RUNS

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1. Introduction. A *relation* is a set of ordered pairs. If R is a relation then it helps our intuition to sometimes think that y comes after x if and only if $(x, y) \in R$. With this in mind we search among relations for directing mechanisms among which are to be not only those familiar ones considered by Moore-Smith, but enough more to handle[1] topological convergence.

We agree that

$$\begin{aligned} \operatorname{dmn} R &= \text{domain } R = \mathrm{E}x\,[(x, y) \in R \text{ for some } y] \\ &= \text{the set of points } x \text{ such that } (x, y) \in R \text{ for some } y, \end{aligned}$$

and that

$$\operatorname{rng} R = \text{range } R = \mathrm{E}y\,[(x, y) \in R \text{ for some } x]\,.$$

Now suppose

$$\Gamma = \mathrm{E}x\,(0 \leqq x < \infty)$$

and

$$\omega = \text{the set of non-negative integers.}$$

Also suppose

$$R_1 = \mathrm{E}x, y(0 \leqq x \leqq y < \infty)$$

and

$$R_2 = \mathrm{E}x, y(x \in \omega \text{ and } 0 \leqq x \leqq y < \infty),$$

so that $(x, y) \in R_1$ if and only if $0 \leqq x \leqq y < \infty$ and $(x, y) \in R_2$ if and only if $x \in \omega$ and $0 \leqq x \leqq y < \infty$.
Clearly

$$\operatorname{rng} R_2 = \operatorname{rng} R_1 = \Gamma$$

but on the other hand

$$\operatorname{dmn} R_2 = \omega \neq \operatorname{dmn} R_1 = \Gamma\,.$$

Nevertheless, R_2 and R_1 are intuitively equivalent directing mechanisms.

Now suppose:

Received February 17, 1958.

[1] See Remark 5.2.

$I = \mathrm{E}t(0 \leqq t \leqq 1)$;

$\Gamma' =$ The set of functions on I to Γ;

$\omega' =$ The set of functions on I to ω;

$R_1' = \mathrm{E}x, y\,[x \in \Gamma'$ and $y \in \Gamma'$ and $x(t) \leqq y(t)$ whenever $t \in I]$;

$R_2' = \mathrm{E}x, y\,[x \in \omega'$ and $y \in \Gamma'$ and $x(t) \leqq y(t)$ whenever $t \in I]$.

Very much as before

$$\operatorname{rng} R_2' = \operatorname{rng} R_1' = \Gamma',$$
$$\operatorname{dmn} R_2' = \omega' \neq \operatorname{dmn} R_1' = \Gamma',$$

but nevertheless R_2' and R_1' are intuitively equivalent directing mechanisms.

Let us look more closely at R_2'. R_2' is clearly transitive. That is, $(x, z) \in R_2'$ whenever (x, y) and (y, z) both belong to R_2'. In other words, if y comes after x and z comes after y, then z comes after x. Moreover if $x \in \operatorname{dmn} R_2'$ and $y \in \operatorname{dmn} R_2'$ then there is a $z \in \operatorname{dmn} R_2'$ which comes after both x and y. That is, corresponding to each $x \in \operatorname{dmn} R_2^1$ and each $y \in \operatorname{dmn} R_2'$ there is a $z \in \operatorname{dmn} R_2'$ for which

$$(x, z) \in R_2' \text{ and } (y, z) \in R_2' .$$

We are thus led to

1.1 DEFINITION. R is a *direction* if and only if R is such a non-vacuous transitive relation that corresponding to each $x \in \operatorname{dmn} R$ and each $y \in \operatorname{dmn} R$ there is a $z \in \operatorname{dmn} R$ for which

$$(x, z) \in R \text{ and } (y, z) \in R .$$

Evidently the directing mechanisms of Moore-Smith are directions, but it turns out that even directions are not topologically adequate.[1]

If R is a direction then clearly for each $x \in \operatorname{dmn} R$ and each $y \in \operatorname{dmn} R$ there is a $z \in \operatorname{dmn} R$ such that anything which comes after z also comes after x and after y. We are now on the right track.

