DERIVATIONS OF JORDAN ALGEBRAS

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Introduction. Let J be a finite-dimensional semisimple Jordan algebra over a field of characteristic zero, and D a derivation of J into a module M. Jacobson has shown, in [6], that D is inner in the sense that there exist elements z_i in J, m_i in M such that for all x in J,

$$D(x) = \sum_i (z_i, x, m_i) \tag{1}$$

(where (z_i, x, m_i) denotes the associator $(z_i \cdot x) \cdot m_i - z_i \cdot (x \cdot m_i)$ and $x \cdot y$ denotes the product in J or the product of an element of J and one of M). This theorem is the analogue for Jordan algebras of the first Whitehead lemma for semisimple Lie algebras of characteristic zero. In this paper we will consider two problems: first to generalize the above theorem to arbitrary characteristic p (excluding $p = 2$ but allowing $p = 3$); second, to express the group of derivations modulo inner derivations of any Jordan algebra (not necessarily finite-dimensional or semisimple) as a cohomology group. The second problem is part of a much more general one: that of developing a cohomology theory for Jordan algebras analogous to the existing theories for associative and Lie algebras (see [9]).

Our results are as follows: with respect to the first problem, we show that if J is finite-dimensional and separable, then every derivation of J into a module is inner (i.e. satisfies (1)) if and only if J satisfies the additional condition that it has no simple ideal which is special and whose degree is divisible by the characteristic of the base field. (This latter condition is directly related to the fact that the Lie algebra of all $n \times n$ matrices over a field of characteristic p cannot be expressed as the direct sum of the derived algebra and the center if p divides n.) For the proof we use the representation theory of Jordan algebras given in [7] and rely to a certain extent on the classification of simple algebras; however, it may be possible to give a proof not relying on the classification by using a Casimir operator (as is done for Lie and alternative algebras in [3]).

As for the second problem, our results cover only special Jordan algebras and certain types of modules. We use the concept of a bimodule with involution (introduced in [7]) for an associative algebra with involution, and introduce cohomology groups which are like the usual cohomology groups of associative algebras but also take into account

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the involution. These cohomology groups may be of interest in themselves, but in this paper we examine only the first cohomology group and its relation to derivations of special Jordan algebras.

Together with the Jordan results we obtain the first Whitehead lemma for separable alternative algebras of characteristic not 2. This result has been proved for characteristic $\neq 2, 3$ by Taft [13], using only the alternative structure theory; previously it has been proved for characteristic zero by Schafer [12] using the result on Jordan algebras. A Casimir operator method is given by Campbell, [3], who does not specify for what characteristic his assumptions are fulfilled. By using an appropriate definition of inner derivations we are able to include characteristic 3 (in [10] Kaplansky announced that the Cayley numbers have outer derivations in characteristic 3, but presumably he used a different definition of inner derivations).

We are indebted to Professor Jacobson and Dr. Taft for conversations on these problems. The derivations of some of the simple Jordan algebras have been worked out independently by them.

Preliminaries. We will consider Jordan algebras over fields of characteristic different from 2. The product in the Jordan algebra, denoted by $x \cdot y$, satisfies

$$x \cdot y = y \cdot x \tag{2}$$

$$((x \cdot x) \cdot y) \cdot x = (x \cdot x) \cdot (y \cdot x) \ . \tag{3}$$

A module M for J is a vector space with a bilinear composition $x \cdot m$ for x in J, m in M such that the vector space direct sum (or "split null extension") $E = J \oplus M$ with the multiplication $(x + m) \cdot (y + n) = (x \cdot y + x \cdot n + y \cdot m)$ is a Jordan algebra, i.e. the product satisfies (2) and (3). In [7] a slightly different definition of module is given: namely, a multilinear identity is used instead of (3); the multilinear identity is implied by (3) for characteristic not 2 and is equivalent to (3) for characteristic not 2 or 3. With the present definition the results of [7] are valid for characteristic 3 also (cf. the footnote on p. 5 of [7], which however should read "$\cdots$ implies the original identity if the characteristic is $\neq 3$" and also footnote 3, p. 18, of [9]). We refer to [7] for all the concepts and results on the module theory of Jordan algebras that we shall use.

