ON THE RADICAL OF A GROUP ALGEBRA

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A basic result in the study of group algebras and characters states that the group algebra $\mathfrak{A}(\mathscr{G})$ of a finite group $\mathscr{G}$ over the field $\mathfrak{F}$ of characteristic $p \neq 0$ has a nonzero radical $\mathfrak{R}$ if and only if p is a divisor of $o(\mathscr{G})$, the order of $\mathscr{G}$. This suggests that $\mathfrak{R}$ is related in some manner to the Sylow p-groups of $\mathscr{G}$ and that it may be possible to define $\mathfrak{R}$ in terms of these subgroups. In [6] Jennings showed that if $o(\mathscr{G}) = p^a$, then $\mathfrak{R}$ is of dimension $p^a - 1$ and has as a basis the set of elements $P_i - 1$. As a generalization of this define $\mathfrak{R}'$ to be the intersection of all the left ideals of $\mathfrak{A}(\mathscr{G})$ generated by the radicals of the group algebras of the Sylow p-groups of $\mathscr{G}$. Then $\mathfrak{R}'$ is a nilpotent ideal of $\mathfrak{A}(\mathscr{G})$ (cf. [2]), and Lombardo-Radici has shown [8] that $\mathfrak{R}' = \mathfrak{R}$ provided $\mathscr{G}$ has a unique Sylow p-group or $o(\mathscr{G}) = pq$ where q is also a prime. Also, in [9] he demonstrated that if $\mathscr{G}$ is the simple group of order 60 and if $p = 2$ or 3 then $\mathfrak{R}'$ is a proper subideal of $\mathfrak{R}$. In this paper it will be shown that $\mathfrak{R}' = \mathfrak{R}$ if one of the following conditions is satisfied:

(A) $\mathscr{G}$ is homomorphic with a Sylow p-group of $\mathscr{G}$.

(B) $\mathscr{G}$ is a super-solvable group.

(C) $\mathscr{G}$ is a solvable group with $(o(\mathscr{G}), p^2) = p$.

In the last section of the paper an application to a related problem is made. If $\mathscr{G}$ contains an invariant p-group then $\mathfrak{A}(\mathscr{G})$ is bound to its radical $\mathfrak{R}$ (i.e., if a in $\mathfrak{A}(\mathscr{G})$ is an element such that $a\mathfrak{R} = \mathfrak{R}a = 0$, then a is in $\mathfrak{R}$). This raises the question: If $\mathfrak{A}(\mathscr{G})$ is bound to its radical $\mathfrak{R}$, does $\mathscr{G}$ contain an invariant p-group? This is equivalent to the question: Does $\mathscr{G}$ contain an invariant p-group if $\mathscr{G}$ possesses no irreducible representation of highest kind? (An irreducible representation of highest kind is one whose dimension is divisible by the highest power of p which divides $o(\mathscr{G})$.) It is shown that if $\mathscr{G}$ is a group such that $\mathfrak{R}' = R$ and if the Sylow p-groups of $\mathscr{G}$ are cyclic, then the above question is answered affirmatively. Also an example is given where the answer is negative.

1. Type A. Let $\mathscr{G}$ be a group of order of order $g = hp^a$, $(h, p) = 1$, with a normal subgroup $\mathscr{H}$ of order h. And let $\mathfrak{F}$ be an algebraically closed field of characteristic p. (The requirement that $\mathfrak{F}$ be algebraically closed is only a convenience since the dimension of $\mathfrak{R}'$ is

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unaffected by any extension of the ground field.)

THEOREM 1. *The radical* $\mathfrak{R}$ *of the group algebra* $\mathfrak{A}(\mathscr{G})$ *of the group* $\mathscr{G}$ *over the field* $\mathfrak{F}$ *equals* $\mathfrak{R}'$, *the intersection of all the left ideals of* $\mathfrak{A}(\mathscr{G})$ *generated by the radicals of the group algebras of the Sylow p-groups of* $\mathscr{G}$.