1.2 DEFINITION. R is a *run* if and only if R is such a non-vacuous relation that corresponding to each $x \in \operatorname{dmn} R$ and each $y \in \operatorname{dmn} R$ there is a $z \in \operatorname{dmn} R$ for which

$$(x, t) \in R \text{ and } (y, t) \in R$$

whenever t is such that $(z, t) \in R$.

From 1.1 and 1.2 follows

1.3 THEOREM. *Every direction is a run.*

As we shall indicate, runs are topologically adequate. For that matter, so are the filter-bases of Cartan, the nets of Kelley, and the syntaxes of McShane. But among these we do not find such an old friend as the Moore-Smith direction R_1.

It is a curious fact that one can come across situations in which the effect of a direction cannot be duplicated by a filter-base.[2] Suppose

$$R_3 = \mathrm{E}a, b\,[a \subset b \text{ and } b \text{ is a finite set}]\ .$$

Clearly, R_3 is a direction. Moreover, it is a direction which has been put to use in defining unordered summation. However, no filter-base can do the work of R_3, since in many set theories the family of all finite supersets of a given finite set is a class incapable of belonging to anything.[3]

The runs which first come to mind are directions. However, some runs are very unlike the directions they generalize. The domain of a run is merely an indexing set of sign-posts which seem to say, "Beyond here is far enough." It may be that many things follow such a sign-post yet no sign-post at all is among them. To savor some possibilities along this line let us examine briefly two more runs.

Assume T topologizes S and that $p \in S$ and check intuitively that

$$\mathrm{E}\beta, x[p \in \beta \in T \text{ and } x \in \beta]$$

is a run which converges to p in the topology T.

Next assume ρ metrizes S and $p \in S$ and check intuitively that

$$\mathrm{E}r, x\,[0 < r < \infty \text{ and } \rho(x, p) \leqq r]$$

is a run which converges to p in the metric ρ.

It must be admitted that filter-bases are less intricate than runs. Moreover, filter-bases handle theoretical limits with less emphasis on inessentials than any other method known to us. What disturbs us and others about filter-bases is that in many specific situations, such as limit by refinement, the filter-bases do not correspond vividly enough to the limiting concept one pictures. Perhaps it is for this reason that directions, though inadequate, are still very much with us. We feel that runs retain the virtues of directions and at the same time remove their inadequacies.

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[2] A filter-base is a non-empty family of non-empty sets such that the intersection of any two of them includes a third.

[3] In this present paper we have in mind a set theory similar to that employed by J.L. Kelley, General Topology, pp. 250 ff. New York, 1953.

2. Some definitions.

2.1 DEFINITIONS.

1. $\sim B =$ The complement of B
2. $\sigma F = \mathrm{E}x\,(x \in \beta$ for some $\beta \in F)$
3. $\pi F = \mathrm{E}x\,(x \in \beta$ for every $\beta \in F)$

2.2 REMARK. Thus σF is the union and πF the intersection of all members of F. If A is the set whose sole member is x, then $\sigma A = x = \pi A$. We assume the integer 0 and the empty set are the same and notice that $\sigma 0 = 0$ and $\pi 0 =$ the universe.

With vertical and horizontal sections in mind we make the following definitions.

2.3 DEFINITIONS.

1. $\mathrm{vs}\, Rx = \mathrm{E}y\,[(x, y) \in R]$.
2. $\mathrm{hs}\, Ry = \mathrm{E}x[(x, y) \in R]$.

When R is a run then we sometimes think:

$y \in \mathrm{vs}\, Rx$ if and only if y comes after x;

$x \in \mathrm{hs}\, Ry$ if and only if x comes before y.

There is no magical significance, as often in analytic geometry, attached to the letters used. Thus we sometimes think:

$x \in \mathrm{vs}\, R\delta$ if and only if x comes after δ;

or even

$x \in \mathrm{vs}\ Ry$ if and only if x comes after y.

2.4 DEFINITIONS.

1. ${}_*RA = \mathrm{E}y\,[(x, y) \in R$ for some $x \in A]$
2. ${}^*RA = \mathrm{E}x[(x, y) \in R$ for some $y \in A]$

2.5 DEFINITION. $\mathrm{inv}\, R = \mathrm{inverse}\ R = \mathrm{E}x, y\,[(y, x) \in R]$.

2.6 DEFINITION. $R : S = \mathrm{E}x, z\,$[There is a y such that $(x, y) \in S$ and $(y, z) \in R$.]

A function is the same as its graph and is hence a special kind of relation. If f and g are functions, then $f : g$ is that function h such that $h(x) = f(g(x))$ for each x.