A module M is called special if

$$(x \cdot y) \cdot m = x \cdot (y \cdot m) + y \cdot (x \cdot m) \text{ for } x, y \text{ in } J, m \text{ in } M \ . \tag{4}$$

Correspondingly we define a universal associative algebra U' such that every special J module is a left module for the associative algebra U'. To each element x of J corresponds an element x' of U' such that

$x \cdot m = x'm$ if M is a special module. If the map $x \to x'$ is one-to-one we call J a special Jordan algebra; in this case we will write x for x' and consider J as a Jordan subalgebra of U' with product $x \cdot y = xy + yx$. U' is generated by the identity and by the elements x', and has an involution which is the identity on the x'.

A module M for J is called a *submodule of a sum of commuting special bimodules* if there is an associative bimodule (i.e. two-sided module) N for U' such that M is a subspace of N and the composition in M is

$$x \cdot m = x'm + mx' \,. \tag{5}$$

If J is special, a module M satisfies this condition if and only if $E = J \oplus M$ is also a special Jordan algebra. The universal associative algebra for such modules is denoted by U'', and we shall call M a U'' module. Every special module M can be considered a U'' module as follows: Let N be the vector space direct sum $M \oplus M$ of M with itself. Define N as U' bimodule by: $x'(a \oplus b) = (x'a) \oplus 0$, $(a \oplus b)x' = 0 \oplus (x'b)$, for $x \in J$, then M is isomorphic to the subspace of elements $m \oplus m$ of N. Using the involution $a \to a^*$ in U', we can speak of left $U' \otimes U'$ (tensor product over the base field) modules N, instead of U' bimodules; let $(a \otimes b)n = anb^*$ for $n \in N$. Then U'' is isomorphic to the subalgebra of $U' \otimes U'$ generated by $1 \otimes 1$ and all $x' \otimes 1 + 1 \otimes x'$ for x in J.

A derivation of a Jordan algebra J into a module M is a linear map D of J into M satisfying

$$D(x \cdot y) = x \cdot D(y) + y \cdot D(x) \,. \tag{6}$$

A definition of inner derivation of any non-associative algebra (into itself) was given by Schafer [11]: a derivation is inner if it belongs to the Lie algebra of linear transformations on the algebra generated by the right and left multiplications. If we extend this definition to derivations into modules (using the extension $E = J \oplus M$), and if we consider Jordan algebras with identity elements, then an inner derivation in this sense has the form (1), and conversely any map of the form (1) is inner. If J is special and M is a U'' module, then, on using (5), (1) becomes

$$D(x) = [x, \sum_i [z_i, m_i]] \tag{7}$$

where $[a, b]$ denotes $ab - ba$. (7) suggests the following more general definition of inner derivation: if J is a special module, M a U'' module contained in a U'-bimodule N, then a derivation is inner if it has the form

$$D(x) = [x, n] \qquad \text{for } n \text{ in } N \,. \tag{8}$$

Thus a derivation of the form (8) is also of form (1) (or (7)) if $n = \sum_i [z_i, m_i]$ for elements z_i in J, m_i in M.

If a module M is a direct sum of two submodules M_1 and M_2 and D is a derivation of J into M, then on writing

$$D(x) = D_1(x) + D_2(x), D_i(x) \in M_i ,$$

the D_i are derivations and are inner in the sense of (1) if and only if D is inner in the sense of (1). Similar remarks apply to inner derivations of the form (8).

Derivations of Special Jordan Algebras

In this section we will consider a situation illustrated by the following example: J is the set of self-adjoint elements of an associative algebra A with involution, and M is J itself. It is natural to call a derivation inner if it is of the form $x \to [x, a]$ where a is a *skew* element of A. We wish to describe the group of derivations mod inner ones as a cohomology group.

We begin with a general discussion of associative algebras and bimodules with involution (see [7], p. 40) and introduce cohomology groups for them (see [4]).

If A is an associative algebra with identity 1 and involution $a \to a^*$ and N is an associative bimodule for A, N is called a bimodule with involution if it possesses a linear transformation $n \to \bar{n}$ such that

$$(9) \qquad \bar{\bar{n}} = n , \quad \overline{an} = \bar{n}a^* , \quad \overline{na} = a^*\bar{n} \quad \text{for } a \in A, n \in N .$$

N may also be considered as left $A \otimes A$ module by $(a \otimes b)n = anb^*$, in which case

$$(9) \qquad \overline{(a \otimes b)n} = (b \otimes a)\bar{n} , \text{ and } \bar{\bar{n}} = n .$$

