)_n$ with suitable integrals and applying Theorem 1*.

Altough the method (27) is equivalent (in the ordinary sense) to $(H, 1)$[3], the two methods behave very differently from the point of view of iteration. The Hölder process has no useful infinite iterate whereas the process (27) has an infinite iterate which contains (and is compatible with) every finite (H^*, k). There exist in fact sequences (both bounded and unbounded) which are summable (H^*, ∞), but not summable by any finite (H^*, k) and (H, k)[4]. On the other hand, there exist (unbounded) sequences which are summable (H, k), but not by any (H^*, k), $k > 0$; for instance $(H^*, 1)$ is not even applicable to $\alpha_n = (-1)^n n(n+1)$. This raises the question of the relative strength of (H^*, k) and (H, k); we shall consider the problem only for the continuous case.

The following theorem is due to R. P. Agnew [1]; it is the continuous analogue of Knopp's equivalence theorem and asserts the equivalence of H and H^* for functions.

THEOREM 3. *$f(x) \in \Phi_{0,1}$ and*

$$\lim_{x\downarrow 0} x \cdot \int_x^\infty t^{-2}f(t)dt = a \tag{28}$$

imply $f(x) \in \Phi_{1,1}$ and

$$\lim_{x\downarrow 0} x^{-1} \cdot \int_0^x f(t)dt = a\,. \tag{29}$$

Converesely, $f(x) \in \Phi_{1,0}$ and the existence of the limit (29) *imply $f(x) \in \Phi_{1,1}$ and* (28).

A similar statement (with $\Phi_{0,1}$ and $\Phi_{1,0}$ interchanged) holds when $x \downarrow 0$ is replaced by $x \uparrow \infty$.

[3] A proof is given in [**6**, p. 487].

[4] Examples for the continuous case will be given below.

In the general case of (C, k) and (C^*, k) or (H, k) and (H^*, k) a stronger assumption on $f(x)$ is necessary. Because of the Theorem 2 and Theorem 2* it is sufficient to consider one of the possible combinations, say C and C^*.

THEOREM 3.* *Let $f(x) \in \Phi_{1,1}$ and suppose that for some $n > 0$,*

$$\lim_{x \downarrow 0} C_n^* f(x) = a \tag{30}$$

then

$$\lim_{x \downarrow 0} C_n f(x) = a \; . \tag{31}$$

Conversely, the existence of the limit (31) *and $f(x) \in \Phi_{1,1}$ imply* (30).

A similar statement holds when $x \downarrow 0$ is replaced by $x \uparrow \infty$.

Theorem 3* shows that (C, n) and (C^*, n) are equivalent within $\Phi_{1,1}$, that is within the class of functions to which both methods are applicable. However, if we disregard the difficulty that $f(x)$ may behave badly at ∞ (0) when we are interested in the limit at 0 (∞), that is, if we restrict ourselves to the essential domain of the two methods, then it appears that C^* includes C for limits at 0, and C includes C^* for limits at ∞. C^* is actually stronger than C for $x \downarrow 0$, as shown by the example $f(x) = 2x^{-3} \sin x^{-2}$. In fact $C[2t^{-3} \sin t^{-2}](x)$ does not exist since the function is not integrable down to 0, but

$$C^*[2t^{-3} \sin t^{-2}](x) = -\frac{1}{x^2} + \sin \frac{1}{x^2}$$

$$C^*\left[-t^{-2} \cos \frac{1}{t^2} + \sin \frac{1}{t^2}\right](x) = O\left(\frac{1}{x}\right) ,$$

hence

$$\lim_{x \downarrow 0} x^2 C^{*2}\left[\frac{2}{t^3} \sin \frac{1}{t^2}\right](x) = 0 \; .$$

This example shows that the condition $f(x) \in \Phi_{1,1}$ cannot be relaxed and for instance $f(x) \in \Phi_{0,1}$ and the existence of $\lim_{x \downarrow 0} C_2^* f(x) = a$ does not imply $f(x) \in \Phi_{1,1}$.

For the proof of Theorem 3 we need the following lemma.