2.7 DEFINITION. $\mathrm{rct}\ AB = \mathrm{rectangle}\ AB = \mathrm{E}x, y(x \in A$ and $y \in B)$.

3. A few properties of relations.

3.1 THEOREM. *If R is a relation and f is a function, then:*

1. ${}_*R(A \cup B) = {}_*RA \cup {}_*RB$
2. $A \subset B$ *implies* ${}_*RA \subset {}_*RB$
3. ${}_*R(A \cap B) \subset {}_*RA \cap {}_*RB$
4. ${}^*RA = {}_*\mathrm{inv}\, RA$
5. ${}^*f(A \cap B) = {}^*fA \cap {}^*fB$
6. ${}_*f{}^*fA = A \cap \mathrm{rng}\, f$
7. ${}^*R_*RA \supset A \cap \mathrm{dmn}\, R.$

3.2 THEOREM. *If R and S are relations and f is a function, then:*

1. $\mathrm{vs}\,(R: S)x = {}_*R\,\mathrm{vs}\; Sx$ *for each* x
2. $B \cap {}_*SA \subset {}_*S(({}^*SB) \cap A)$
3. $B \cap {}_*fA = {}_*f(({}^*fB) \cap A)$
4. $B \cap {}_*fA \neq 0$ *implies* $({}^*fB) \cap A \neq 0.$

4. Properties of runs.

4.1 THEOREM. *R is a run if and only if R is such a non-vacuous relation that for each x and y in the domain of R there exists a z in the domain of R for which* $\mathrm{vs}\, Rz \subset \mathrm{vs}\, Rx \cap \mathrm{vs}\, Ry$.

Accordingly if some vertical section of R is a set belonging to the universe,[3] then the vertical sections form a filter-base *theoretically* as useful as R itself. Only in the peripheral situation that every vertical section of R is a class incapable of belonging to anything are runs more effective than filter-bases. However, runs do operate on an essentially different and, we feel, more convenient level.

The passage from a filter-base W to a run R can always be successfully accomplished by putting $R = \mathrm{E}\beta, x(x \in \beta \in W)$.

4.2 DEFINITIONS.

1. R *runs* in A if and only if R is a run and $\mathrm{rng}\, R \subset A$.
2. R is *eventually* in A if and only if R is a run and vs $Rx \subset A$ for some $x \in \mathrm{dmn}\, R$.
3. R is *frequently* in A if and only if R is a run and vs $Rx \cap A \neq 0$ for each $x \in \mathrm{dmn}\, R$.

4.3 THEOREMS.

1. *If R runs in A then R is eventually in A.*
2. *If R is eventually in A, then R is frequently in A.*

4.4 DEFINITIONS.

1. S is a *subrun* of R if and only if S is a run, R is a run,

and for each $x \in \operatorname{dmn} R$ there exists such a $y \in \operatorname{dmn} S$ that $\operatorname{vs} Sy \subset \operatorname{vs} Rx$.

2. *R runs the same* as S if and only if R is a subrun of S and S is a subrun of R.

We agree that S is a *corun* of R if and only if there exists such an A that R is frequently in A and $S = R \cap \operatorname{E}x, y(y \in A)$.

If S is a corun of R then S is a subrun of R. However coruns are often inadequate in that S may be a subrun of R and yet no corun of R will run the same as S.