Let $\mathscr{P}$ be a Sylow p-group of $\mathscr{G}$: then $\mathscr{G}/\mathscr{H}$ is isomorphic with $\mathscr{P}$ and $\mathscr{G}$ is an extension of $\mathscr{H}$ by $\mathscr{P}$. Now $\mathfrak{A}(\mathscr{P})$, the group algebra of $\mathscr{P}$ over $\mathfrak{F}$, has the radical $\mathfrak{N}$ which is of dimension $p^a - 1$ over $\mathfrak{F}$ and has as a basis the differences $P_i - 1$, all $P_i \in P$. Form $\mathfrak{M}$, the left ideal of $\mathfrak{A}(\mathscr{G})$ generated by $\mathfrak{N}$. The ideal $\mathfrak{M}$ is of dimension $h(p^a - 1)$ over $\mathfrak{F}$, and we propose to show that $\mathfrak{R}$, the radical of $\mathfrak{A}(\mathscr{G})$, is contained in $\mathfrak{M}$.

Now $\mathfrak{A}(\mathscr{H})$, the group algebra of $\mathscr{H}$ over $\mathfrak{F}$, is expressible as $\mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_n$ where $\mathfrak{B}_i$ is a simple ideal of $\mathfrak{A}(\mathscr{H})$. Let $\mathfrak{B}$ be one of these, and let $\mathscr{P}'$ be the subgroup of $\mathscr{P}$ consisting of elements P_i such that $P_i\mathfrak{B}P_i^{-1} = \mathfrak{B}$, with $o(\mathscr{P}') = r = p^c$, $o \leqq c \leqq a$. The elements H of $\mathscr{H}$ are represented by $\bar{H}$ in $\mathfrak{B}$ and the $\bar{H}$ form a group $\bar{H}$ homomorphic with $\mathscr{H}$. Furthermore the elements of $\mathfrak{B}$ can be expressed linearly in terms of the elements of $\bar{\mathscr{H}}$.

If $P \in \mathscr{P}'$, then P corresponds to an automorphism of $\mathfrak{B}$ since $P\mathfrak{B}P^{-1} = \mathfrak{B}$, and since $\mathfrak{B}$ is central simple this automorphism is an inner automorphism of $\mathfrak{B}$. Thus P corresponds to a sum of elements of $\bar{\mathscr{H}}$ and so leaves the conjugate classes of $\bar{\mathscr{H}}$ invariant since these classes commute with the individual elements of $\bar{\mathscr{H}}$. Basically, therefore, we are dealing with an extension $\bar{\mathscr{G}}$ of $\bar{\mathscr{H}}$ by a p-group $\mathscr{P}'$ in which each element of $\mathscr{P}'$ induces an automorphism A of $\bar{\mathscr{H}}$ which leaves the conjugate classes invariant. Since the order of $\bar{\mathscr{H}}$ is prime to p it is well-known [**11**, p. 123] that A is an inner automorphism of $\bar{\mathscr{H}}$. Now a result due to M. Hall [**4**, Theorem 6.1] implies that $\bar{\mathscr{G}}$ is a direct product of $\mathscr{P}'$ and $\bar{\mathscr{H}}$, and this leads to the conclusion that the elements of $\mathscr{P}'$ commute elementwise with $\mathfrak{B}$. If $\mathfrak{Q} = \sum_{P_i \in \mathscr{P}'} P_i\mathfrak{B}$, then the radical $\mathfrak{Q}'$ of $\mathfrak{Q}$ equals $\mathfrak{B}$ times the radical of $\mathfrak{A}(\mathscr{P}_{\prime})$, and therefore $\mathfrak{Q}'$ is contained in $\mathfrak{M}$.