A bimodule with involution can also be described as a left module over an algebra K constructed as follows: in $A \otimes A$ consider the automorphism $\sigma: a \otimes b \to b \otimes a$, and let K be the crossed product algebra consisting of all pairs (α, β) with $\alpha, \beta \in A \otimes A$, with componentwise addition, and multiplication $(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma + \beta\delta^\sigma, \alpha\delta + \beta\gamma^\sigma)$ where $\alpha \to \alpha^\sigma$ is the automorphism just described. We can also represent K as the set of all elements of the form $\alpha + \beta\sigma, \alpha, \beta$ in $A \otimes A$, with multiplication rules $\sigma\alpha = \alpha^\sigma\sigma, \sigma^2 = 1$. If N is a bimodule with involution, setting $\sigma n = \bar{n}$ makes N a left K module, and conversely if N is a left K module, the map $n \to \bar{n} = \sigma n$ makes N a bimodule with involution.

A itself is a left $A \otimes A$ module under $(a \otimes b)c = acb^*$. It can be made a bimodule with involution in two ways: either by letting $\bar{a} = a^*$, or by letting $\bar{a} = -a^*$.

We define the nth cohomology group (see [4]) as $\mathrm{Ext}_K^n(A, N)$ where we consider A as left K module with $\sigma a = -a^*$. We shall now exhibit a specific projective resolution of A as K module.

Let

$$X_0 = K(1-\sigma),$$
$$X_n = \{\sum \alpha \otimes a_1 \otimes \cdots \otimes a_n + \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_1^* \mid \alpha \in A \otimes A \subseteqq K,\ a_i \in A\}$$

for $n \geqq 0$, where $\omega_n = (-1)^{\frac{1}{2}(n-1)(n-2)}$.

Setting

$$e = \frac{1+\sigma}{2}, \quad f = \frac{1-\sigma}{2},$$

we see that $X_n = Ke \otimes H_n \oplus Kf \otimes S_n$ where H_n is the subspace of $A_n = A \otimes \cdots \otimes A$ (n factors) of elements x such that $Tx = x$ and S_n is the subspace of x such that $Tx = -x$, where $T(a_1 \otimes \cdots \otimes a_n) = (-1)^{\frac{1}{2}(n-1)(n-2)} a_n^* \otimes \cdots \otimes a_1^*$. We make X_n a left K module by defining $k(k_1 e \otimes h_n + k_2 f \otimes s_n) = kk_1 e \otimes h_n + kk_2 f \otimes s_n$ for $k \in K$. Since e, f are idempotents, X_n is K-projective.

We define K-homomorphisms $d_n : X_n \to X_{n-1}$ as follows: we first define a map $d_n : K \otimes A_n \to K \otimes A_{n-1}$ by

$$\begin{aligned} d_n(k \otimes a_1 \otimes \cdots \otimes a_n) = {} & k(a_1 \otimes 1) \otimes a_2 \otimes \cdots \otimes a_n \\ & + \textstyle\sum_{i=1}^{n-1} (-1)^i k \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n k(1 \otimes a_n^*) \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

and note that $d_{n-1} \cdot d_n = 0$. Next we verify that d_n maps $X_n \to X_{n-1}$:

$$\begin{aligned} & d_n(\alpha \otimes a_1 \otimes \cdots \otimes a_n + \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_1^*) \\ & = \alpha(a_1 \otimes 1) \otimes a_2 \otimes \cdots \otimes a_n + \textstyle\sum_{i=1}^{n-1} (-1)^i \alpha \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & \quad + (-1)^n \alpha(1 \otimes a_n^*) \otimes a_1 \otimes \cdots \otimes a_{n-1} + \omega_n \alpha\sigma(a_n^* \otimes 1) \otimes a_{n-1}^* \otimes \cdots \otimes a_1^* \\ & \quad + \textstyle\sum_{i=1}^{n-1} (-1)^{n-i} \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_{i+1}^* a_i^* \otimes \cdots \otimes a_1^* \\ & \quad + (-1)^n \omega_n \alpha\sigma(1 \otimes a_1) \otimes a_n^* \otimes \cdots \otimes a_2^* . \\ & = [\alpha(a_1 \otimes 1)] \otimes a_2 \otimes \cdots \otimes a_n + (-1)^n \omega_n [\alpha(a_1 \otimes 1)]\sigma \otimes a_n^* \otimes \cdots \otimes a_2^* \\ & \quad + \textstyle\sum_{i=1}^{n-1} (-1)^i (\alpha \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & \quad + (-1)^n \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes (a_i a_{i+1})^* \otimes \cdots \otimes a_1^*) \\ & \quad + (-1)^n (\alpha(1 \otimes a_n^*) \otimes a_1 \otimes \cdots \otimes a_{n-1} \\ & \quad + (-1)^n \omega_n \alpha(1 \otimes a_n^*)\sigma \otimes a_{n-1}^* \otimes \cdots \otimes a_1^*) . \end{aligned}$$

