A MAXIMAL PROBLEM IN HARMONIC ANALYSIS, II

I. I. Hirschman, Jr.

1. Introduction. Let G be a compact topological group with elements x, x_0, etc. We denote by dx the Haar measure of G normalized by the condition that the measure of G is 1. Let the matrices

$$(1) \qquad [g(\alpha, i, j, x)]_{i,j=1}^{r(\alpha)} \qquad\qquad \alpha \in A$$

be a complete set[1] of inequivalent unitary representations of G. We recall that this implies that[2]

$$\int_G g(\alpha, i, j, x) g(\beta, k, l, x)^* dx = \frac{\delta(\alpha, i,j; \beta, k, l)}{r(\alpha)} .$$

Here $\delta(\alpha, i, j; \beta, k, l)$ is 1 if $\alpha = \beta, i = k$ and $j = l$; otherwise it is zero. Further if $f(x) \in L^2(G)$ and if

$$c(\alpha, i, j, f) = \int_G f(x) g(\alpha, i, j, x)^* dx ,$$

then

$$(2) \qquad \left\{ \sum_\alpha r(\alpha) \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2 \right\}^{1/2} = \|f\|_2 .$$

Let $1 < p \leqq 2, 1/p + 1/q = 1$. The object of the present paper is to demonstrate the inequalities

$$(3') \qquad \left\{ \sum_\alpha r(\alpha)^{2-q/2} \left[\sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2 \right]^{q/2} \right\}^{1/q} \leqq \|f\|_p ,$$

$$(3'') \qquad \left\{ \sum_\alpha r(\alpha)^{2-p/2} \left[\sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2 \right]^{p/2} \right\}^{1/p} \geqq \|f\|_q ,$$

and to determine for $p \neq 2$ all cases in which equality occurs. (If $p = q = 2$ then $(3')$ and $(3'')$ reduce to (2) and equality holds for every f). The inequalities $(3')$ and $(3'')$ are an extension to compact groups of the Young-Hausdorff-Riesz inequalities for Fourier series. The corresponding problem for locally compact Abelian groups has been discussed by E. Hewitt and the author in [2], and the present paper may be considered as a continuation of [2]. Closely related results are also contained in a paper of A. Calderón and A. Zygmund [1].

Note that the $r(\alpha) \times r(\alpha)$ matrix $[g(\alpha, i, j, x)]$ is not uniquely

[1] For the definitions of the group theoretic terms used here see [3].

[2] If γ is a complex number then γ^* denotes its conjugate.

specified. Indeed if $[t(i, j)]$ is any $r(\alpha) \times r(\alpha)$ unitary matrix then $[g(\alpha, i, j. x)]$ may be replaced by

$$[g_1(\alpha, i, j, x)] = [t(i, j)][g(\alpha, i, j, x)][t(i, j)]^{-1} .$$

Let

$$c_1(\alpha, i, j, f) = \int_G f(x)g_1(\alpha, i, j, x)^* dx .$$

We assert that

$$\sum_{i,j=1}^{r(\alpha)} |c_1(\alpha, i, j, f)|^2 = \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2 ;$$

that is the left hand side of (3') and (3'') remains unchanged if some representations are replaced by equivalent representations. This is an immediate consequence of the easily verified matrix equation

$$[c_1(\alpha, i, j, f)] = [t(i, j)][c(\alpha, i, j, f)][t(i, j)]^{-1} .$$

This property, which is clearly essential if the extremal functions are to have group theoretic significance, explains the appearance of the terms

$$\sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2$$

in the inequalities (3') and (3'').

2. The inequalities.
We begin by demonstrating the inequality (3') of § 1. Essentially this result is a consequence of the Riesz-Thorin convexity theorem. However we will give the demonstration in full, first because the proof differs slightly from the proofs of the previously known special cases, and secondly because we shall need the apparatus of the proof in order later to identify the extremal functions.

THEOREM 2a. *If* $f(x) \in L^p(G) \, 1 < p \leq 2$ *then the inequality* (3') *of* § 1 *holds.*

Let sgn $re^{i\theta} = e^{i\theta}$ if $r \neq 0$ and let sgn $0 = 0$. Let w be a complex number. We define[3]

$$T_1(w)f(x) = |f(x)|^{(p/2)(1+w)} \operatorname{sgn} f(x)$$

$$c(\alpha, i, j, T_1(w)f) = \int_G [T_1(w)f(x)]g(\alpha, i, j, x)^* dx .$$

Let

$$s(\alpha, f) = \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, f)|^2 ,$$

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[3] If $f(x) \neq 0$ we define $|f(x)|^{(p/2)(1+w)}$ to be $\exp\left[\log|f(x)|\frac{1}{2}p(1+w)\right]$ where $\log|f(x)|$ is taken as real.

$$C(\alpha, i, j, f) = [s(\alpha, f)/r(\alpha)]^{(q-1)/2} \cdot c(\alpha, i, j, f)^* \ ,$$

$$S(\alpha, f) = \sum_{i,j=1}^{r(\alpha)} | \, C\,(\alpha, i, j, f)|^2 = s(\alpha, f)^{q-1} \cdot r(\alpha)^{2-q} \ .$$