If $t = p^{a-c}$ is the index of $\mathscr{P}'$ in $\mathscr{P}$, then there are t distinct ideals $\mathfrak{B}_i$ in the decomposition of $\mathfrak{A}(\mathscr{H})$ which form a set of transitivity $\mathbf{T}$ for $\mathscr{P}$, with $\mathfrak{B}_1 = \mathfrak{B}$. That is, $P_i\mathfrak{B}_jP_i^{-1} \in \mathbf{T}$ if $\mathfrak{B}_j \in \mathbf{T}$ and $P_i \in \mathscr{P}$, and furthermore, if $\mathfrak{B}_i, \mathfrak{B}_j \in \mathbf{T}$, then there is a $P_k \in \mathscr{P}$ such that $\mathfrak{B}_i = P_k\mathfrak{B}_jP_k^{-1}$. Then the algebra $\mathfrak{T} = \sum P_i\mathfrak{B}_j$, all $P_i \in \mathscr{P}$ and $\mathfrak{B}_j \in \mathbf{T}$, is an ideal of $\mathfrak{A}(\mathscr{G})$, and we assert that its radical is contained in $\mathfrak{M}$. To see this consider the coset expansion of $\mathscr{P}$ relative to $\mathscr{P}'$, $\mathscr{P} = \sum S_i \mathscr{P}' = \sum \mathscr{P}' S_i$. Then clearly the algebra $\mathfrak{T}' = \sum_{i,j}^t S_i \mathfrak{Q}' S_j$ is a nilpotent ideal of $\mathfrak{T}$, while the transitivity of $\mathbf{T}$ implies that $\mathfrak{T} - \mathfrak{T}'$ is a simple algebra. Thus $\mathfrak{T}'$ is the radical of $\mathfrak{T}$ and obviously is contained in $\mathfrak{M}$.

As the choice of $\mathfrak{B}$ was arbitrary in the decomposition of $\mathfrak{A}(\mathscr{H})$, clearly the process above leads to the conclusion that $\mathfrak{R}$ is contained in $\mathfrak{M}$. Since the choice of $\mathscr{P}$ was arbitrary this enables us to conclude that $\mathfrak{R}' \supseteqq \mathfrak{R}$. However $\mathfrak{R}'$ is known to be nilpotent (cf [2]), hence $\mathfrak{R}' = \mathfrak{R}$.

2. Type B. A group $\mathscr{G}$ is defined to be *super-solvable* if it possesses a sequence of subgroups $\mathscr{G}_0 = \mathscr{G} \supset \mathscr{G}_1 \supset \cdots \supset \mathscr{G}_s = 1$ such that $\mathscr{G}_i$ is normal in $\mathscr{G}$ and $\mathscr{G}_i/\mathscr{G}_{i+1}$ is cyclic. If in addition each $\mathscr{G}_i/\mathscr{G}_{i+1}$ is contained in the center of $\mathscr{G}/\mathscr{G}_{i+1}$, then $\mathscr{G}$ is called a *nilpotent* group. A basic result concerning nilpotent groups states that a nilpotent group is a direct product of its Sylow groups. And a principal theorem on super-solvable groups states that a super-solvable group is an extension of a nilpotent group by a nilpotent group. (For these results see Kurosch [7, pp. 216 and 228])

THEOREM 2. *The radical $\mathfrak{R}$ of the group algebra $\mathfrak{A}(\mathscr{G})$ of a super-solvable group $\mathscr{G}$ over the field $\mathfrak{F}$ equals $\mathfrak{R}'$.*

By the theorems quoted above $\mathscr{G}$ contains a normal nilpotent subgroup $\mathscr{G}_1$ such that $\mathscr{G}/\mathscr{G}_1$ is nilpotent while $\mathscr{G}_1$ has a normal Sylow p-group $\mathscr{P}_1$. Evidently $\mathscr{P}_1$ is normal in $\mathscr{G}$ since $\mathscr{G}_1$ is a direct product of its Sylow groups. Then the radical of $\mathfrak{A}(\mathscr{P}_1)$ generates a nilpotent ideal $\mathfrak{R}_1$ of $\mathfrak{A}(\mathscr{G})$ and $\mathfrak{A}(\mathscr{G}) - \mathfrak{R}_1$ is isomorphic with the group algebra $\mathfrak{A}(\mathscr{G}/\mathscr{P}_1)$ of $\mathscr{G}/\mathscr{P}_1$. Now the group $\mathscr{G}/\mathscr{P}_1$ is a group of Type A which was discussed in the preceding section. So if $\mathfrak{J}$ is a left ideal of $\mathfrak{A}(\mathscr{G})$ generated by the radical of the group algebra of $\mathscr{P}$, a Sylow p-group of $\mathscr{G}$, then $\mathfrak{A}(\mathscr{G}) - \mathfrak{J}$ is a completely reducible left $\mathfrak{A}(\mathscr{G})$-module since $\mathscr{P}/\mathscr{P}_1$ is a Sylow p-group of $\mathscr{G}/\mathscr{P}_1$. Hence $\mathfrak{R} = \mathfrak{R}'$.

