DIRICHLET MULTIPLICATION IN LATTICE POINT PROBLEMS. II

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1. The author [8]* has given a theorem in which we assume two functions A and B have asymptotic formulae of the form

$$(1) \qquad A(x) = \sum_{\mu=1}^{h} x^{\alpha_\mu} P_\mu(\log x) + O\{x^\alpha \log^l(x+1)\}$$

and upper estimates $O\{x^\rho \log^m(x+1)\}$ on their total variations. We then conclude that their Stieltjes resultant C satisfies a formula similar to (1). The α_μ are complex numbers, the P_μ are polynomial functions, and we give an explicit formula for the error term in the resultant in terms of the given parameters.

In this paper we shall give a generalization of the above-mentioned result which will cover a wider class of lattice point problems.

2. Given two functions A and B defined for $x \geqq 1$, of bounded variation on each bounded interval, we call the Stieltjes resultant of A by B any function C such that

$$(1) \qquad C(x) = \int_1^x A(x/u)dB(u)$$

wherever the integral exists and for all x either.

$$(2) \qquad \lim_{h\to 0+} C(x-h) \leqq C(x) \leqq \lim_{h\to 0+} C(x+h)$$

or

$$(3) \qquad \lim_{h\to 0+} C(x+h) \leqq C(x) \leqq \lim_{h\to 0+} C(x-h) .$$

Note that there are at most countably many x for which the integral (1) does not exist, namely those $x = ab$ where a is a discontinuity of A and b is a discontinuity of B. Note further that if $A(1) = B(1) = 0$ then the Stieltjes resultant is a commutative binary operation:

$$(4) \qquad \int_1^x A(x/u)dB(u) = \int_1^x B(x/u)dA(u) .$$

Widder [9] gives a slightly more restrictive definition of Stieltjes resultant, however, his requirement that A, B, and C be "normalized" is unnecessary.

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* Note that in the proof of [8] Theorem 2, no use was made of the assumption α, β, ρ, τ are non-negative. Thus that assumption may be deleted from the theorem.

We shall denote the total variation of a function A on $(1, x)$ by $V_A(x)$.

3. In place of the functions $P_\mu(\log x)$ in the asymptotic formulae we shall use the more general slowly oscillating functions. By "slowly oscillating function" we mean a function L which is positive-valued and continuous on $x \geqq x_0$, for some $x_0 \geqq 0$, such that

$$(1) \qquad \lim_{x\to\infty} L(cx)/L(x) = 1$$

for every $c > 0$. Karamata [4] has characterized such functions in the form

$$(2) \qquad L(x) = a\rho(x)e^{\int_{x_0}^{x} t^{-1}\delta(t)dt}$$

where $a > 0$, $\rho(x) \to 1$, $\delta(x) \to 0$ as $x \to \infty$; ρ and δ being continuous on $x \geqq x_0$. (See also a proof by Korevaar et al [5]).

From (2) it follows that the limit in (1) is uniform in c on each bounded interval

$$(3) \qquad 0 < k_1 \leqq c \leqq k_2 .$$

Korevaar et al actually proved the uniformity first then used this property in proving the characterization (2).

Now a wide class of slowly oscillating functions can be written in the special form

$$(4) \qquad L(x) = ae^{\int_1^x t^{-1}\delta(t)dt} .$$

We shall give such functions the name "special slowly oscillating function". However, we require in (4) only that $a > 0$ and $\delta(x) \to 0$ as $x \to \infty$, where δ is bounded and Lebesgue integrable on each bounded interval. The function δ need not be continuous.

In order to maintain smoothness and to get a definite result we find it necessary—and quite reasonable—to deal mainly with special slowly oscillating functions.

For applications see Bateman [1], [2] and Bateman and Grosswald [3].