We further define

$$T_2(w)C(\alpha, i, j, f) = [r(\alpha)/S(\alpha, f)]^{1/2-1/4p(w+1)} \cdot C(\alpha, i, j, f) \ ,$$

if $s(\alpha, f) \neq 0$. If $s(\alpha, f) = 0$ we set $T_2(w)C(\alpha, i, j, f) = 0$. Suppose first that f is a finite simple function; that is, there exist disjoint measurable sets $J_1, \cdots, J_m$ in G such that $f(x) = a_k \neq 0$ for $x \in J_k$ $k = 1, \cdots, m$, while $f(x) = 0$ for $x \in G - (J_1 \cup J_2 \cup \cdots \cup J_m)$. Then

$$c(\alpha, i, j, T_1(w)f) = \sum_{k=1}^{m} | \, a_k \, |^{(p/2)(1+w)} \operatorname{sgn} \alpha_k \int_{J_k} g(\alpha, i, j, x)^* dx \ .$$

Let A_0 be a finite subset of A and define

$$\mathbf{F}_1(w, A_0) = \sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [c(\alpha, i, j, T_1(w)f)][T_2(w)C(\alpha, i, j, f] \ .$$

Clearly $\mathbf{F}_1(w, A_0)$ is an exponential polynomial with real exponents; that is $\mathbf{F}_1(w, A_0)$ is of the form $\sum_{\nu=1}^{N} \Phi_\nu e^{\varphi_\nu w}$ with the φ_ν real. Thus $\mathbf{F}_1(w, A_0)$ is an entire function of w bounded in every vertical strip $u_1 \leq \Re w \leq u_2$. By Schwarz's inequality

$$| \, \mathbf{F}_1(w, A_0)| \leq \left(\sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} | \, c(\alpha, i, j, T_1(w)f)^2 \right)^{1/2}$$

$$\left(\sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} | \, T_2(w)C(\alpha, i, j, f)|^2 \right)^{1/2} \ .$$

Suppose $\Re w = 0$; that is $w = 0 + iv$. Then we have

$$| \, T_1(iv)f(x)| = | f(x)|^{(1/2) \, p} \ .$$

By Bessel's inequality

$$\sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} | \, c(\alpha, i, j, T_1(iv)f)|^2 \leq \int_G | \, T_1(iv)f(x)|^2 dx = \| f \|_p^p \ .$$

On the other hand

$$| \, T_2(iv)C(\alpha, i, j, f)| = | \, C(\alpha, i, j, f)|[r(\alpha)/S(\alpha, f)]^{\frac{1}{2} - (p/4)}$$

and hence

$$\sum_{i,j=1}^{r(\alpha)} | \, T_2(iv)C(\alpha, i, j, f)|^2 = [r(\alpha)/S(\alpha, f)]^{1-p/2} \sum_{i,j=1}^{r(\alpha)} | \, C(\alpha, i, j, f)|^2$$

$$= r(\alpha)^{1-p/2}S(\alpha, f)^{p/2} = r(\alpha)^{1-q/2}s(\alpha, f)^{q/2} \ .$$

Thus

$$| \mathbf{F}_1(iv, A_0)| \leqq \|f\|_p^{p/2} \Big[\sum_{\alpha \in A_0} r(\alpha)^{2-q/2} s(\alpha, f)^{q/2} \Big]^{1/2} \ .$$

Next suppose $\Re w = 1$, that is $w = 1 + iv$. Then we have

$$| T_1(1 + iv)f(x))| = |f(x)|^p \ .$$

Now

$$c(\alpha, i, j, T_1(1 + iv)f) = \int_G [T_1(1 + iv)f(x)]g(\alpha, i, j, x)^* dx \ ,$$

$$| c(\alpha, i, j, T_1(1 + iv)f)|^2$$
$$\leqq \int_G | T_1(1 + iv)f(x)| \, | g(\alpha, i, j, x)|^2 dx \int_G | T_1(1+iv)f(x)| \, dx \ ,$$

and thus

$$\sum_{i,j=1}^{r(\alpha)} | c(\alpha, i, j, T_1(1 + iv)f)|^2 \leqq \|f\|_p^p \int_G |f(x)|^p \sum_{i,j=1}^{r(\alpha)} | g(\alpha, i, j, x)|^2 dx \ .$$

Since

$$\sum_{i,j=1}^{r(\alpha)} | g(\alpha, i, j, x)|^2 = r(\alpha)$$

this implies that

$$\sum_{i,j=1}^{r(\alpha)} | c(\alpha, i, j, T_1(1 + iv, f)|^2 \leqq \|f\|_p^{2p} r(\alpha) \ .$$

On the other hand we have

$$| T_2(1 + iv)C(\alpha, i, j, f)| = r(\alpha)/S(\alpha, f)]^{1/2-p/2} | C(\alpha, i, j, f)| \ ,$$

$$\sum_{i,j=1}^{r(\alpha)} | T_2(1 + iv)C(\alpha, i, j f)|^2 = r(\alpha)^{1-p} S(\alpha, f)^p = r(\alpha)^{1-q} s(\alpha, f)^q \ .$$

By Schwarz's inequality

$$| \mathbf{F}_1(w, A_0)| \leqq \sum_{\alpha \in A_0} \Big\{ r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, T_1(w)f|^2 \Big\}^{1/2}$$
$$\times \Big\{ r(\alpha)^3 \sum_{i,j=1}^{r(\alpha)} | T_2(w)C(\alpha, i, j, f)|^2 \Big\}^{1/2} \ ,$$

and from this it follows that

$$\mathbf{F}_1(w, A_0)| \leqq \Big\{ \underset{\alpha \in A_0}{\text{l.u.b.}} \ r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, T_1(w)f)|^2 \Big\}^{1/2}$$
$$\times \sum_{\alpha \in A_0} \Big\{ r(\alpha)^3 \sum_{i,j=1}^{r(\alpha)} | T_2(w)C(\alpha, i, j, f)|^2 \Big\}^{1/2} \ .$$