LEMMA 2. *Given $\infty \geqq a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$, $n > 0$, and $f(x) \in \Phi_{1,1}$. Let $f_k(x)$ for $k = 0, 1, \cdots, n$ be defined by*

$$f_0(x) = f(x) \; , \;\; f_k(x) = \int_x^{a_k} t^{-2} f_{k-1}(t) dt \quad \textit{for } k > 0 \; .$$

Then

$$\lim_{x \downarrow 0} x^{k+1} f_k(x) = 0 \qquad \textit{for } k = 1, 2, \cdots, n \; .$$

Proof. $\int_0^x f(t)dt$ exists by assumption. Therefore given $\varepsilon > 0$ we can choose a positive δ such that

$$\left| \int_\xi^\eta f(t)dt \right| < \varepsilon$$

for every $0 < \xi < \eta < \delta$. Let $\xi < \delta$. By the second mean value theorem

$$\xi^2 \int_\xi^\delta t^{-2} f(t)dt = \int_\xi^\eta f(t)dt$$

for some η in the interval (ξ, δ). Hence

$$\xi^2 \left| \int_\xi^\delta t^{-2} f(t)dt \right| < \varepsilon \qquad \text{for every } 0 < \xi < \delta .$$

Also

$$\xi^2 \left| \int_\delta^{a_1} t^{-2} f(t)dt \right| < \varepsilon \qquad \text{for every } 0 < \xi < \xi_0 \leqq \delta$$

provided that ξ_0 is sufficiently small. Therefore

$$\left| \xi^2 \int_\xi^{a_1} t^{-2} f(t)dt \right| < 2\varepsilon \qquad \text{for every } \xi \text{ in } (0, \xi_0).$$

This proves the lemma for $k = 1$. Suppose now that $k > 1$ and $f_{k-1}(x) = o(x^{-k})$, as $x \downarrow 0$. Given $\varepsilon > 0$ choose $\delta \leqq a_k$ so that $|f_{k-1}(t)| < \varepsilon t^{-k}$ for $0 < t < \delta$. For $0 < x < \delta$

$$|f_k(x)| \leqq \left| \int_x^\delta t^{-2} f_{k-1}(t)dt \right| + \left| \int_\delta^{a_k} t^{-2} f_{k-1}(t)dt \right| .$$

But

$$\left| \int_x^\delta t^{-2} f_{k-1}(t)dt \right| < \varepsilon \int_x^\delta t^{-k-2}dt < \frac{\varepsilon}{k+1} x^{-k-1}$$

and

$$\left| \int_\delta^{a_k} t^{-2} f_{k-1}(t)dt \right| < \frac{\varepsilon k}{k+1} x^{-k-1}$$

provided that x is sufficiently small, $0 < x < x_0 \leqq \delta$, say. Therefore $|f_k(x)| < \varepsilon x^{-k-1}$ for all x, $0 < x < x_0 = x_0(\varepsilon)$.

COROLLARY. *For $k > 0$ and $f(x) \in \Phi_{1,1}$ we have $C^{*k} f(x) = o(x^{-k-1})$ as $x \downarrow 0$.*

Proof of Theorem 3*. For simplicity we shall write $D = C^*$ throughout the proof. It is convenient to prove the first statement of the theorem in the following more general form: Suppose that $f(x) \in \Phi_{1,1}$ and for some $n > 0$ and $p \geqq 0$,

$$\lim_{x \downarrow 0} (n + p)!\, x^{n+p} D^{n+p} f(x) = a , \tag{32}$$

where $Df(x) = \int_x^\infty t^{-2}f(t)dt$; then for every $r \geqq 0$, $s \geqq 0$

$$\lim_{x \downarrow 0} \frac{(s+n)!\,(p+r)!}{s!} x^{-(n+s)} C^n[t^{p+r+s} D^{p+r} f(t)](x) = a . \tag{33}$$

The proof is by induction on n. We first note that (32) implies

$$D^m f(x) = \frac{1}{m!} \cdot x^{-m} \cdot a + o(x^{-m}) , \quad \text{as } x \downarrow 0 , \tag{34}$$

for every $m \geqq n + p$. Now let $k \geqq 0$ and $m \geqq n + p$. We have, by partial integration,

$$\begin{aligned} k(m-1)!\, x^{-k} & \int_0^x t^{m+k-2} D^{m-1} f(t)dt \\ &= -k(m-1)!\, x^{-k}[t^{m+k} D^m f(t)]_0^x \\ &\qquad + k(m-1)!\,(m+k)x^{-k}\int_0^x t^{m+k-1} D^m f(t)dt . \end{aligned}$$