Since

$$(-1)^n \omega_n = (-1)^n (-1)^{\frac{1}{2}(n-1)(n-2)} = (-1)^{\frac{1}{2}(n^2-n+2)} = (-1)^{\frac{1}{2}(n^2-5n+6)} = \omega_{n-1}$$

we see that the last term is indeed in X_{n-1}.

We also define $\varepsilon: X_0 \to A$ by $(a \otimes b)\left(\frac{1}{2}(1-\sigma)\right) \to ab^*$. Then ε and the d_n are all k-homomorphisms, and $\varepsilon \cdot d_1 = 0$.

Finally, we have to show the sequence $\cdots \to X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ is exact. We use the fact that the corresponding sequence used to define the usual homology groups for A (i.e. using $(A \otimes A) \otimes A_n$ instead of X_n) is exact, and that $\alpha_1 + \alpha_2\sigma = 0$ in K for α_1, α_2 in $A \otimes A$ implies $\alpha_1 = 0 = \alpha_2$. Let now

$$d_n(\alpha \otimes a_1 \otimes \cdots \otimes a_n + \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_1^*) = 0.$$

Then $d_n(\alpha \otimes a_1 \otimes \cdots \otimes a_n) = 0$ in $(A \otimes A) \otimes A_{n-1}$, and so

$$\begin{aligned}\alpha \otimes a_1 \otimes \cdots \otimes a_n &= d_{n+1}(\textstyle\sum \beta \otimes b_1 \otimes \cdots \otimes b_{n+1}) \\ &= \textstyle\sum(\beta(b_1 \otimes 1) \otimes b_2 \otimes \cdots \otimes b_{n+1} \\ &\quad + \textstyle\sum_{i=1}^n (-1)^i \beta \otimes b_i \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{n+1} \\ &\quad + (-1)^{n+1}\beta(1 \otimes b_{n+1}^*) \otimes b_1 \otimes \cdots \otimes b_n)\end{aligned}$$

therefore

$$\begin{aligned}\alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_1^* &= \textstyle\sum(\beta(b_1 \otimes 1)\sigma \otimes b_{n+1}^* \otimes \cdots \otimes b_2^* \\ &\quad + \textstyle\sum_{i=1}^n (-1)^i \beta\sigma \otimes b_{n+1}^* \otimes \cdots \otimes b_{i+1}^* b_1^* \otimes \cdots \otimes b_1^* \\ &\quad + (-1)^{n+1}\beta(1 \otimes b_{n+1}^*)\sigma \otimes b_n^* \otimes \cdots \otimes b_1^*) \\ &= (-1)^{n+1} d_{n+1}(\textstyle\sum \beta\sigma \otimes b_{n+1}^* \otimes \cdots \otimes b_1^*)\,.\end{aligned}$$

Thus $\alpha \otimes a_1 \otimes \cdots \otimes a_n + \omega_n \alpha\sigma \otimes a_n^* \otimes \cdots \otimes a_1^*$

$$= d_{n+1}(\textstyle\sum \beta \otimes b_1 \otimes \cdots \otimes b_{n+1} + (-1)^{n+1}\omega_n \beta\sigma \otimes b_{m+1}^* \otimes \cdots \otimes b_1^*)$$

but $(-1)^{n+1}\omega_n = \omega_{n+1}$, and so we have shown that kernel of $d_n =$ image of d_{n+1} and in exactly the same way we see that kernel of $\varepsilon =$ image of d_1. It is clear that the image of ε is A. Thus we have a projective resolution of A.