Using the above estimates we obtain

$$|\mathbf{F}_1(1 + iv, A_0)| \leqq \|f\|_p^p \Big\{ \sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2} \Big\} .$$

Finally since

$$T_1\Big(\frac{2}{p} - 1\Big) f(x) = f(x), \quad T_2\Big(\frac{2}{p} - 1\Big) C(\alpha, i, j, f) = C(\alpha, i, j, f)$$

we have

$$\mathbf{F}_1\Big(\frac{2}{p} - 1, A_0\Big) = \sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} c(\alpha, i, j, f) C(\alpha, i, j, f) ,$$

$$= \sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2}$$

By the three lines theorem, see [1 p. 169],

$$\Big| \mathbf{F}_1\Big(\frac{2}{p} - 1, A_0\Big) \Big| \leqq \Big[\operatorname*{l.u.b.}_{v} |\mathbf{F}_1(iv, A_0)| \Big]^{2-2/p} \Big[\operatorname*{l.u.b.}_{v} |\mathbf{F}_1(1 + iv, A_0)| \Big]^{2/p-1} .$$

Thus if

$$x = \Big\{ \sum_{\alpha \in A_0} r(\alpha)^{2-q/2} s(\alpha, f)^{q/2} \Big\}^{1/q}$$

we find that

$$x^q \leqq \{ \|f\|_p^{p/2} x^{q/2} \}^{2-2/p} \{ \|f\|_p^p x^q \}^{2/p-1}$$

which implies that $x \leqq \|f\|_p$. This has been established under the assumption that f is a simple function ; however this restriction can be removed as follows. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of simple functions such that $\|f - f_n\|_p \to 0$ as $n \to \infty$. We have

$$\Big\{ \sum_{\alpha \in A_0} r(\alpha)^{2-q/2} s(\alpha, f_n)^{(1/2)q} \Big\}^{1/q} \leqq \|f_n\|_p \qquad (n = 1, 2, \cdots) .$$

Since $c(\alpha, i, j, f_n) \to c(\alpha, i, j, f)$ as $n \to \infty$ we obtain in the limit

$$\Big\{ \sum_{\alpha \in A_0} r(\alpha)^{2-q/2} s(\alpha, f)^{(1/2)q} \Big\}^{1/q} \leqq \|f\|_p .$$

Finally, A_0 being an arbitrary finite subset of A this implies the validity of (3') of § 1.

The second of our two inequalities can be deduced from the first by a familiar duality argument.

THEOREM 2b.   *If $f(x) \in L^q(G) 2 \leqq q < \infty$ then (3'') of § 1 holds.*

Let $\mathbf{L}^p$ be the class of functions $g(x)$ for which $c(\alpha, i, j, g)$ is different from 0 for only a finite number of indices and for which $\| g(x) \|_p \leqq 1$. We have if $g \in \mathbf{L}^p$

$$\int_G f(x) g(x)^* dx = \sum_{\alpha \in A} r(\alpha) \sum_{i,j=1}^{r(\alpha)} c(\alpha, i, j, f) c(\alpha, i, j, g)^* ,$$

and hence

$$\left| \int_G f(x) g(x)^* dx \right| \leqq \left\{ \sum_{\alpha \in A} r(\alpha)^{2-p/2} s(\alpha, f)^{p/2} \right\}^{1/p} \left\{ \sum_{\alpha \in A} r(\alpha)^{2-q/2} s(\alpha, g)^{q/2} \right\}^{1/q} ,$$

$$\leqq \left\{ \sum_{\alpha \in A} r(\alpha)^{2-p/2} s(\alpha, f)^{p/2} \right\}^{1/p} ,$$

since by Theorem 2a

$$\left\{ \sum_{\alpha \in A} r(\alpha)^{2-q/2} s(\alpha, g)^{q/2} \right\}^{1/q} \leqq \| g \|_p \leqq 1 .$$

Since $\mathbf{L}^p$ is dense in the unit ball of $L^p(G)$

$$\| f \|_q = \underset{g \in \mathbf{L}^P}{\mathrm{l.u.b.}} \left| \int_G f(x) g(x)^* \right| dx ;$$

it follows that

$$\| f \|_q \leqq \left\{ \sum_{\alpha \in A} r(\alpha)^{2-p/2} s(\alpha, f)^{p/2} \right\}^{1/p}$$

as desired.

It should be noted that the inequality is valid (3′) for $p = 1$ in the form

$$\mathrm{l.u.b.}_{\alpha} \left[r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} | c(\alpha, i, j, f)|^2 \right]^{1/2} \leqq \| f \|_1 ,$$

the (elementary) proof of this result being one of the steps in the demonstration of Theorem 2a. Similarly (3″) is valid for $q = \infty$ in the form

$$\| f \|_\infty \leqq \sum_{\alpha} r(\alpha)^{3/2} \left[\sum_{i,j=1}^{r(\alpha)} | c(\alpha, i, j, f)|^2 \right]^{1/2}$$

Here $\| f \|_\infty = \mathrm{ess. \ sup.} \ | f(x)|$.