The first expression on the right is $-k(m-1)!\, x^m D^m f(x)$ which, by (34), tends to $-(k/m)a$ when $x \downarrow 0$; similary, the second term tends to $[(m+k)/m] \cdot a$. Hence

$$\lim_{x \downarrow 0} k(m-1)!\, x^{-k} \cdot \int_0^x t^{m+k-2} D^{m-1} f(t)dt = a . \tag{35}$$

This proves (33) for $n = 1$ (with $k = s + 1$, $m = p + r + 1$). Note that $f(x) \in \Phi_{0,1}$ and the existence of the limit (32) implies the existence of the integral in (35); therefore in particular $f(x) \in \Phi_{0,1}$ and (28) in Theorem 3 implies $f(x) \in \Phi_{1,1}$ and (29). Suppose now that $n > 1$, and write $m = n + p + r$, $r \geqq 0$. We have

$$\begin{aligned} k(m-1)!\, x^{-x} & \cdot \int_0^x t^{m+k-2} D^{m-1} f(t)dt \\ &= k(m-1)!\, x^{-k}\int_0^x t_1^{m+k-2}dt_1 \int_{t_1}^\infty t_2^{-2}dt_2 \cdots \int_{t_{n-2}}^\infty t_{n-1}^{-2}dt_{n-1}\int_{t_{n-1}}^\infty t_n^{-2} D^{p+r} f(t_n)dt_n \\ &= k(m-1)!\, x^{-k}\Big\{D^{m-1}f(x)\int_0^x t_1^{m+k-2}dt_1 + D^{m-2}f(x)\int_0^x t_1^{m+k-2}dt_1 \int_{t_1}^x t_2^{-2}dt_2 \\ &\qquad + \cdots + D^{p+r+1}f(x)\int_0^x t_1^{m+k-2}dt_1 \cdots \int_{t_1}^x t_2^{-2}dt_2 \int_{t_{n-2}}^x t_{n-1}^{-2}dt_{n-1} \\ &\qquad\qquad + \int_0^x t_1^{m+k-2}dt_1 \int_{t_1}^x t_2^{-2}dt_2 \cdots \int_{t_{n-1}}^x t_n^{-2} D^{p+r} f(t_n)dt_n \Big\} \\ &= \frac{k(m-1)!}{(m+k-1)!}\Big\{\sum_{j=1}^{n-1} (j+k+p+r-1)!\, x^{j+p+r} D^{j+p+r} f(x) \\ &\qquad + (k+p+r)!\, x^{-k}\int_0^x t^{k+p+r+1} D^{p+r} f(t)\,dt\Big\} . \end{aligned} \tag{36}$$

The last expression in the brackets is obtained by repeated partial integration, and using Lemma 2 Equations (35) and (36) give

$$x^{-k}\int_0^x t^{k+p+r-1}D^{p+r}f(t)dt = \frac{(m+k-1)!}{k(m-1)!(k+p+r)!}a$$
$$-\sum_{j=1}^{m-1}\frac{(j+k+p+r-1)!}{(k+p+r)!}x^{j+p+r}D^{j+p+r}f(x)+o(1)\,.$$

Hence

$$\frac{(n+k-1)!}{(k-1)!}x^{-n-k+1}C^n[t^{k+p+r-1}D^{p+r}f(t)](x)$$
$$=\frac{(n+k-1)!(p+r)!}{(k-1)!}x^{-n-k+1}C^{n-1}\left[\int_0^t u^{k+p+r-1}D^{p+r}f(u)du\right](x)$$
$$=\frac{(p+r)!(n+k+p+r-1)!}{(k+p+r)!(n+p+r-1)!}a$$
$$-\sum_{j=1}^{n-1}\frac{(n+k-1)!(k+j+p+r-1)!(p+r)!}{(k+p+r)!(k-1)!}x^{-n-k+1}$$
$$\times C^{n-1}[t^{j+k+p+r}D^{j+p+r}f(t)](x)+o(1)\,.$$