3. Type C. Let $\mathscr{G}$ be a solvable group whose order is divisible by p to the first power only. Then $\mathscr{G}$ possesses a sequence of subgroups $\mathscr{G}_0 = \mathscr{G} \supset \mathscr{G}_1 \supset \cdots \supset \mathscr{G}_n = 1$ such that $\mathscr{G}_{i+1}$ is normal in $\mathscr{G}_i$ and $\mathscr{G}_i/\mathscr{G}_{i+1}$ is a group of order q where q is a prime.

THEOREM 3. *The radical $\mathfrak{R}$ of the group algebra $\mathfrak{A}(\mathscr{G})$ of the group $\mathscr{G}$ over the field $\mathfrak{F}$ equals $\mathfrak{R}'$.*

The proof will be by induction on n, the length of the series defined above. If $n = 1$ the theorem is trivially true; so assume the result to be true for groups of length less than n. Now consider $\mathscr{G}_1$, which is of length $n-1$. If $\mathscr{G}/\mathscr{G}_1$ is of order p, then the order of $\mathscr{G}_1$ is prime to p and the result follows by Theorem 1. So we shall restrict our attention to the case where $\mathscr{G}/\mathscr{G}_1$ is of order q, $(p, q) = 1$.

Now by a theorem due to P. Hall [5] $\mathscr{G}$ contains a group $\mathscr{H}$ of order t, where $pt = g$, the order of $\mathscr{G}$. If $\mathscr{P}$ is a Sylow p-group $\mathscr{G}$ of form $\mathfrak{J}$, the left ideal of $\mathfrak{A}(\mathscr{G})$ generated by the radical of $\mathfrak{A}(\mathscr{P})$. Then $\mathfrak{A}(\mathscr{G}) - \mathfrak{J} = \mathfrak{Q}$ is a left $\mathscr{G}$-module representable by $\mathfrak{A}(\mathscr{H})$ and is a completely reducible $\mathfrak{A}(\mathscr{G}_1)$-module. For $\mathfrak{R}_1$, the radical of $\mathfrak{A}(\mathscr{G}_1)$, is such that $\mathfrak{R}_1\mathfrak{A}(\mathscr{G})$ is contained in $\mathfrak{J}$ and so $\mathfrak{R}_1\mathfrak{Q} = 0$. So let $\mathfrak{Q}_1$ be an irreducible left $\mathscr{G}$-submodule of $\mathfrak{Q}$. Then $\mathfrak{Q}$ may be written $\mathfrak{Q} = \mathfrak{Q}_1 + \mathfrak{Q}_2$ where $\mathfrak{Q}_2$ is a left $\mathfrak{A}(\mathscr{G}_1)$-module and $\mathfrak{Q}_1 \cap \mathfrak{Q}_2 = 0$. Therefore a projection T of $\mathfrak{Q}$ onto $\mathfrak{Q}_2$ exists such that T annihilates the elements of $\mathfrak{Q}_1$ and is the identity operator on $\mathfrak{Q}_2$ and such that T commutes with (the representations of) the elements of $\mathfrak{A}(\mathscr{G}_1)$. Now form the projection $T' = t^{-1} \sum H_i T H_i^{-1}$, summed over the t elements of $\mathscr{H}$. Then T' commutes with all the elements of $\mathscr{G}$ and hence the submodule $\mathfrak{Q}_1' = T'\mathfrak{Q}$ of $\mathfrak{Q}$ is a left $\mathfrak{A}(\mathscr{G})$-module. Furthermore $\mathfrak{Q} = \mathfrak{Q}_1 + \mathfrak{Q}_1'$ where $\mathfrak{Q}_1 \cap \mathfrak{Q}_1' = 0$. Thus $\mathfrak{Q}$ is a completely reducible left $\mathfrak{A}(\mathscr{G})$-module and so $\mathfrak{J}$ contains the radical of $\mathfrak{A}(\mathscr{G})$. This proves Theorem 3.

4. A related problem. An algebra having the property that only elements of the radical can be both left and right annihilators of the radical has been termed a *bound algebra* by M. Hall [3].