3. An order relation. If the group G is commutative then the following result reduces to the Riemann-Lebesgue lemma.

THEOREM 3.   *If* $f(x) \in L^1(G)$ *then* $s(\alpha, f) = o[r(\alpha)]$.

This means, of course, that given any $\delta > 0$ the set of values α for which $s(\alpha, f) > \delta r(\alpha)$ is finite. We can write $f(x) = f_1(x) + f_2(x)$

where $f_2(x) \in L^2(G)$ and where $\| f_1 \|_1 \leqq \varepsilon$, ε being any positive number given in advance. A familiar inequality implies that $s(\alpha, f) \leqq 2s(\alpha, f_1) + 2s(\alpha, f_2)$. Since

$$\sum_{\alpha \in A} r(\alpha) s(\alpha, f_2) = \| f_2 \|_2^2$$

it follows that $s(\alpha, f_2) = o[r(\alpha)]^{-1}$ and thus a fortiori

$$s(\alpha, f_2) = o[r(\alpha)] \ .$$

On the other hand

$$c(\alpha, i, j, f_1) = \int_G f_1(x) g(\alpha, i, j, x)^* dx \ ,$$

$$| c(\alpha, i, j, f_1) |^2 \leqq \int_G | f_1(x) | | g(\alpha, i, j, x) |^2 dx \int_G | f_1(x) | \, dx \ ,$$

$$s(\alpha, f_1) \leqq \| f_1 \|_1 \int_G | f_1(x) | \sum_{i,j=1}^{r(\alpha)} | g(\alpha, i, j, x) |^2 \leqq \| f_1 \|_1^2 \, r(\alpha) \leqq \varepsilon^2 r(\alpha) \ .$$

Thus

$$s(\alpha, f) \leqq o[r(\alpha)] + 2\varepsilon^2 r(\alpha) \ .$$

4. A duality property. A function $f(x) \in L^p(G)$ $(1 < p < 2)$ is said to be maximal in $L^p(G)$ if equality obtains in the inequality (3′) of §1; a function $f(x) \in L^q(G)$ $(2 < q < \infty)$ is said to be maximal in $L^q(G)$ if equality obtains in (3″) of §1.

THEOREM 4a. *Let $\| f \|_p = 1$ where $1 < p < 2$ and let $F(x) = | f(x) |^{p-1}$ $\mathrm{sgn} \, f(x)$. If f is maximal in $L^p(G)$ then F is maximal in $L^q(G)$ and*

$$c(\alpha, i, j, F) = C(\alpha, i, j, f)^* \ .$$

Let **B** be the collection of all finite subsets B of A, partially ordered by inclusion. For $B \in \mathbf{B}$ we define

$$\phi(x, B) = \sum_{\alpha \in B} r(\alpha) \sum_{i,j=1}^{r(\alpha)} C(\alpha, i, j, f)^* g(\alpha, i, j, x) \ .$$

By Theorem 2b

$$\| \phi(x, B) \|_q \leqq \left\{ \sum_{\alpha \in B} r(\alpha)^{2-(1/2)p} S(\alpha, f)^{(1/2)p} \right\}^{1/p} \ .$$

Now

$$r(\alpha)^{2-(1/2)p} S(\alpha, f)^{(1/2)p} = r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2}$$

from which it follows that

$$\| \varphi(x, B) \|_q \leq \left\{ \sum_{\alpha \in B} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2} \right\}^{1/p} \leq \| f \|_p^{q/p} = 1 \ .$$

Also if $B_1 \in \mathbf{B}$ and if $B \Delta B_1$ denotes the symmetric difference of B and B_1 then

$$\| \varphi(x, B) - \varphi(x, B_1) \|_q \leq \left\{ \sum_{\alpha \in B \Delta B_1} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2} \right\}^{1/p} \ .$$

It follows that the limit in the mean of order q of $\phi(x, B)$ exists. Let us denote this limit by $\varphi(x)$; then

$$\lim_{B \in \mathbf{B}} \| \phi(x) - \phi(x, B) \|_q = 0 \ ,$$

$$\| \varphi(x) \|_q \leq 1 \ ,$$

and

$$c(\alpha, i, j, \varphi) = C(\alpha, i, j, f)^* \ .$$

We have

$$\int_G f(x) \phi(x, B)^* dx = \sum_{\alpha \in B} r(\alpha) \sum_{i,j=1}^{r(\alpha)} c(\alpha, i, j, f) C(\alpha, i, j, f)$$

$$= \sum_{\alpha = B} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \ .$$

It follows that

$$\int_G f(x) \varphi(x)^* dx = \lim_{B \in \mathbf{B}} \int_G f(x) \phi(x, B)^* dx$$

$$= \lim_{B \in \mathbf{B}} \left\{ \sum_{\alpha \in B} r(\alpha)^{2-q/2} s(\alpha, f)^{(1/2)q} \right\} = \| f \|_p^q = 1 \ .$$

Here we have made use of the fact that f is maximal in L^p. We now have

$$1 = \int_G f(x) \varphi(x)^* dx \leq \| f \|_p \| \phi \|_q \leq 1 \ ;$$

that is, equality obtains in Hölder's inequality. Using the fact that $\| f \|_p = 1$ this implies, see [2], that

$$\phi(x) = | f(x) |^{p-1} \operatorname{sgn} f(x)$$

Thus $\phi(x) = F(x)$ and the statement of our theorem is proved. That $F(x)$ is maximal in $L^q(G)$ follows from an evident computation.