4.5 THEOREMS.

1. *If R is a run, then R is a subrun of R.*
2. *If R'' is a subrun of R' and R' is a subrun of R, then R'' is a subrun of R.*
3. *If R is frequently in A and $S = R \cap \operatorname{E}x, y(y \in A)$, then S is a subrun of R,* $\operatorname{dmn} S = \operatorname{dmn} R$, *and* $\operatorname{vs} Sx = A \cap \operatorname{vs} Rx$ *for each x.*
4. *If R' is a subrun of R and R is eventually in A, then R' is eventually in A.*

4.6 THEOREMS.

1. *If S is a relation and R is frequently in* $\operatorname{dmn} S$, *then $S : R$ is a run,* $\operatorname{dmn}(S : R) = \operatorname{dmn} R$, *and* $\operatorname{vs}(S : R)x = {}_*S \operatorname{vs} Rx$ *for each x.*
2. *If S is a relation and R is a run, then R is frequently in* $\operatorname{dmn} S$ *if and only if* $\operatorname{dmn}(S : R) = \operatorname{dmn} R$.
3. *If S is a relation, R' is a subrun of R, and R' is frequently in* $\operatorname{dmn} S$, *then $S : R'$ is a subrun of $S : R$.*
4. *If S is a relation and R is eventually in* $\operatorname{dmn} S$, *then R is a subrun of* $(\operatorname{inv} S) : (S : R)$.
5. *If f is a function and R is eventually in* $\operatorname{rng} f$, *then* $f : (\operatorname{inv} f) : R$ *runs the same as R.*

4.7 DEFINITIONS.

1. merger RS = the set of points of the form $((x, y), z)$, where x, y and z are such that R and S are runs, $(x, z) \in R$, and $(y, z) \in S$.
2. *R merges* with S if and only if R and S are such runs that $\operatorname{vs} Rx \cap \operatorname{vs} Sy \neq 0$ whenever $x \in \operatorname{dmn} R$ and $y \in \operatorname{dmn} S$.

4.8 THEOREMS.

1. *If R and S are runs and $V =$* merger RS, *then V is a relation,* $\operatorname{dmn} V \subset \operatorname{rct} \operatorname{dmn} R \operatorname{dmn} S$, *and* $\operatorname{vs} V(x, y) = \operatorname{vs} Rx \cap \operatorname{vs} Sy$ *whenever* $(x, y) \in \operatorname{dmn} V$.
2. *If R merges with S and $V =$* merger RS, *then dmn $V =$* $\operatorname{rct} \operatorname{dmn} R \operatorname{dmn} S$, *and V is a subrun of both R and S.*

3. *If W is a subrun of both R and S, then R and S merge and W is a subrun of* merger RS.

4.9 THEOREM. *If f is a function, R is frequently in* dmn f, *and W is a subrun of $f : R$, then there exists such a subrun V of R that W runs the same as $f : V$.*

Proof. *Let* $V' = (\text{inv} f) : W$, *let* $V =$ merger $V'R$, *and let* $W' = f : V$. Use 4.6.1 to see that

$$\text{dmn}\,(f : R) = \text{dmn}\,R \quad \text{and} \quad \text{dmn}\,V' = \text{dmn}\,W.$$

We complete the proof in three parts by showing that W' and W are subruns of each other.

Part 1. *V' merges with R, V is a subrun of V' and R, W' is a run, and* dmn $W' =$ dmn $V =$ rct dmn W dmn R.

Proof. Let $x \in \text{dmn}\,W$ and $y \in \text{dmn}\,R$. Since W is a subrun of $f : R$ we have

$$0 \neq \text{vs}\,Wx \cap \text{vs}\,(f :\, R)y = \text{vs}\,Wx \cap {}_*f\,\text{vs}\,Ry.$$

Hence, using 3.2.4, 3.2.1, and 3.1.4, we find that

$$0 \neq ({}^*f\text{vs}\,Wx) \cap \text{vs}\,Ry = \text{vs}\,V'x \cap \text{vs}\,Ry.$$

Use of 4.6.1 and 4.8.2 completes the proof.

Part 2. *W' is a subrun of W.*

Proof. Use Part 1, 4.6.3, and 4.6.5 to see that $W' = f :\, V$ is a subrun of $f :\, V' = f :\, (\text{inv}\, f) :\, W$, which runs the same as W.