If N is a left K module, the vector space $\mathrm{Hom}_K(X_n, N)$ can be identified with the space of linear function g on A_n to N which satisfy the condition $\sigma g(a_1, \cdots, a_n) = \omega_n g(a_n^*, \cdots, a_1^*)$, by identifying a K homomorphism h of X_n into K and the linear function

$$g(a_1, \cdots, a_n) = h(1 \otimes a_1 \otimes \cdots \otimes a_n + \omega_n \sigma \otimes a_n^* \otimes \cdots a_1^*)\,.$$

The functions g which correspond to cocycles or coboundaries are then cocycles or coboundaries in the usual sense which also satisfy the additional condition above.

In particular the 1-cocycles are functions g on A to N which satisfy $g(ab) = ag(b) + g(a)b$ and $g(a^*) = \overline{g(a)}$ i.e. derivations mapping self-adjoint

elements of A into self-adjoint elements of N, and skew elements into skew elements; the 1-coboundaries are functions of the form $g(a) = [a, n]$ where $n \in fN =$ skew elements of N.

We will now apply these considerations to the algebra U' of a special Jordan algebra J. (Actually everything we will do applies to any Jordan algebra J, provided we replace J by its image in U', which is a homomorphic image of J). U' has an involution $a \to a^*$ such that $x = x^*$ for x in J. We form the algebra K as above, taking U' for A, and let $\sigma a = -a^*$ for $a \in U'$. Our theorem is:

THEOREM 1. *Let J be a special Jordan algebra, M a U''-module such that $M = H(N)$, the set of self-adjoint elements of N, where N is an associative bimodule with involution for U'. Then* $\mathrm{Ext}^1_K(U', N)$ *is isomorphic to the vector space of derivations of J into M modulo inner derivations of the form $x \to [x, n]$ where n is a skew element of N.*

Proof. We have to show that the derivations D of J into M may be identified with the derivations D_0 of U' into N which satisfy $D_0\sigma = -\sigma D_0$: Let D be a derivation of J into M. For x in J, the map h: $x \to x + D(x)$ of J into $E = J \oplus M$ is a homomorphism (actually an isomorphism) of Jordan algebras. E may be considered as a Jordan subalgebra of $B = U' \oplus N$, and so h can be extended to a homomorphism, also denoted by h, of U' into B taking $1 \to 1$. For a in U', $h(a) - a$ lies in N, since this is so for $a \in F_1 \oplus J$ which is a set of generators for U' and since N is an ideal in B. The map $D_0: a \to h(a) - a$ is a derivation of U' into N, and $D_0(x) = D(x)$ for x in J. $D_0(\sigma a) = -\sigma D_0(a)$ since this equation holds for $a \in F1 \oplus J$. The map $D \to D_0$ is clearly one-to-one, since D and D_0 coincide on a set of generators for U', and it is onto the vector space of derivations D_0 satisfying $D_0\sigma = -\sigma D_0$ since the restriction of such a D_0 to J is a derivation D of J into M. Under this identification, the inner derivations D_0 of U' clearly correspond to inner derivations D of J.

The above theorem applies only to a certain type of module for a special Jordan algebra. However, there exists a large class of special Jordan algebras for which all modules satisfy the conditions of the theorem, namely the Jordan algebras of all $n \times n$ hermitian matrices (with $n \geqq 4$) over an arbitrary involutorial associative algebra with identity (this is explicitly stated in [7], Th. 10.2, 10.4, 7.1 for unital modules; however every module for a Jordan algebra with identity is a direct sum of a unital module and a special module, and special modules also satisfy the condition, by the construction in the previous section). This class of Jordan algebras includes the algebras of all $n \times n$ matrices; the algebra of hermitian bounded operators (or of all bounded operators) on

a Hilbert space, or more generally the hermitian operators (or all operators) in a factor, also belongs to the class. For these algebras of matrices, consideration of modules for the Jordan algebra can de reduced to consideration of bimodules with involution for the coefficient algebras ([7]), however in our theorem the hypotheses are much more general than this.

Derivations of Separable Jordan and Alternative Algebras

In this section we will consider finite dimensional separable Jordan and alternative algebras over fields of characteristic not 2. (A separable algebra is one which is semisimple and remains semisimple under any extension of the base field.) We will obtain the first Whitehead lemma for both classes of algebras as follows: first we will prove it for simple Jordan algebras of degree two, then for alternative algebras, and lastly for the other Jordan algebras.