Here we have, by the induction hypothesis (33), applied to $Df(x)$ instead of $f(x)$ and $n-1$, $p+1$,

$$\frac{(n+k-1)!(j+p+r)!}{k!}x^{-n-k+1}C^{n-1}[t^{j+k+p+r}D^{j+p+r}f(t)](x)=a+o(1)\,.$$

Hence by (37)

$$\frac{(n+k-1)!(p+r)!}{(k-1)!}x^{-n-k+1}C^n[t^{k+p+r-1}D^{p+r}f(t)](x)$$
$$=\frac{(p+r)!k!}{(k+p+r)!}\left\{\binom{n+k+p+r-1}{k}-\sum_{j=1}^{n-1}\binom{j+k+p+r-1}{k-1}\right\}a+o(1)$$
$$=a+o(1)\,.$$

This proves (33). The proof of the converse is very similar. With the notation $s(x)\equiv f(1/x)$ the converse statement can be formulated as follows:

(30*) $$\lim_{x\uparrow\infty} n!\cdot x^nD^ns(x)=a$$

implies

(31*) $$\lim_{x\uparrow\infty} n!x^{-n}C^ns(x)=a\,.$$

The proof is identical with the derivation of (31) from (30) except that $f(x)$ has to be replaced everywhere by $s(x)$ (which is also in $\Phi_{1,1}$) and $x\downarrow 0$ by $x\uparrow\infty$.

3. The relative strengh of the (H,∞) and (C,∞) methods. So far we did not consider the relative strengh of the (H,∞) and (C,∞) methods. We know from Theorem 1 and Theorem 1* that both these methods include the finite (H, k) and (C, k) methods for $x \downarrow 0$, more precisely, if

$$\lim_{x \downarrow 0} C_k f(x) = \lim_{x \downarrow 0} H_k f(x) = a$$

exists for some $k \geqq 0$ then

$$H_\infty f(x) = \lim_{m \uparrow \infty} H_m f(x) \quad \text{and} \quad C_\infty f(x) = \lim_{m \uparrow \infty} C_m f(x)$$

exist for every $x > 0$ and are in fact the constant function $H_\infty f(x) = C_\infty f(x) = a$.

Now the following theorems show that this is always so: whenever $C_\infty f(x)$ and $H_\infty f(x)$ exist at all, they are a constant.

THEOREM 4. *Let $f(x) \in \Phi_I$ and suppose that for some fixed $\xi > 0$*

(38) $$\lim_{n \to \infty} C_n f(\xi) = a$$

then

(39) $$\lim_{\sigma \uparrow \infty} \sigma \int_0^\infty e^{-\sigma t} f(t) dt = a \, .$$

Conversely, $f(x) \in \Phi_I$ and (39) *imply*

$$\lim_{n \to \infty} C_n f(x) = a$$

for every $x > 0$.

THEOREM 4*. *Let $f(x) \in \Phi_{\infty,0}$ and suppose that for a fixed $\xi > 0$,*

(40) $$\lim_{n \to \infty} H^n f(\xi) = a \, ;$$

then for every $x > 0$

$$\lim_{n \to \infty} H^n f(x) = a \, .$$

Theorem 4 shows that (C, ∞) is essentially equivalent to the Abel-Poisson method L:

(41) $$Lf(x) = \int_0^\infty e^{-t} f(xt) dt$$

in the sense that $\lim_{x \downarrow 0} C_\infty f(x) = a$ if, and only if, $\lim_{x \downarrow 0} Lf(x) = a$ provided that $f(x)$ is in the essential domain of the two methods. As a corollary we find that L includes every (C, k)[5]; but we know of no example to show that (C, ∞) or L is actually stronger than the collection of every (C, k). For bounded functions (C, ∞) is equivalent to $(C, 1)$; more generally the following is true.

[5] A dual of this statement, referring to $x \to \infty$, is proved by G. Doetsch in [3, p. 204].