Part 3. *W is a subrun of W'.*

Proof. Let $x \in \text{dmn}\,W$ and $y \in \text{dmn}\,R$. Select $x' \in \text{dmn}\,W$ so that vs $Wx' \subset \text{vs}(f : R)y$, and select $x'' \in \text{dmn}\,W$ so that vs $Wx'' \subset$ vs $Wx \cap$ vs Wx'.
Then

$$\text{vs}\,Wx'' \subset \text{vs}\,Wx \cap \text{vs}(f :\, R)y = \text{vs}\,Wx \cap {}_*f\,\text{vs}\,Ry,$$

which in accordance with 3.2.3 equals

$$\begin{aligned} {}_*f(({}^*f\,\text{vs}\,Wx) \cap \text{vs}\,Ry) &= {}_*f\,[\text{vs}\,((\text{inv}\,f) :\, W)x \cap \text{vs}\,Ry] \\ &= {}_*f\,\text{vs}\,V(x,\,y) = \text{vs}\,W'(x,\,y). \end{aligned}$$

In view of Part 1 the proof is complete.

4.10. REMARK. Theorems 4.6.1, 4.6.3, and 4.9 show us that under any properly chosen function f, a run R is mapped into a run $S = f : R$, subruns of R are mapped into subruns of S, and any subrun of S runs the same as the map of some subrun of R.

4.11 DEFINITION. indexrun $R = \mathrm{E}x, y[R$ is a run, $x \in \operatorname{dmn} R$, $y \in \operatorname{dmn} R$, and $\operatorname{vs} Ry \subset \operatorname{vs} Rx]$.

4.12 THEOREM. *If* R *is a run and* $D =$ indexrun R, *then* D *is a direction,* $\operatorname{dmn} D = \operatorname{rng} D = \operatorname{dmn} R$, $(x, x) \in D$ *whenever* $x \in \operatorname{dmn} D$, *and* $R = R : D$.

4.13 REMARK. According to Theorem 4.12, every run is the composition of a relation with a reflexive direction. In fact, every run runs the same as the composition of a *function* with a reflexive direction. Suppose R is a run and D is the set of pairs of the form $((x, y), (x', y'))$, where x, y, x', and y' are such that $(x, y) \in R$, $(x', y') \in R$, and $\operatorname{vs} Rx' \subset \operatorname{vs} Rx$. Let f be such a function that $f(x, y) = y$ whenever $(x, y) \in \operatorname{dmn} D$. It is easy to check that D is a reflexive direction and that R runs the same as $f : D$.

In this connection it should be remarked that if (f, D) is a net in the sense of Kelley (op. cit.) then $f : D$ is a corresponding run. The above construction gives a method for passing from a run back to a corresponding net.

4.14 DEFINITIONS.

1. R is a *full* run if and only if R is a run which runs the same as all of its subruns.
2. R is *fillable* if and only if there exists a full subrun of R.

4.15 THEOREMS.

1. *If* R *is a full run and* R *is frequently in* A, *then* R *is eventually in* A.

Proof. Note that R is a subrun of $R \cap \mathrm{E}x, y(y \in A)$.

2. *R is a full run if and only if for every A, R is either eventually in A or eventually in $\sim A$.*
3. *If R is a full run, f is a function, and R is frequently in* $\operatorname{dmn} f$, *then* $f : R$ *is a full run.*

Proof. Use 2.

4. *If S is a full run which merges with R, then S is a subrun of R.*

4.16 DEFINITIONS.

1. N is R *nested* if and only if R is a relation and either or $y = x(x, y) \in R \cup \operatorname{inv} R$ whenever x and y are in N.
2. N is *nested* if and only if either $\alpha \subset \beta$ or $\beta \subset \alpha$ whenever α and β are in N.

4.17 DEFINITIONS.

1. F is R *capped* if and only if R is a relation and corresponding to each R nested subfamily N of F there is a $z \in F$ such that $(x, z) \in R$ whenever $x \in N$.
2. F is *capped* if and only if corresponding to each nested subfamily N of F there is $\gamma \in F$ such that $\sigma N \subset \gamma$.

We have found quite useful the following inductive variants of Zorn's

4.18 LEMMAS.

1. *If R is transitive and F is R capped, and if corresponding to each $x \in F \sim K$ there is a $y \in F$ for which $(x, y) \in R \sim \operatorname{inv} R$ then $F \cap K \neq 0$.*
2. *If F is capped and if each member of $F \sim K$ is a proper subset of some member of F, then $F \cap K \neq 0$.*

4.19 REMARK. In accordance with the terminology used by Kelley (op. cit.), we agree that a *set* is a class which is small enough to belong to the universe.