THEOREM 4b. *Let* $\| f \|_p = 1$ *where* $1 < p < 2$ *and let* $F(x) = | f(x) |^{p-1} \operatorname{sgn} f(x)$. *If* $F(x)$ *maximal in* $L^q(G)$ *than* $f(x)$ *is maximal in* $L^p(G)$ *and*

$$C(\alpha, i, j, f)^* = c(\alpha, i, j, F) .$$

Let us set

$$F(x, B) = \sum_{\alpha \in B} r(\alpha) \sum_{i,j=1}^{r(\alpha)} c(\alpha, i, j, F) g(\alpha, i, j, x)$$

where $B \in \mathbf{B}$. Since F is maximal

$$\left\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)p} s(\alpha, F)^{(1/2)p} \right\}^{1/p} = \| F \|_q = 1 ,$$

and from this it follows easily using Theorem 2b that

$$\lim_{B \in \mathbf{B}} \| F(x) - F(x, B) \|_q = 0 .$$

We have

$$\sum_{\alpha \in B} r(\alpha) \sum_{i,j=1}^{r(\alpha)} c(\alpha, i, j, F) c(\alpha, i, j, f)^* = \int_G F(x, B) f^*(x) dx ,$$

and thus passing to the limit

$$\sum_{\alpha \in A} r(\alpha) c(\alpha, i, j, F) c(\alpha, i, j, f)^* = \int_G F(x) f^*(x) dx = 1 .$$

On the other hand

$$1 = \sum_{\alpha \in A} r(\alpha) c(\alpha, i, j, F) c(\alpha, i, j, f)^* \leqq \left\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \right\}^{1/q}$$
$$\cdot \left\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)p} s(\alpha, F)^{(1/2)p} \right\}^{1/p} .$$

Now

$$\left\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)p} s(\alpha, F)^{(1/2)p} \right\}^{1/p} = \| F \|_q = 1$$

by assumption, and this implies that

$$\left\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \right\}^{1/q} \geqq 1 = \| f \|_p .$$

Thus f is maximal in L^p. Applying Theorem 4a we see that $c(\alpha, i, j, F) = C(\alpha, i, j, f)^*$.

5. Necessary conditions.

LEMMA 5. *If $f(x)$ is maximal then so is $f(x_1^{-1} x)$ for any x_1 in G.*

The matrix relation

$$[c(\alpha, i, j, f(x_1^{-1} x))] = [g(\alpha, i, j, x_1)^*][c(\alpha, i, j, f(x))]$$

is easily verified, and because $[g(\alpha, i, j, x_1)^*]$ is unitary this implies that $s(\alpha, f(x_1^{-1} x) = s(\alpha, f(x))$. Since $\| f(x_1^{-1} x) \|_p = \| f(x) \|_p$, the desired result follows.

If $f(x)$ is any function on G and if G_1 is any subset of G then by $f_{G_1}(x)$ we denote the function which is equal to $f(x)$ for $x \in G_1$ and to 0 for $x \notin G_1$.

THEOREM 5. *If* $f(x)$ *is maximal in* $L^p (1 < p < 2)$ *or* $L^q (2 < q < \infty)$ *then* $f(x)$ *is of the form*

$$f(x) = \gamma_0 \chi_H(\alpha_0, x_1^{-1} x)$$

where γ_0 *is a complex constant,* H *is an open and closed normal subgroup of* G, $\chi(\alpha_0, x)$ *is a character of rank* $r(\alpha_0)$ *such that*

$$| \chi(\alpha_0, x) | = r(\alpha_0) \qquad\qquad x \in H ,$$

and x_0 *is an arbitrary element of* G.

For $B \in \mathbf{B}$ let us define as in § 2

$$\mathbf{F}_1(w, B) = \sum_{\alpha \in B} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [c(\alpha, i, j, T_1(w)f)][T_2(w)C(\alpha, i, j, f)] .$$

At the moment we suppose only that $f \in L(G)$ and not that f is maximal. If f is simple then $\mathbf{F}_1(w, B)$ is analytic for $0 \leqq \Re w \leqq 1$. If f is not simple then approximating f by a sequence of simple functions it is easy to show that $\mathbf{F}_1(w, B)$ is analytic for $0 < \Re w < 1$ and continuous and bounded for $0 \leqq \Re w \leqq 1$. Let

$$\mathbf{p}(u + iv) = \frac{2}{1 + u} , \quad \mathbf{q}(u + iv) = \frac{2}{1 - u} .$$

By Hölder's inequality $| \mathbf{F}_1(w, B) | \leqq I_1 I_2$ where

$$I_1 = \left\{ \sum_{o \in B} r(\alpha)^{2 - (1/2)\mathbf{q}(w)} s(\alpha, T_1(w)f)^{(1/2)\mathbf{q}(w)} \right\}^{1/\mathbf{q}(w)} ,$$

$$I_2 = \left\{ \sum_{\alpha \in B} r(\alpha)^{2 - (1/2)\mathbf{p}(w)} \left[\sum_{i,j=1}^{r(\alpha)} | T_2(w)C(\alpha, i, j, f) |^2 \right]^{(1/2)\mathbf{p}(w)} \right\}^{1/\mathbf{p}(w)} .$$

By Theorem 2a

$$I_1 \leqq \| T_1(w)f \|_{\mathbf{p}(w)} = \| f(x) \|_p^{p/\mathbf{p}(w)} ,$$

while direct computation gives

$$I_2 = \left\{ \sum_{\alpha \in B} r(\alpha)^{2 - (1/2)q} s(\alpha, f)^{(1/2)q} \right\}^{1/\mathbf{p}(w)} .$$