THEOREM 5. *$f(x) \in \Phi_{1,0}$, $f(x) = 0$ for $x \geqq x_0$, $f(x) = O_L(1)$ and $C_\infty f(x) = a$, $x > 0$, imply*

$$\lim_{x \downarrow 0} C_1 f(x) = a .$$

Proof of Theorem 4. By Lemma 1 we can replace the integral variable n by the continuous variable σ. The assumption $f(x) \in \Phi_I$ implies that $F(x) \equiv Cf(x)$ is bounded for $x > 0$ and $\lim_{x\downarrow 0} F(x) = 0$. Therefore we obtain for $\sigma > 1$

$$\begin{aligned}(42) \quad C_\sigma f(\xi) &= \sigma \int_0^1 (1-t)^{\sigma - t} f(\xi t) dt \\ &= \frac{\sigma - 1}{\xi} \int_0^\sigma \left(1 - \frac{u}{\sigma}\right)^{\sigma - 2} F\left(\frac{\xi u}{\sigma}\right) du \\ &= \frac{\sigma - 1}{\xi} \int_0^{\sigma^{1/2}} \left(1 - \frac{u}{\sigma}\right)^{\sigma - 2} F\left(\frac{\xi u}{\sigma}\right) du + O(\sigma e^{-\sigma/2}) \\ &= \frac{\sigma - 1}{\xi} \int_0^{\sigma^{1/2}} e^{-u} F\left(\frac{\xi u}{\sigma}\right) du + O\left(\int_0^{\sigma^{1/2}} u^2 e^{-2}\left|F\left(\frac{\xi u}{\sigma}\right)\right| du\right) \\ &\qquad + O(\sigma e^{-\sigma^{1/2}}) \\ &= \frac{\sigma(\sigma - 1)}{\xi} \int_0^\infty e^{-\sigma t} F(\xi t) dt + o(1) \\ &= (\sigma - 1) \int_0^\infty e^{-\sigma t} f(\xi t) dt + o(1)\end{aligned}$$

where all O and o symbols refer to fixed ξ and $\sigma \uparrow \infty$. Hence $\lim_{\sigma\uparrow\infty} C_\sigma f(\xi) = a$ implies $\lim_{\sigma\uparrow\infty} \sigma \int_0^\infty e^{-\sigma t} f(\xi t) dt = a$. Therefore

$$\lim_{\rho \uparrow \infty} \rho \int_0^\infty e^{-\rho u} f(xu) du = a$$

for any fixed $x > 0$. By (42) we see that $\lim_{\rho\uparrow\infty} C_\rho f(x) = a$.

Proof of Theorem 5. Theorem 5 is an immediate consequence of Theorem 4 and the following lemma, which is a special case of a well-known Tauberian theorem for the Laplace transform (see [3, p. 210, Satz 3]).

LEMMA 3. *Suppose that $g(x) = O_L(1)$, $\int_0^\infty e^{-\sigma t} g(t) dt$ converges for all $\sigma > 0$ and*

$$\lim_{\sigma \uparrow \infty} \sigma \int_0^\infty e^{-\sigma t} g(t) dt = a .$$

Then

$$\lim_{x \downarrow 0} \frac{1}{x} \int_0^x g(t) dt = a .$$

The statement remains true if $\sigma \uparrow \infty$ *is replaced by* $\sigma \downarrow 0$ *and* $x \downarrow 0$ *by* $x \uparrow \infty$.

Proof of Theorem 4*. Without loss of generality we may assume $a = 0$; otherwise consider $f(x) - a$ instead of $f(x)$. Let x_1, x_2 be two fixed positive numbers $x_1 < \xi < x_2$. Write $g(x) = Hf(x)$. Then $H^n f(x) = H^{n-1}g(x)$, for $n > 0$, and $\lim_{n\to\infty} H^n g(\xi) = 0$. We shall show that

$$(40^*) \qquad \lim_{n\to\infty} H^n g(x) \equiv \lim_{n\to\infty} H^{n+1} f(x) = 0, \text{ uniformly for } x_1 \leqq x \leqq x_2.$$

Denote

$$(43) \qquad \begin{cases} \varepsilon_0 = \dfrac{x_1}{\xi} \cdot \text{upper bound } \{|g(x)|;\ x_1 \leqq x \leqq x_2\} < +\infty, \\ \varepsilon_k = |H^k g(\xi)| \qquad \text{for } k > 0. \end{cases}$$