We shall need the following facts on the structure of these algebras ([1], [2]): a simple Jordan algebra over an algebraically closed field is said to be of degree n if the identity element is a sum of n, but not more, mutually orthogonal idempotents. A simple Jordan algebra over any field will be defined to be of degree n if on extending its *center* to its algebraic closure it becomes an algebra of degree n. Over an algebraically closed field, the simple algebras are as follows: An algebra of degree one is just the base field, [(8]); all derivations are zero. An algebra of degree 2 is the vector space direct sum $F1 + V$ of the space $F1$ spanned by its identity element 1 and a vector space V of dimension at least two on which is defined a non-degenerate symmetric bilinear form (x, y) and the multiplication in the algebra is

$$(\alpha 1 + x)\cdot(\beta 1 + y) = [\alpha\beta + (x, y)]1 + (\alpha y + \beta x)$$

for $\alpha, \beta \in F$, $x, y \in V$. An algebra of degree $n \geq 3$ consists of all $n \times n$ hermitian matrices with coefficients in an involutorial alternative algebra D which has no proper self-adjoint ideals. D is thus F, E, Q or C where F is the base field, E is two dimensional over F, Q and C are the quaternion and Cayley algebra respectively. J is denoted by $H(D_n)$; if D is C then $n = 3$. The Cayley algebra C is generated by Q and one other element v with $v^2 = \alpha 1, \alpha$ in F (and $\alpha \neq 0$), $C = Q + Qv$ (vector space direct sum) and the multiplication is

$$(q_1 + q_2 v)(q_3 + q_4 v) = (q_1 q_3 + \alpha \bar{q}_4 q_2) + (q_4 q_1 + q_2 \bar{q}_3)v \ . \tag{10}$$

Our results are as follows:

THEOREM 2. *Let J be a finite dimensional separable Jordan algebra. Then every derivation D of J into a module M is inner, in the sense*

that there exist elements z_i in J and m_i in M such that $D(x) = \sum_i (z_i x, m_i)$ if and only if J contains no simple ideal which is special and whose degree is divisible by the characteristic of the base field.

THEOREM 3. *Let A be a finite dimensional separable alternative algebra, M a bimodule, and D a derivation of A into M. Then there exist elements a_i in A, m_i in M, and g in the nucleus of M, such that*

$$D(x) = [x, g] + \sum_i [a_i, x, m_i] \text{ and } \sum_i [a_i, m_i] = 0 \tag{11}$$

where $[a, b, c]$ is the associator $(ab)c - a(bc)$. If the characteristic is not two or three, then

$$D(x) = [x, g] + \sum_i D_{a_i, m_i}(x) \tag{12}$$

a_i in A, m_i in M, g in the nucleus of M, where

$$D_{a,b}(x) = (xa)b - (xb)a + (ax)b - a(xb) + b(ax) - a(bx) .$$

We remark that in any Jordan the map $x \to \sum (z_i, x, m_i)$ is a derivation, and in any alternative algebra the maps $x \to [x, g]$ with g in the nucleus, $x \to \sum_i D_{a_i, m_i}(x)$, and $x \to \sum [a_i, x, m_i]$ with $\sum [a_i, m_i] = 0$ are derivations if the characteristic is not two (see [12], formulas (12), (19)).

The proof of the theorems is in several steps:

(a) Let J be a semi-simple Jordan algebra over an algebraically closed field, which satisfies the condition of the theorem concerning degrees. We want to show a derivation D of J into a module M is of the form (1). A module M is called unital if $e \cdot m = m$ for all m in M, where e is the identity element of J. We first show:

1. M may be assumed unital. In general, $M = M_1 + M_{1/2} + M_0$ (direct sum) where $e \cdot m_i = im_i$ for m_i in M_i, M_1 is unital, and $M_{1/2} + M_0$ is a special module. Let $D(x) = D_1(x) + D_2(x)$, $D_1(x) \in M_1$, $D_2(x) \in M_{1/2} + M_0$. Then D_1, D_2 are also derivations, and are of the form (1) if and only if D is. Since we are assuming the result for D_1, we have to prove it for D_2. However, it is true, more generally, that if J is any Jordan algebra with identity e and N is a special module then any derivation D is of form (1): Let $N = N_{1/2} + N_0$, then $D = D_{1/2} + D_0$, and $D_0(x) = D_0(e \cdot x) = e \cdot D_0(x) + x \cdot D_0(e) = 0$ so D maps J into $N_{1/2}$.

$$D(x) = D(e \cdot x) = e \cdot D(x) + x \cdot D(e) = \frac{1}{2} D(x) + x \cdot D(e) ,$$

or $D(x) = 2x \cdot D(e) = 4(e, x, D(e))$.