Thus

$$|\mathbf{F}_1(w, B)| \leqq f(x) \|_p^{p/\mathbf{p}(w)} \Big\{ \sum_{\alpha \in B} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \Big\}^{1/\mathbf{p}(w)} ,$$

and

$$|\mathbf{F}_1(w, B) - \mathbf{F}_1(w, B_1)| \leqq \| f(x) \|_p^{p/\mathbf{p}(w)} \Big\{ \sum_{\alpha \in B \triangle B_1} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \Big\}^{1/\mathbf{p}(w)}$$

It is easily deduced from this that if

$$\mathbf{F}_1(w) = \lim_{B \in \mathbf{B}} \mathbf{F}_1(w, B) = \sum_{\alpha \in A} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [c(\alpha, i, j, T_1(w)f)][T_2(w)C(\alpha, i, j, f)]$$

then $\mathbf{F}_1(w)$ is defined for $0 \leqq \Re w \leqq 1$, and that $\mathbf{F}_1(w, B) \to \mathbf{F}_1(w)$ uniformly in this strip. $\mathbf{F}_1(w)$ is therefore analytic for $0 < \Re w < 1$ and continuous for $0 \leqq \Re w \leqq 1$. In addition

$$(1) \qquad |\mathbf{F}_1(w)| \leqq \| f(x) \|_p^{p/\mathbf{p}(w)} \Big\{ \sum_{\alpha \in A} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \Big\}^{1/\mathbf{p}(w)} .$$

We may assume without loss of generality that $\| f \|_p = 1$. Then

$$(2) \qquad\qquad\qquad |\mathbf{F}_1(w)| \leqq 1 \qquad\qquad 0 \leqq \Re w \leqq 1 .$$

Let us now use the fact that f is maximal in L^p. This implies that

$$(3) \qquad\qquad \mathbf{F}_1\Big(\frac{2}{p} - 1\Big) = \sum_{\alpha \in A} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{q/2} = 1 ,$$

The relations (2) and (3) together imply, using the maximum modulus principle, that

$$\mathbf{F}_1(w) \equiv 1 \qquad\qquad 0 \leqq \Re w \leqq 1 .$$

Let us analyze this relation for $w = 1$. We have

$$1 = \mathbf{F}_1(1) \leqq \sum_{\alpha \in A} r(\alpha) \Big[\sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, T_1(1)f)|^2 \Big]^{1/2} \Big[\sum_{i,j=1}^{r(\alpha)} |T_2(1)C(\alpha, i, j, f)|^2 \Big]^{1/2} ,$$

$$(4) \qquad \leqq \Big\{ \underset{\alpha}{\text{l.u.b.}} \Big[r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, T_1(1)f)|^2 \Big]^{1/2} \Big\}$$

$$\cdot \Big\{ \sum_{\alpha \in A} \Big[r(\alpha)^3 \sum_{i,j=1}^{r(\alpha)} |T_2(1)C(\alpha, i, j, f)|^2 \Big]^{1/2} \Big\} .$$

Now

$$\underset{\alpha}{\text{l.u.b.}} \Big[r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} |c(\alpha, i, j, T_1(1)f)|^2 \Big]^{1/2} \leqq \| T(1)f \|_1 = \| f \|_p^p = 1$$

by Theorem 2a with $p = 1$, while since f is maximal in L^p

$$\sum_{\alpha \in A} r(\alpha)^{3/2} \Big[\sum_{i,j=1}^{r(\alpha)} |T_2(1)C(\alpha, i, j, f)|^2 \Big]^{1/2} = \sum_{\alpha \in A} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} = 1 .$$

Thus equality obtains in the inequality (4) and this is possible only if

$$r(\alpha)^3 \sum_{i,j=1}^{r(\alpha)} | T_2(1)C(\alpha, i, j, f) |^2 = 0$$

except for those values of α for which

$$r(\alpha)^{-1} \sum_{i,j=1}^{r(\alpha)} | c(\alpha, i, j, T(1)f) |^2 = 1 \; .$$

By Theorem 3 there are only a finite number such α's. Thus the set A_0 of indices α for which $s(\alpha, f) \neq 0$ is finite. In particular this implies that if $f(x)$ is maximal in L^p it is almost everywhere equal to a linear combination of the functions $g(\alpha, i, j, x)(\alpha \in A_0 \; ; \; i, j = 1, \cdots r(\alpha))$. Consequently redefining $f(x)$ if necessary on a set of measure 0 we may suppose that it is continuous. Because of Lemma 5 it is no restriction to assume that $f(e) \neq 0$.