We prove that

$$(44) \qquad |H^n g(x)| \leqq \frac{\xi}{x_1} \sum_{p=0}^{n} \frac{1}{p!} \left| \frac{x-\xi}{x_1} \right|^p \varepsilon_{n-p} \quad \text{for } x_1 \leqq x \leqq x_2.$$

For $n = 0$ the statement follows from (43); suppose therefore that $n > 0$ and that (44) is true for $n - 1$.

$$\begin{aligned} |H^n g(x)| &\leqq \left| \frac{1}{x} \int_0^{\xi} H^{n-1} g(t)dt \right| + \left| \frac{1}{x} \int_{\xi}^{x} H^{n-1} f(t)dt \right| \\ &\leqq \left| \frac{\xi}{x_1} \cdot \frac{1}{\xi} \int_0^{\xi} H^{n-1} g(t)dt \right| + \frac{\xi}{x_1^2} \sum_{p=0}^{n-1} \int_{\xi}^{p} \frac{1}{p!} \left| \frac{t-\xi}{x_1} \right|^p \varepsilon_{n-p-1} dt \\ &\leqq \frac{\xi}{x_1} \left\{ \varepsilon_n + \sum_{p=0}^{n-1} \frac{1}{(p+1)!} \left| \frac{x-\xi}{x_1} \right|^{p+1} \varepsilon_{n-p-1} \right\} \end{aligned}$$

which proves the statement for n. From (44) we obtain, by writing $\lambda = \max\{\xi - x_1, x_2 - \xi\}$,

$$(45) \qquad |H^n g(x)| \leqq \frac{\xi}{x_1} \sum_{p=0}^{n} \frac{1}{(n-p)!} \lambda^{n-p} \varepsilon_p, \qquad x_1 \leqq x \leqq x_2.$$

But for any $\lambda > 0$,

$$e^{-\lambda} \sum_{p=0}^{n} \frac{1}{(n-p)!} \lambda^{n-p} s_p$$

is a regular transform of the sequence $\{s_k\}$, $k \geqq 0$; it follows therefore that the expression on the right side of (45) tends to zero when $n \to \infty$.

By Theorem 5 (C, ∞) does not extend the range of $(C, 1)$ for bounded functions. This is in striking contrast with (H, ∞) which is decidedly more powerful for bounded functions than (H, 1). An example is furnished by $\cos \log x$, or more conveniently by $e^{-i \log x}$. We find by induction

$$H^k[e^{-i\log t}](x) = \left(\frac{1+i}{2}\right)^k e^{-i\log x}$$

which has no limit at $x = 0$; by Theorem 2* and Theorem 5 it has therefore no limit by (C, ∞). On the other hand it has limit zero by (H, ∞):

$$\lim_{k\to\infty} H^k[e^{-i\log t}](x) = 0 \qquad \text{for every } x > 0 .$$

Also for unbounded functions (H, ∞) appears to be more effective than (C, ∞); a suitable example is $-x^{-1/2}\cdot\cos\log x$ or $x^{-1/2}e^{-i\log x}$ which can be shown to have no limit by (C, ∞), and limit zero by (H, ∞). These examples reveal a remarkable difference (in favour of the Hölder process) between the Cesàro and Hölder processes which remains completely hidden when finite iterations alone are considered.

For bounded functions repeated partial integration gives

$$H^{k+1}f(x) = \frac{1}{k\,!}\int_0^1\left(\log\frac{1}{t}\right)^k f(xt)dt\ ; \tag{46}$$

and we find the following analogue of the equivalence of (C, ∞) and L for (H, ∞):

THEOREM 6. *Let $f(x)\in\Phi_B$ and*

$$I_0(x) \equiv J_0(ix) \equiv \sum_{n=0}^{\infty}\frac{1}{(n\,!)^2}\left(\frac{t}{2}\right)^{2n} .$$

Then

$$\lim_{n\to\infty}\int_0^1\frac{1}{n\,!}\left(\log\frac{1}{t}\right)^n f(t)dt = \lim_{v\to\infty} e^{-v}\int_0^1 f(t)I_0\left(2\left(v\log\frac{1}{t}\right)^{1/2}\right)dt \tag{47}$$

in the sense that if one side exists then the other side exists too and they are equal.