4.20 THEOREMS.

1. *If R is a full run, then R is eventually in some set.*

Outline of proof. Otherwise according to 4.15.2 R is eventually in $\sim A$ whenever A is a set. Advantage may be taken of this fact to construct by transfinite induction two classes B and C for which $B \cap C = 0$, R is frequently in B, and R is frequently in C. In view of 4.15.1 this is impossible.

2. *R is fillable if and only if R is frequently in some set.*

Proof. If R is fillable it is easy to check with the help of 1. that R is frequently in some set. We now assume that R is frequently in some set A and show that R is fillable.

We agree that sng x is the family whose sole member is x, and that $G \cap \cap H = \mathrm{E}\gamma\,[\gamma = \alpha \cap \beta$ for some $\alpha \in G$ and $\beta \in H]$.

Let $B = \mathrm{E}\alpha\,[\alpha \subset A$ and R is frequently in $\alpha]$, let $F = \mathrm{E}W[W$ is a filter-base and $W \subset B]$, and let $K = \mathrm{E}W$ [for each $\alpha \subset A$ there exists a $\beta \in W$ for which either $\beta \subset \alpha$ or $\beta \subset A \sim \alpha$. If N is a nested subfamily of F then : if $N = 0$ then $\sigma N = 0 \subset \operatorname{sng} A \in F$; if $N \neq 0$, then $\sigma N \subset \sigma N \in F$. Accordingly F is capped.

Now suppose $W \in F \sim K$, and select such a set α that $\beta \cap \alpha \ni B$ and $\beta \sim \alpha \neq 0$ whenever $\beta \in W$. Let $W' = W \cup (W \cap \bigcap \text{sng } \alpha)$ and check that $W' \in F$ and that W is a proper subfamily of W'. According to 4.18.2 we conclude that $F \cap K \neq 0$ and select $V \in F \cap K$, so that V is a filter base, R is frequently in every member of V, and for each $\alpha \subset A$ there exists such a $\beta \in V$ that $\beta \subset \alpha$ or $\beta \subset A \sim \alpha$.

Let $S = \mathrm{E}\beta, x[x \in \beta \in V]$ and notice that S is a full run which merges with R. According to 4.15.4, S is a full subrun of R. This completes the proof.

4.19 REMARK. The run R_3 is not fillable.

5. Topological convergence.

5.1 DEFINITIONS.

1. R *clusters* about p in the topology T if and only if T is a topology, $p \in \sigma T$, and R is frequently in every T neighborhood of p.
2. R *converges* to p in the topology T if and only if T is a topology, $p \in \sigma T$, and R is eventually in every T neighborhood of p.
3. R *converges* in the topology T if and only if there exists such a point p that R converges to p in the topology T.
4. nhbdrun pT = the neighborhood run of p in the topology $T = \mathrm{E}\beta, x[T$ is a topology, $p \in \beta \in T$, and $x \in \beta]$.
5. nhbdrun$'$ $pT = \mathrm{E}\beta, x[T$ is a topology, $p \in \beta \in T, x \in \beta$, and $p \neq x]$.

5.2 REMARK. If T is a topology, $A \subset \sigma T$, and p is a point in the T closure of A, then $\mathrm{E}\beta, x(p \in \beta \in T$ and $x \in \beta \cap A)$ runs in A and converges to p in the topology T. It is possible that no run which runs in A and converges to p in the topology T can also be a direction. This can be seen by making use of the topology defined in Problem E on page 77 of Kelley (op. cit.).

5.3 THEOREMS.

1. *R clusters about p in the topology T if and only if R merges with* nhbdrun pT.
2. *R converges to p in the topology T if and only if R is a subrun of* nhbdrun pT.

As an application of the foregoing we offer the following characterizations of compactness.

5.4 THEOREM. *Each of the following is a necessary and sufficient condition that a topology T be compact.*

1. *Whenever R runs in σT, then for some point p, R clusters about p in the topology T.*
2. *Whenever R runs in σT, then there exists such a subrun R' of R that R' converges in the topology T.*
3. *Whenever R is a full run which runs in σT, then R converges in the topology T.*

5.5 REMARK. The Tychonoff theorem,[4] which assures us that the topological product of compact topologies is compact, we will now prove following a well-known pattern. Suppose T is the product topology[5] in question and that R is a full run which runs in σT. Considering any coordinate, let P be the usual projection which maps σT into the corresponding coordinate space. According to 4.15.3 and 5.4.3, $P:R$ is a full run which converges in the topology of the coordinate space. Consequently R converges coordinatewise and hence converges in the topology[5] T.