Let us set

$$\psi(x, w) = \sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [T_2(w)C(\alpha, i, j, f)]^* g(\alpha, i, j, x) \; .$$

We define

$$\mathbf{F}_2(w) = \int_G \psi(x, w)^* [T_1(w)f(x)]dx \; .$$

It is easily seen that $\mathbf{F}_2(w)$ is analytic for $0 < \Re w < 1$ and continuous for $0 \leqq \Re w \leqq 1$. By Hölder's inequality

$$(5) \qquad\qquad | \mathbf{F}_2(w) | \leqq \| \psi(x, w) \|_{q(w)} \| T_1(w)f(x) \|_{p(w)} \; .$$

It is easily checked that

$$\| T_1(w)f(x) \|_{p(w)} = \| f(x) \|_p^{p/p(w)} = 1 \; ,$$

while Theorem 2b gives

$$\| \psi(x, w) \|_{q(w)} \leqq \left\{ \sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)p(w)} \left[\sum_{i,j=1}^{r(\alpha)} | T_2(w)C(\alpha, i, j, f) |^2 \right]^{(1/2)p(w)} \right\}^{1/p(w)} ,$$

from which it follows that

$$\| \psi(x, w) \|_{q(w)} \leqq \left[\sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \right]^{1/p(w)} \leqq 1 \; .$$

We have

$$| \mathbf{F}_2(w) | \leqq 1 \qquad\qquad\qquad 0 \leqq \Re w \leqq 1 \; .$$

Let us put $w = 2/p - 1$. Then $T_1(2/p - 1)f(x) = f(x)$ while $\psi(x, 2/p - 1) = F(x)$ by Theorem 4a. Consequently we have

$$\mathbf{F}_2\left(\frac{2}{p} - 1\right) = 1$$

and thus

$$\mathbf{F}_2(w) \equiv 1 \qquad\qquad 0 \leqq \Re w \leqq 1 \; .$$

It follows that in the application of Hölder's inequality (5) there is equality. This implies that if $0 < \Re w < 1$

$$\psi(x, w) = |\, T_1(w)f(x)\,|^{\mathrm{p}(w)-1} \operatorname{sgn}\,[\,T_1(w)f(x)\,] \; .$$

Thus

$$\psi(x, w) = \sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [T_2(w)C(\alpha, i, j, f)]^* g(\alpha, i, j, x)$$

$$= |\, T_1(w)f(x)\,|^{\mathrm{p}(w)-1} \operatorname{sgn}\,[\,T_1(w)f(x)\,] \; .$$

Letting $w \to 1 + iv$ we obtain

$$(6) \quad \psi(x, 1 + iv) = \sum_{\alpha \in A_0} r(\alpha) \sum_{i,j=1}^{r(\alpha)} [T_2(1 + iv)C(\alpha, i, j, f)]^* g(\alpha, i, j, x)$$

$$= \operatorname{sgn}\,[T_1(1 + iv)f(x)] \; .$$

It follows that $|\, \psi(x, 1 + iv)\,|$ is either 1 or 0 and since $|\, \psi(x, 1 + iv)\,|$ is a continuous function of v it follows that the set X of values x where it is 1 is independent of v. Note that X is both open and closed. If $x \in X$ then

$$1 = |\, \psi(x, 1)\,|$$

$$\leqq \sum_{\alpha \in A_0} r(\alpha)\left[\sum_{i,j=1}^{r(\alpha)} |\, T_2(1)C(\alpha, i, j, f)\,|^2\right]^{1/2}\left[\sum_{i,j=1}^{r(\alpha)} |\, g(\alpha, i, j, x)\,|^2\right]^{1/2}$$

$$= \sum_{\alpha \in A_0} r(\alpha)^{3/2}\left[\sum_{i,j=1}^{r(\alpha)} |\, T_2(1)C(\alpha, i, j, f)\,|^2\right]^{1/2} = 1 \; .$$

Thus equality obtains in the above application of Schwarz's inequality and from this it follows that if $x \in X, \alpha \in A_0$, then

$$(7) \qquad T_2(1)C(\alpha, i, j, f) = \tau(\alpha)\delta(x)r(\alpha)^{-1/2}g(\alpha, i, j, x)$$

where $|\, \delta(x)\,| = 1$ and where

$$\tau(\alpha) = \left[\sum_{i,j=1}^{r(\alpha)} |\, T_2(1)C(\alpha, i, j, f)^2\right]^{1/2} \; .$$

Since $f(e) \neq 0$ we have $e \in X$. Setting $x = e$ in (7) we deduce the matrix relation

$$(8) \qquad [T_2(1)C(\alpha, i, j, f)] = \tau(\alpha)\delta(e)r(\alpha)^{-1/2}I$$

where I is the $r(\alpha) \times r(\alpha)$ identity matrix.   Let

$$\chi(\alpha, x) = \sum_{i=1}^{r(\alpha)} g(\alpha, i, i, x)$$

be the character associated with the representation $[g(\alpha, i, j, x)]$.   From (7) and (8) we see that if $x \in X, \alpha \in A_0$, then $\delta(x)\chi(\alpha, x) = \delta(e)r(\alpha)$. Thus we find that

$$(9) \qquad\qquad |\chi(\alpha, x)| = r(\alpha) \qquad\qquad (x \in X;\ \alpha \in A_0)\,,$$

$$\chi(\alpha, x)r(\alpha)^{-1} = \chi(\beta, x)r(\beta)^{-1} \qquad (x \in X;\ \alpha, \beta \in A_0)\,.$$

Let H be the set of all x in G such that (9) holds for $\alpha, \beta \in A_0$.   We have proved $X \subset H$.   Since it is easily verified that if $x \in H$ then $|\psi(x, 1)| = 1$ it follows that $H \subset X$, and thus $H = X$.   If $x \in H$ then $[g(\alpha, i, j, x)] = r(\alpha)^{-1}\chi(\alpha, x)\ I$ from which it is evident that H is a normal subgroup of G and we have previously noted that $H = X$ is open and closed.   Inserting (7) in (6) we find, using the relation

$$T_2(1 + iv)C(\alpha, i, j, f) = [T_2(1)C(\alpha, i, j, f)] \cdot [r(\alpha)/S(\alpha, f)]^{-(1/4)\,piv}\,,$$

that if $x \in H$ then

$$\delta(x)^* \sum_{\alpha \in A_0} \tau(\alpha)r(\alpha)^{3/2}[r(\alpha)/S(\alpha, f)]^{+(1/4)\,piv} = [\operatorname{sgn} f(x)]\,|f(x)|^{+piv/2}\,.$$