By making use of the well-known asymptotic expression

$$I_0(x) = (2\pi)^{-1/2}x^{-1/4}e^x\left(1 + O\left(\frac{1}{x}\right)\right), \qquad \text{as } x\uparrow\infty ,$$

Theorem 6 can be put in a more convenient form. For bounded functions we have

$$\begin{aligned}\lim_{v\uparrow\infty} & e^{-v}\int_0^1 f(t)I_0\left(2\left(v\log\frac{1}{t}\right)^{1/2}\right)dt \\ &= \lim_{v\uparrow\infty}\frac{1}{2}\pi^{-1/2}e^{-v}\int_0^1 f(t)\left(v\log\frac{1}{t}\right)^{-1/4}\exp\left[2\left(v\log\frac{1}{t}\right)^{1/2}\right]dt \\ &= \lim_{\sigma\uparrow\infty}\pi^{-1/2}\int_0^\infty f(e^{-u^2})\cdot e^{-(u-\sigma)^2}\cdot\left(\frac{u}{\sigma}\right)^{1/2}du\end{aligned}$$

(by the substitution $v = \sigma^2$, $t = e^{-u^2}$), and the latter is easily seen to be equal to

$$\lim_{\sigma\uparrow\infty} \pi^{-1/2}\int_0^\infty f(e^{-u^2})e^{-(u-\sigma)^2}du .$$

This gives

THEOREM 6*. *Let $f(x) \in \Phi_B$; then $\lim_{x\downarrow 0} f(x) = a$ by (H, ∞) if, and only if,*

$$\lim_{\sigma\uparrow\infty} \pi^{-1/2}\int_0^\infty f(e^{-u^2})e^{-(u-\sigma)^2}du = \lim_{v\uparrow\infty} \frac{1}{2}\pi^{-1/2}e^{-v}\int_0^1 f(t)\frac{\exp(2(v\log 1/t)^{1/2}}{(\log 1/t)^{1/2}}dt = a .$$

The following estimate of (H, ∞) for bounded functions is weaker, but it has the advantage of great formal simplicity.

THEOREM 7. *Le $f(x) \in \Phi_B$ and suppose that for a fixed $\xi > 0$*

$$\lim_{n\to\infty} H_n f(\xi) = a ,$$

then

$$\lim_{x\downarrow 0} H[f(e^{-1/t})](x) = \lim_{x\uparrow\infty} H[f(e^{-t})](x) = a .$$

Theorem 6, Theorem 6* and Theorem 7 do not remain valid for unbounded functions; a suitable counter example is $x^{-1/2}\cos\log(1/x)$. Also the converse of Theorem 7 is not true; a counter example is furnished by $f(x) = \exp(i(\log 1/x)^{1/2})$. Clearly

$$\frac{1}{x}\int_0^x f(e^{-1/t})dt = \frac{1}{x}\int_0^x \exp(it^{-1/2})dt = O(x^{1/2}) \to 0 \quad \text{when } x\downarrow 0 .$$

On the other hand

$$\lim_{n\to\infty}\int_0^\infty \frac{1}{n!}\left(\log\frac{1}{t}\right)^n \exp\left[i\left(\log\frac{1}{t}\right)^{1/2}\right]dt$$

does not exist; for otherwise by Theorem 6*

$$\begin{aligned}&\lim_{\sigma\uparrow\infty} \pi^{-1/2}\int_0^\infty \exp(iu - (u-\sigma)^2)du \\ &\quad = \lim_{\sigma\uparrow\infty} \pi^{-1/2}\exp\left(-\frac{1}{u} - i\sigma\right)\int_0^\infty \exp\left[-\left(u-\sigma-\frac{1}{2}\right)^2\right]du\end{aligned}$$

existed, But the last expression is asymptotically equal to $e^{-1/1\to i\sigma}$ when $\sigma\uparrow\infty$.