6. Limits.

6.1 DEFINITIONS.

1. far RxP if and only if R is eventually in $\mathrm{E}xP$.

In 1. above we allow "P" to be replaced by an arbitrary formula such as, for example,

$$\text{“}[y < x < x^2]\text{”}.$$

2. $f(x)$ tends to p in the topology T as x runs along R if and only if T is a topology, $p \in \sigma T$, and far $Rx(f(x) \in \beta)$ whenever β is a T neighborhood of p.
3. $f(x)$ tends uniquely to p in the topology T as x runs along R if and only if for every q ($p = q$ if and only if $f(x)$ tends to q in the topology T as x runs along R).
4. $\mathrm{lmt}\, TxRf(x) =$ the limit in the topology T as x runs along R of $f(x) = \pi \mathrm{E}p\,[f(x)$ tends uniquely to p in the topology T as x runs along $R]$.[6]

Thus if $f(x)$ tends uniquely to p in the topology T as x runs along R we know that $\mathrm{lmt}\, TxRf(x) = p$.

6.2 THEOREM. *If T is a topology, $p \in \sigma T$, f is a function, and R is eventually in the domain of f, then*

1. *$f(x)$ tends to p in the topology T as x runs along R if and only if $f:R$ converges to p in the topology T; and*
2. *$\mathrm{lmt}\, TxRf(x) = p$ if and only if $f{:}R$ converges to p in the*

[4] See Kelley (op. cit.) p. 143.

[5] See Kelley (op. cit.) pp. 88–92.

[6] See Remark 2.2.

topology T, and $q = p$ for every q such that $f{:}R$ converges to q in the topology T.

Very elementary but of considerable use is the

6.3 THEOREM. *If* far $Rx(u(x) = v(x))$ *then* $\text{lmt}\ TxRu(x) = \text{lmt}\ T\,x\,Rv(x)$.

6.4 REMARK. As examples of specialized limit notations in which either the run or the topology or both are suppressed, we give the following definitions. We agree that $\mathscr{T}$ is the usual topology for the extended real number system, and that

$$\mathscr{R} = \text{E}m, n[m \in \omega \text{ and } m \leqq n \in \omega].$$

6.5 DEFINITIONS.

1. $\text{lnt}\ T\,n\,u(n) = \text{lmt}\ \mathscr{T}\,n\,\mathscr{R}\,u(n)$
2. $\text{lm}\,x\,R\,f(x) = \text{lmt}\ \mathscr{T}\,x\,R\,f(x)$
3. $\overline{\text{lm}}\,x\,R\,f(x) = \text{lmt}\ \mathscr{T}\,t$ indexrun R (sup $x \in$ (vs Rt) $f(x)$)
4. $\underline{\text{lm}}\,x\,R\,f(x) = \text{lmt}\ \mathscr{T}\,t$ indexrun R (inf $x \in$ (vs Rt) $f(x)$)
5. $\text{lim}\,x$ a $f(x) = \text{lm}\,x$(nhbdrun$'$ $a\mathscr{T}$) $f(x)$
6. $\text{lin}\,n\,u(n) = \text{lnt}\,\mathscr{T}\,nu(n)$

6.6 REMARK. In 6.5.1 we have a limit notation for ordinary sequences. If u is a sequence, T is a topology, and $p \in \sigma T$, then $\text{lnt}\,T\,n\,u(n) = p$ if and only if p is the unique point such that $u : \mathscr{R}$ converges to p in the topology T.

We give a few more simple but useful theorems.

6.7 THEOREMS.

1. *If* $\delta \in$ dmn R *and if far* Rx $\{f(x \geqq f(y)\}$ *whenever* $y \in$ vs $R\delta$,

   then $-\infty \leqq \text{lm}\,x\,R\,f(x) = \sup x \in \text{vs}\,R\delta\ f(x) \leqq \infty$.