4. THEOREM. *Suppose A and B are complex valued function on $(1, \infty)$ of bounded variation on each bounded interval and with $A(1) = B(1) = 0$. Suppose*

$$(1) \qquad R(x) = \sum_{\mu=1}^{m} c_\mu x^{\alpha_\mu} L_\mu(x)$$

and

$$S(x) = \sum_{\nu=1}^{n} d_\nu x^{\beta_\nu} M_\nu(x)$$

with

$$V_R(x) = O\{V_A(x)\}, \ V_S(x) = O\{V_B(x)\} \tag{2}$$

where the $c_\mu, \alpha_\mu, d_\nu, \beta_\nu$ *are complex numbers such that if* $\Re(\alpha_\mu) = \Re(\beta_\nu)$ *then* $\alpha_\mu = \beta_\nu$, *and the* L_μ *and* M_ν *are special slowly oscillating functions.*

We assume R *and* S *so chosen that there exist* α, β *with* $\alpha \leqq \Re(\alpha_\mu)$ *for each* $\mu, \beta \leqq \Re(\beta_\nu)$ *for each* ν, *and there exist slowly oscillating functions* L_0, M_0 *such that*

$$\begin{aligned} \Delta_1(x) &= A(x) - R(x) = O\{x^\alpha L_0(x)\} , \\ \Delta_2(x) &= B(x) - S(x) = O\{x^\beta M_0(x)\} . \end{aligned} \tag{3}$$

Further, we assume that there exist $\rho \geqq \alpha, \tau \geqq \beta$ *and slowly oscillating functions* P_0 *and* Q_0 *such that*

$$V_A(x) = O\{x^\rho P_0(x)\}, \ V_B(x) = O\{x^\tau Q_0(x)\} . \tag{4}$$

If C *is the Stieltjes resultant of* A *and* B *then there is a function* T *of the form* (1) *such that if* $\rho > \beta$ *and* $\tau > \alpha, 1 \leqq y \leqq x, z = x/y$,

$$\begin{aligned} C(x) = {} & T(x) + O\{x^\alpha L_0(x)\} + O\{x^\beta M_0(x)\} \\ & + O\Big\{x^\alpha \sum_{\Re(\beta_\nu)=\alpha}{}_\nu \int_1^x u^{-1} M_\nu(u) L_0(x/u) du\Big\} + O\Big\{x^\beta \sum_{\Re(\alpha_\mu)=\beta}{}_\mu u^{-1} L_\mu(u) M_0(x/u) du\Big\} \\ & + O\{z^\alpha y^\tau L_0(z) Q_0(y)\} + O\{z^\rho y^\beta M_0(y) P_0(z)\} , \end{aligned} \tag{5}$$

the O-constants being independent of x, y *and* z. *More specifically,*

$$\begin{aligned} T(x) = {} & \int_1^x R(x/u) dS(u) + \sum_{\Re(\beta_\nu)>\alpha}{}_\nu d_\nu x^{\beta_\nu} N_\nu(x) \\ & + \sum_{\Re(\alpha_\mu)>\beta}{}_\mu c_\mu x^{\alpha_\mu} N'_\mu(x) - \Delta_2(1) \sum_{\Re(\alpha_\mu)>\beta} c_\mu x^{\alpha_\mu} L_\mu(x) \end{aligned} \tag{6}$$

where

$$\int_1^x R(x/u) dS(u) = \sum_{\mu,\nu} c_\mu d_\nu \int_1^x (x/u)^{\alpha_\mu} L_\mu(x/u) d(u^{\beta_\nu} M_\nu(u))$$

is a complex linear combination of functions

$$x^{\gamma_{\mu\nu}} L_{\mu\nu}(x)$$

with $\gamma_{\mu\nu} = \alpha_\mu$ *or* β_ν, *whichever has the greater real part,* $L_{\mu\nu}$ *a slowly oscillating function; and*

$$N_\nu(x) = x^{-\beta_\nu} \int_1^x \Delta_1(x/u) d(u^{\beta_\nu} M_\nu(u))$$

$$N'_\mu(x) = x^{-\alpha_\mu} \int_1^x \Delta_2(x/u) d(u^{\alpha_\mu} L_\mu(u))$$

are linear combinations of slowly oscillating functions. Further, the functions $L_{\mu\nu}$, N'_{μ}, N_{ν} are absolutely continuous on each bounded interval $(1, x)$.