If follows from this on setting $v = 0$ that $\operatorname{sgn} f(x)^* = \delta(x)$.   Dividing this out we see that $|f(x)|^{+piv/2}$ is independent of x for $x \in H$; that is, $|f(x)|$ is constant on H.   Thus if α_0 is some index in A_0 and if γ_0 is a suitable complex constant we have proved that $f(x) = \gamma_0\,\chi_H(\alpha_0, x)$ where $|\chi(\alpha_0, x)| = r(\alpha_0)$ on H.   (Note that since we assumed earlier that $\|f(x)\|_p = 1$ the absolute value of γ_0 is determined.)   Thus our theorem is proved if f is maximal in $L^p, 1 < p < 2$.   The case where f is maximal in $L^q, 2 < q < \infty$, follows from that treated above and Theorem 4b.

6. Sufficient conditions.

THEOREM 6.   *If $f(x)$ is of the form*

$$(1) \qquad\qquad\qquad f(x) = \gamma_0\chi_H(\alpha_0, x)$$

where γ_0 is a complex constant, H an open and closed normal subgroup of G, $\chi(\alpha_0, x)$ a character such that $|\chi(\alpha_0, x)| = r(\alpha_0)\ x \in H$, or if f is a left translate of such a function, then f is maximal in $L^p(1 < p < 2)$ and in $L^q(2 < q < \infty)$.

Let $f_1(x)$ and $f_2(x) \in L^1(G)$.   We define

$$f_0(x) = f_1 * f_2 = \int_G f_1(xy^{-1})f_2(y)dy\,.$$

It is well known and easily verified that this implies the matrix relation

$$[c(\alpha, i, j, f_0)] = [c(\alpha, i, j, f_1)][c(\alpha, i, j, f_2)] .$$

Let us put

$$m(\alpha, f) = \int_G f(x)\chi(\alpha, x)^* dx .$$

If $f_1(x)$ and $f_2(x)$ are central functions then as a special case we have

(2) $$r(\alpha)m(\alpha, f_0) = m(\alpha, f_1)m(\alpha, f_2) .$$

Consider the function (1), except that we may without loss of generality suppose $\gamma_0 = 1$. We have

$$f(xy^{-1})f(y) = r(\alpha_0)f(x) \qquad\qquad (x \in G, y \in H) .$$

It follows from this that

$$f * f = r(\alpha_0) \,|\, H \,|\, f$$

where $|\,H\,|$ is the Haar measure of H. Applying (2) we see that

$$m(\alpha, f)^2 = r(\alpha)m(\alpha, f * f) = r(\alpha)r(\alpha_0) \,|\, H \,|\, m(\alpha, f)$$

and thus either

(3′) $$m(\alpha, f) = r(\alpha)r(\alpha_0) \,|\, H \,|$$

or

(3″) $$m(\alpha, f) = 0 .$$

A simple computation shows that if A_0 is the set of indices for which (3′) holds then

$$s(\alpha, f) = r(\alpha)r\alpha_0)^2 \,|\, H \,|^2 \qquad\qquad \alpha \in A_0 .$$

Note that $\|\, f(x)\|_p = r(\alpha_0) \,|\, H \,|^{1/p}$. By Parseval's equality

$$\sum_{\alpha \in A_0} r(\alpha)^2 r(\alpha_0)^2 \,|\, H \,|^2 = \|\, f \,\|_2^2 = r(\alpha_0)^2 \,|\, H \,| ,$$

and thus

$$\left[\sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)q} s(\alpha, f)^{(1/2)q} \right]^{1/q} = \left[\sum_{\alpha \in A_0} r(\alpha)^{2-(1/2)q} r(\alpha)^{(1/2)q} r(\alpha_0)^q \,|\, H \,|^q \right]^{1/q} ,$$

$$= \left[r(\alpha_0)^{q-2} \,|\, H \,|^{q-2} \sum_{\alpha \in A_0} r(\alpha)^2 r(\alpha_0)^2 \,|\, H \,|^2 \right]^{1/q} ,$$

$$= [r(\alpha_0)^q \,|\, H \,|^{q-1}]^{1/q} = r(\alpha_0) \,|\, H \,|^{1/p} = \|\, f \,\|_p ,$$

and we have proved that f is maximal in $L^p, 1 < p < 2$. Exactly the same argument shows that f is maximal in $L^q, 2 < q < \infty$. That the translates of f are also maximal follows from Lemma 5.

7. Since the present paper was written, there has appeared "L^p Fourier transforms on locally compact unimodular groups" by R. A. Kunze, Trans. Amer. Math. Soc., Vol. 89 (1958), pp. 519–540. Together with more general results Kunze establishes inequalities like (3') and (3'') of § 1. Actully Kunze's inequalities are somewhat sharper than (3') and (3''). However since the bulk of the present paper is largely denoted to finding the extremal functions in (3') and (3'') and since these are a fortiori the extremal functions for Kunze's inequalities, the two papers are in a certain degree complementary.

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