2. J may be assumed simple. Let $J = J_1 \oplus \cdots \oplus J_m$, J_i simple ideals satisfying the condition on degrees, and let M be a unital J module. Let e_i be the identity of J_i, then M decomposes into a direct sum of subspaces M_{ii}, $1 \leqq i \leqq m$, and $M_{ij} = M_{ji}$, $i \neq j$, $1 \leqq i, j \leqq m$, where $M_{ii} = \{m \in M / e_i \cdot m = m\}$ and $M_{ij} = \left\{m \in M / e_i \cdot m = \frac{1}{2} m = e_j \cdot m\right\}$ for $i \neq j$. Then M_{ii} is unital for J_i and annihilated by J_j for $j \neq i$, M_{ij} is special for J_i and J_j for $i \neq j$ and $x_i \cdot (x_j \cdot m_{ij}) = x_j \cdot (x_i \cdot m_{ij})$ for

$$x_i \in J_i, x_j \in J_j, m_{ij} \in M_{ij},$$

and M_{ij} is annihilated by J_k for $k \neq i, j$.

Let D be a derivation of J into M. We want to show D satisfies (1), assuming this for J simple. Let $D(x) = \sum_i D_{ii}(x) + \sum_{i<j} D_{ij}(x)$ for $x \in J$, where $D_{ii}(x) \in M_{ii}$, $D_{ij}(x) \in M_{ij}$. For $x_k, y_k \in J_k$,

$$\begin{aligned}
D(x_k \cdot y_k) &= \textstyle\sum_i D_{ii}(x_k \cdot y_k) + \sum_{i<j} D_{ij}(x_k \cdot y_k) \\
&= x_k \cdot D(y_k) + y_k \cdot D(x_k) = x_k \cdot (\textstyle\sum_i D_{ii}(y_k) + \sum_{i<j} D_{ij}(y_k)) \\
&\qquad + y_k \cdot (\textstyle\sum_i D_{ii}(x_k) + \sum_{i<j} D_{ij}(x_k)) \\
&= x_k \cdot D_{kk}(y_k) + y_k \cdot D_{kk}(x_k) + x_k \cdot (\textstyle\sum_{i<k} D_{ik}(y_k) + \sum_{k<j} D_{kj}(y_k)) \\
&\qquad + y_k \cdot (\textstyle\sum_{i<k} D_{ik}(x_k) + \sum_{k<j} D_{kj}(y_k)) .
\end{aligned}$$

Thus
$$\begin{aligned}
&D_{kk}(x_k \cdot y_k) = x_k \cdot D_{kk}(y_k) + y_k \cdot D_{kk}(x_k) \\
&D_{ii}(x_k \cdot y_k) = 0 \text{ for } i \neq k \\
&D_{ij}(x_k \cdot y_k) = x_k \cdot D_{ij}(y_k) + y_k \cdot D_{ij}(x_k) \text{ if } i < j \text{ and } i = k \text{ or } j = k , \\
&D_{ij}(x_k \cdot y_k) = 0 \text{ if } k \neq i \text{ and } k \neq j.
\end{aligned}$$

In particular, putting $y_k = e_k$, $D_{ii}(x_k) = 0 = D_{ij}(x_k)$ if $k \neq i, k \neq j$.

Since D_{ii} is a derivation of J_i into M_{ii}, there exist $z_r \in J_i$, $m_r \in M_{ii}$ such that $D_{ii}(x_i) = \sum_r (z_r, x_i, m_r)$ for $x_i \in J_i$. Also, $D_{ii}(x_k) = 0 = \sum_r (z_r, x_k, m_r)$ for $x_k \in J_k$, $k \neq i$. Thus $D_{ii}(x) = \sum_r (z_r, x, m_r)$ for all $x \in J$.