THEOREM 4. *If the group $\mathscr{G}$ contains an invariant p-subgroup $\mathscr{P}$, then the group algebra $\mathfrak{A}(\mathscr{G})$ of $\mathscr{G}$ over a field of characteristic p is a bound algebra.*

If $\mathscr{P}$ is of order $p^a = x$ and of index y, then the radical of $\mathfrak{A}(\mathscr{P})$ generates a nilpotent ideal $\mathfrak{J}$ of $\mathfrak{A}(\mathscr{G})$ of dimension $y(x-1)$. Now the element $P_1 + \cdots + P_x$, where P_i is in $\mathscr{P}$, annihilates $\mathfrak{J}$ and is also in the center of $\mathfrak{A}(\mathscr{G})$. Hence it generates an ideal J of order y which is contained in $\mathfrak{J}$ and $\mathfrak{J}J = J\mathfrak{J} = 0$. Since $\mathfrak{A}(\mathscr{G})$ is a Frobenius algebra, a result due to Nakayama [10] states that the set of all right annihilators of $\mathfrak{J}$ in $\mathfrak{A}(\mathscr{G})$ forms an ideal of dimension y. Hence $\mathfrak{J}$ contains all of the right annihilators of $\mathfrak{J}$. Since $\mathfrak{J} \subseteqq \mathfrak{R}$, $\mathfrak{J}$ contains the right annihilators of $\mathfrak{R}$, and so $\mathfrak{A}(\mathscr{G})$ is bound to $\mathfrak{R}$.

This raises the question: If $\mathfrak{A}(\mathscr{G})$ is bound to its radical $\mathfrak{R} \neq 0$, does $\mathscr{G}$ contain an invariant p-subgroup? A partial answer is provided by

THEOREM 5. *If the Sylow p-groups of $\mathscr{G}$ are cyclic and if the* *radical* $\mathfrak{R}$ *of* $\mathfrak{A}(\mathscr{G})$ *equals* $\mathfrak{R}'$ *then* $\mathscr{G}$ *contains an invariant p-subgroup if* $\mathfrak{A}(\mathscr{G})$ *is bound to* $\mathfrak{R}$.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two Sylow p-groups of $\mathscr{G}$ and let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be the two left ideals of $\mathfrak{A}(\mathscr{G})$ generated by the radicals of $\mathfrak{A}(\mathscr{P}_1)$ and $\mathfrak{A}(\mathscr{P}_2)$ respectively. Denote by $r(\mathfrak{J}_1)$ and $r(\mathfrak{J}_2)$ the right ideals of $\mathfrak{A}(\mathscr{G})$ consisting of all elements which annihilate $\mathfrak{J}_1$ and $\mathfrak{J}_2$, respectively, on the right. Then since $\mathfrak{R} \subseteqq \bigcap \mathfrak{J}_i$ and since $r(\mathfrak{R}) \subseteqq \mathfrak{R}$ it follows readily that $r(\mathfrak{J}_1)$ and $r(\mathfrak{J}_2)$ are contained in $\mathfrak{R} = \mathfrak{R}'$. In particular, the sum S of the elements of $\mathscr{P}_1$ is contained in $\mathfrak{J}_2$. Now the only elements of $\mathfrak{J}_2$ which involve 1, the identity of $\mathscr{G}$, also involve other elements of $\mathscr{P}_2$, so that the belonging of S to $\mathfrak{J}_2$ implies that $\mathscr{P}_1 \cap \mathscr{P}_2$ is a group containing more than one element. Then, since the $\mathscr{P}_i$ are all cyclic, it follows readily that the p-subgroup $\mathscr{P}_1 \cap \mathscr{P}_2$ is normal in $\mathscr{G}$.

Now $\mathfrak{A}(\mathscr{G})$ is bound to $\mathfrak{R}$ if and only if $\mathscr{G}$ possesses no representation of highest kind (see [1]). If $\mathscr{G}$ is S_5, the symmetric group of order 120 and if $p = 2$, then the table of ordinary characters readily demonstrates that $\mathscr{G}$ has no representation of highest kind. Yet S_5 has no invariant 2-subgroup. It may be noteworthy that this example is related to the one given by Lombardo-Radici [9] to show that $\mathfrak{R}$ is not always equal to $\mathfrak{R}'$.

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