2. *If* $\delta \in$ dmn R *and if far* $Rx\{f(x) \leqq f(y)\}$ *whenever* $y \in$ vs $R\delta$,

   then $-\infty \leqq \text{lm}\,x\,R\,f(x) = \inf x \in \text{vs}\,R\delta\ f(x) \leqq \infty$.

From 1. and 2. we infer 3. and 4. below. These results are generalizations of the fact that non-decreasing and non-increasing functions have limits.

3. *If* far Ry far $Rx\{f(x) \geqq f(y)\}$ then

   $$-\infty \leqq \text{lm}\,x\,Rf(x) \leqq \infty.$$

4. *If* far Ry far $Rx\{f(x) \leqq f(y)\}$ then

   $$-\infty \leqq \text{lm}\,x\,Rf(x) \leqq \infty.$$

6.8 THEOREM. *If R is a run and $-\infty \leqq a \leqq \infty$, then*

$$\operatorname{lm} x\, R\, a = a\,.$$

6.9 THEOREM. *If $A = \operatorname{lm} x\, R\, u(x)$ and $B = \operatorname{lm} x\, R\, v(x)$, then:*
1. *if $-\infty \leqq A + B \leqq \infty$, then $\operatorname{lm} x\, R\{u(x) + v(x)\} = A + B$;*
2. *if $-\infty \leqq A \cdot B \leqq \infty$, then $\operatorname{lm} x\, R\{u(x) \cdot v(x)\} = A \cdot B$.*

In connection with 6.9 above and 6.10 below it is understood that $\infty - \infty$, $0 \cdot \infty$, and $1/0$ are not real numbers.

6.10 THEOREM. *If $A = \operatorname{lm} x\, R\, u(x)$ and $-\infty \leqq 1/A \leqq \infty$, then*

$$\operatorname{lm} x\, R\{1/u(x)\} = 1/A\,.$$

From 4.15.3 and 5.4.3 we infer

6.11 THEOREM. *If R' is a full subrun of R and* far

$$R\, x\,(-\infty \leqq u(x) \leqq \infty), \textit{ then}:$$

$$-\infty \leqq \operatorname{lm} x\, R\, u(x) \leqq \operatorname{lm} x\, R'\, u(x) \leqq \overline{\operatorname{lm}}\, x\, R\, u(x) \leqq \infty\,.$$

If R is fillable, then Theorem 6.11 furnishes us with a generalized limit which, since it is expressed as an actual limit, automatically enjoys the properties found in Theorems 6.8, 6.9, and 6.10. In the event u is bounded, it does not at first glance seem too unreasonable to hope that a similar generalized limit could be arrived at by some Hahn-Banach technique. We, however, are inclined to think this impossible.[7]

We close with an application of limits to integration which expresses the Lebesgue integral as a genuine limit of Riemann-like sums.

6.12 REMARK. Suppose that $\mathscr{L}$ is Lebesgue measure and $\mathfrak{P} = \mathrm{E}P$ [P is a countable disjointed family of non-empty $\mathscr{L}$ measurable subsets of the unit interval for which $\sigma P =$ the unit interval]. We agree that Q is a *refinement* of P if and only if every member of Q is included in some member of P, and agree that ξ is a selector function if and only if $\xi(\beta) \in \beta$ whenever $\beta \in \operatorname{dmn} \xi$. Let

$R_4 = \mathrm{E}P, \xi$[$P \in \mathfrak{P}$ and ξ is a selector function whose domain is a member of $\mathfrak{P}$ and a refinement of P].

We now have the

THEOREM. $$\int_0^1 f(x)\, dx = \operatorname{lm} \xi\, R_4 \sum \beta \in \operatorname{dmn} \xi \{f(\xi(\beta)) \cdot \mathscr{L}(\beta)\}$$

[7] See R. P. Agnew and A. P. Morse, *Extensions of linear functionals with applications to limits, integrals, measures, and densities*, Ann. Math. Stat. **39**, no. 1, January. 1938. Notice especially the first two lines on page 24.

whenever f *is a finite-value* $\mathscr{L}$ *measurable function defined on the unit interval.*

We think it noteworthy that R_4 runs in the selector functions.

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