If $\tau = \alpha$ the error is then

$$O\{x^{\beta}M_0(x)\} + O\left\{x^{\tau}\int_1^x u^{-1}Q_0(u)L_0(x/u)du\right\} \tag{7}$$
$$+ O\left\{x^{\beta}\sum_{\Re(\alpha_\mu)=\beta}{}_{\mu}\int_1^x u^{-1}L_{\mu}(u)M_0(x/u)du\right\}$$

the integrals in the error terms being absolutely continuous slowly ocsillating functions.

If $\alpha > \tau$ we have simply

$$C(x) = T(x) + O\{x^{\alpha}L_0(x)\} \tag{8}$$

$T(x)$ being, in these latter two cases, the same as in (6) *excepting that the term*

$$\sum_{\Re(\beta_\nu)>\alpha}{}_{\nu} d_{\nu}x^{\sigma_\nu}N_{\nu}(x)$$

is absorbed in the error term. The cases $\rho \leqq \beta$ are similar.

Finally,

$$V_C(x) = O\{x^{\max(\rho,\tau)}S_0(x)\} \tag{9}$$

with

$$S_0(x) = \begin{cases} P_0(x)\ , & if\ \rho > \tau\ , \\ Q_0(x)\ , & if\ \rho < \tau\ , \\ \int_1^x u^{-1}P_0(u)Q_0(x/u)du, & if\ \rho = \tau\ . \end{cases}$$

Note that if there are different α_μ, β_ν with equal real parts, then the theorem still holds with the exception that the resulting integral

$$\int_1^x (x/u)^{\alpha_\mu}L_{\mu}(x/u)d(u^{\beta_\nu}M_{\nu}(u))$$

in $T(x)$ is not necessarily a combination of powers of x times slowly oscillating functions.

5. It suffices to consider only those values of $x > 1$ for which the integral defining the Stieltjes resultant converges. For any such x,

$$C(x) = \int_1^x A(x/u)dB(u) = I_1 + I_2 + I_3 + I_4\ ,$$

where

$$I_1 = \int_1^x R(x/u)dS(u) ,$$
$$I_2 = \int_1^x \Delta_1(x/u)dS(u) ,$$
$$I_3 = \int_1^x R(x/u)d\Delta_2(u) ,$$
$$I_4 = \int_1^x \Delta_1(x/u)d\Delta_2(u) .$$

The proof now consists of demonstrating several lemmas which amount to the fact that if I_1 and I_2 are split up suitably, as indicated in formula (4.6), then the right hand side of (4.6) is a sum of functions $ax^{\gamma}N(x)$ where N is an absolutely continuous slowly oscillating function, and the remaining terms are of the order of magnitude stated in (4.5), (4.7), or (4.8) as the case may be. We shall give here the two lemmas used to demonstrate the estimates on the error terms given in (4.5, 7, 8, 9). The remainder of the proof then involves juggling terms by means of integration by parts and considering various cases according to whether

$$\Re(\beta_\nu) >, =, \text{ or } > \alpha \text{ and } \Re(\alpha_\mu) >, =, < \beta .$$

LEMMA 1. *If L and M are slowly oscillating functions (on $x \geqq 1$) and $\gamma > 0$, then for $1 \leqq y \leqq x$, $q \geqq 1$,*

$$\int_q^y u^{\gamma-1} M(u)L(x/u)du = O\{y^{\gamma}M(y)L(x/y)\} , \tag{1}$$

the O-constant depending only on L, M, and γ.