In the proof of Theorem 6 we use

LEMMA 5. *Let $f(x) \in \Phi_B$. Then for every fixed $\xi > 0$,*

$$H^n f(\xi) - H^{n+1}f(\xi) = O(n^{-1/2}) \qquad \text{as } n\to\infty .$$

Proof. Observing the relations

$$\frac{1}{(n+1)!}\int_x^\infty u^{n+1}e^{-u}du - \frac{1}{n!}\int_x^\infty u^n e^{-u}du$$
$$= \frac{1}{n!}\int_0^x u^n e^{-u}du - \frac{1}{(n+1)!}\int_0^x u^{n+1}e^{-u}du$$
$$= \frac{x^{n+1}e^{-x}}{(n+1)!},$$

$$n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n}, \qquad \text{as } n \to \infty,$$

$$|f(t)| \leqq K \qquad \text{for a suitable constant } K > 0,$$

we obtain

$$\left|\int_0^1 f(\xi t)\frac{1}{n!}\left(\log\frac{1}{t}\right)^n dt - \int_0^1 f(\xi t)\frac{1}{(n+1)!}\left(\log\frac{1}{t}\right)^{n+1}dt\right|$$
$$\leqq K\int_0^1\left|\frac{1}{n!}\left(\log\frac{1}{t}\right)^n - \frac{1}{(n+1)!}\left(\log\frac{1}{t}\right)^{n+1}\right|dt$$
$$= K\int_0^\infty\left|\frac{u^n}{n!} - \frac{u^{n+1}}{(n+1)!}\right|e^{-u}du$$
$$= 2K\frac{(n+1)^{n+1}}{(n+1)!}e^{-(n+1)}$$
$$\sim \frac{2K}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{n+1}}.$$

Proof of Theorem 6. By the regularity of the Borel transform, $\lim_{n\to\infty}s_n = a$ implies

$$\lim_{v\uparrow\infty} e^{-v}\sum_{n=0}^\infty \frac{s_n}{n!}\cdot v^n = a.$$

Hence

$$\lim_{n\to\infty}\int_0^1 f(t)\frac{1}{n!}\left(\log\frac{1}{t}\right)^n dt = a \tag{48}$$

implies

$$\lim_{v\uparrow\infty} e^{-v}\sum_{n=0}^\infty\int_0^1 f(t)\frac{1}{(n!)^2}\left(\log\frac{1}{t}\right)^n v^n dt$$
$$= \lim_{v\uparrow\infty} e^{-v}\int_0^1 f(t)\sum_{n=0}^\infty\frac{1}{(n!)^2}\left(v\log\frac{1}{t}\right)^n dt$$
$$= \lim_{v\uparrow\infty} e^{-v}\int_0^1 f(t)I_0\left(2\left(v\log\frac{1}{t}\right)^{1/2}\right)dt = a; \tag{49}$$

the interchange of the order of summation and integration is clearly permissible if $f(t)$ is bounded. Conversely, from Lemma 5 and the Tauberian theorem of Hardy-Littlewood for the Borel transform [4, p.

220, Theorem 156] we conclude that (49) implies (47).

Proof of Theorem 7. If in the proof of Theorem 6 we use the Abel transform instead of the Borel transform, Theorem 7 is obtained. First

$$\lim_{n\to\infty}\int_0^1 f(t)\frac{1}{n\,!}\left(\log\frac{1}{t}\right)^n dt = a$$

implies

$$\begin{aligned} a &= \lim_{v\uparrow 1}(1-v)\sum_{n=0}^{\infty}\int_0^1 f(t)\frac{1}{n\,!}\left(v\log\frac{1}{t}\right)^n dt \\ &= \lim_{v\uparrow 1}(1-v)\int_0^1 f(t)t^{-v}dt \\ &= \lim_{\sigma\downarrow 0}\int_1^\infty f\left(\frac{1}{u}\right)u^{-\sigma+1}du \\ &= \lim_{\sigma\downarrow 0}\sigma\int_0^\infty f(e^{-t})e^{-\sigma t}dt \end{aligned}$$

and this implies by Lemma 3

$$\lim_{x\uparrow\infty}\frac{1}{x}\int_0^x f(e^{-t})dt = \lim_{x\uparrow\infty} H[f(e^{-t})](x) = a\ .$$

By Theorem 3 the last equation is equivalent to

$$\lim_{x\downarrow 0}\frac{1}{x}\int_0^x f(e^{-1/t})dt = \lim_{x\downarrow 0} H[f(e^{-1/t})](x) = a\ .$$

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