Since L and M are positive functions we may assume $q = 1$. Now if

$$\begin{aligned} L(x) &= a\rho(x) \exp \int_1^x t^{-1}\delta(t)dt \\ M(x) &= b\tau(x) \exp \int_1^x t^{-1}\varepsilon(t)dt \end{aligned} \tag{2}$$

where $\rho(x) \to 1$, $\tau(x) \to 1$, $\delta(x) \to 0$, $\varepsilon(x) \to 0$, as $x \to \infty$, $a > 0$, $b > 0$, then let

$$J(x) = L(x)/\rho(x) , \quad K(x) = M(x)/\tau(x)$$

so that

$$L(x) \sim J(x) , \quad M(x) \sim K(x)$$

as $x \to \infty$.

Without loss of generality, we may assume that for all $t \geqq 1$,

$$|\delta(t)| \leqq \gamma/4 , \quad |\varepsilon(t)| \leqq \gamma/4 \tag{3}$$

for otherwise, the excess, which occurs only on a bounded interval, can be obsorbed into the coefficient functions $a\rho(x)$ and $b\tau(x)$.

Now clearly

$$\int_1^y u^{\gamma-1} M(u)L(x/u)du = O\left\{\int_1^y u^{\gamma-1} K(u)J(x/u)du\right\} .$$

Upon integrating by parts we have

$$\begin{aligned}\int_1^y & u^{\gamma-1} K(u)J(x/u)du \\ &= \frac{1}{\gamma}\{y^\gamma K(y)J(x/y) - K(1)J(x)\} \\ &\quad - \frac{1}{\gamma}\int_1^y u^{\gamma-1} K(u)\varepsilon(u)J(x/u)du \\ &\quad + \frac{1}{\gamma}\int_1^y u^{\gamma-1} K(u)J(x/u)\delta(x/u)du \\ &\leqq \frac{1}{\gamma}\{y^\gamma K(y)J(x/y) - K(1)J(x)\} \\ &\quad + \frac{1}{2}\int_1^y u^{\gamma-1} K(u)J(x/u)du\end{aligned}$$

by (3). Thus

$$\begin{aligned}\int_1^y u^{\gamma-1} K(u)J(x/u)du &\leqq \frac{2}{\gamma}\{y^\gamma K(y)J(x/y) - K(1)J(x)\} \\ &\leqq \frac{2}{\gamma} y^\gamma K(y)J(x/y) = O\{y^\gamma M(y)L(x/y)\} .\end{aligned}$$

This completes the proof.

With the aid of lemma 1 and some simple inequalities one can readily prove the following.

LEMMA 2. *If A and B are functions of bounded variation on* $(1, x)$ *for each $x > 1$ and if for $x \geqq 1$,*

$$A(x) = O(x^\alpha L(x)) , \quad V_B(x) = O(x^\tau Q(x)) ,$$

where L and Q are slowly oscillating functions on $x \geqq 1$ and α and τ are real numbers, then for $2 \leqq y \leqq x, z = x/y$, if x is not a product of a discontinuity of A by a discontinuity of B

$$\int_1^y A(x/u)dB(u) = \begin{cases} O\{z^\alpha y^\tau L(z)Q(y)\} & (\tau > \alpha) \\ O\left\{x^\alpha \int_1^y u^{-1} Q(u)L(x/u)du\right\} & (\tau = \alpha) \\ O\{z^\alpha y^\tau L(z)Q(y)\} + O\{x^\alpha L(x)\} & (\tau < \alpha) \end{cases} \tag{4}$$

uniformly in x and y

6. It is fairly easy to show that Theorem 2 of [8] is a corollary to the theorem of the present paper.

As was mentioned in paper [8] the author has given an example [7] which shows that Landau's theorem [6] on the multiplication of Dirichlet series gives the best possible result. Further, we showed that the same example applies to the result of [8] so that the exponent of x in the error term is as a general theorem best possible if we restrict the main terms to the form in (1.1). This gives some indication of the strength of our result, however, we are as yet unable to decide whether the result with the more general main terms of the present paper is improvable.

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