Now let $i \neq j$. Then D_{ij} is a derivation on $J_i + J_j$ into M_{ij}, and is zero on J_k for $k \neq i$ or j. Since M_{ij} is a unital module for $J_i + J_j$, D_{ij} annihilates $e_i + e_j$. $D_{ij}(e_i) = -D_{ij}(e_j)$. M_{ij} is a special module for J_i, so (as shown in 1.) $D_{ij}(x_i) = 4(e_i, x_i, D_{ij}(e_i)) = 2(e_i, x_i, D_{ij}(e_i - e_j)) = (e_i - e_j, x_i, D_{ij}(e_i - e_j))$ for $x_i \in J_i$, and similarly $D_{ij}(x_j) = 4(e_j, x_j, D_{ij}(e_j)) = (e_i - e_j, x_j, D_{ij}(e_i - e_j))$ for $x_j \in J_j$. Thus $D_{ij}(x) = (e_i - e_j, x, D_{ij}(e_i - e_j))$ for $x \in J_i + J_j$, and $D_{ij}(x_k) = 0 = (e_i - e_j, x_k, D_{ij}(e_i - e_j))$ for $k \neq i$ or j, so $D_{ij}(x) = (e_i - e_j, x, D_{ij}(e_i - e_j))$ for all $x \in J$.

Thus all D_{ii} and D_{ij} satisfy (1), and so D does also.

3. Let J be simple and special, M unital, and the base field algebraically closed. Assume that the degree of J is not divisible by the characteristic, and further that J is not $H(Q_3)$. We may also assume the degree of J is at least two.

Since J has an identity, the algebra U' is the direct sum of an ideal U_s and a one dimensional ideal Fe_0, where U_s is the ideal in U' generated by the identity element e of J. U_s is the universal algebra for special representations such that $2e$ is the identity operator, that is, $2e$ is the identity element of U_s. The ideal in U'' generated by the elements $x \otimes e + e \otimes x$ with x in J is denoted by $U_s^{(2)}$: this is the universal algebra for U'' representations which are also unital for J. From now on we shall consider only U_s and $U_s^{(2)}$ instead of U' and U'', and denoted by 1 the identity element $2e$ of U_s.

For the special simple algebras we are considering (i.e. excluding $H(Q_3)$), every unital module is equivalent to a $U_s^{(2)}$ module. Further, U_s and $U_s^{(2)}$ are separable if J is, so all unital module are completely reducible. The irreducible unital modules for each of these algebras are all obtained by decomposing a certain unital module which is also a U_s bimodule, and we may assume that the unital module M we are considering is this one in order to show the derivations satisfy (1). Thus in M, $x \cdot m = xm + mx$. Let D be a derivation of J into M and let B be the associative algebra $U_s \oplus M$ with multiplication $(a + m)(b + n) = ab + (an + mb)$. Then the map: $x \to x + D(x)$ of J into B satisfies

$$h(x \cdot y) = h(x)h(y) + h(y)h(x) ,$$

that is, h is a Jordan homomorphism. Also, $h(e) = e = \frac{1}{2} \cdot$(identity element of B). By the definition of U_s, h extends to a homomorphism, also denoted by h, of U_s into B. Since U_s is generated by J and since $h(x) = x + D(x)$ for x in J, $h(a) = a + d(a)$ for a in U_s, where $d(a) \in M$. Since h is a homomorphism, d is a derivation.

Since U_s is a separable associative algebra, $d(a) = [a, m]$ for some $m \in M$, and $D(x) = [x, m]$ for x in J. By (7), we see that D is of the form (1) if $m = \sum_i [z_i, m_i]$ for elements z_i in J, m_i in M. It is clear that m is determined only modulo $Z(M) = \{n \in M \mid [x, n] = 0$ for all $x \in J\}$. Thus we only have to show that $M = Z(M) + [J, M]$, and for this purpose we will examine separately the various types of simple Jordan algebras.

First, let $J = Fe + V$ be of degree two. U_s is the Clifford algebra of the space V and its inner product, and M may be taken as U_s itself if V has even dimension, while if V has odd dimension M is the Clifford algebra of a space W of dimension one larger: $W = V + Fu$ where $(u, u) = 1, (u, v) = 0$ for all $v \in V$. Assume now that V has dimension r, and let $x_1, \cdots, x_r$ be a basis for V satisfying $(x_i, x_i) = 1, (x_i, x_j) = 0$ for $i \neq j$. A basis for $M = U_s$ consists of 1 and all monomials

$$x_{i_1}x_{i_2} \cdots x_{i_k}, i_1 < i_2 < \cdots < i_k .$$

$Z(M)$ is just $F1$. We have $x_ix_j = -x_jx_i$ for $i \neq j$, $x_i^2 = 1$, so

$$\frac{1}{2}[x_{i_1}, x_{i_1}x_{i_2}\cdots x_{i_k}] = \frac{1}{2}(1-(-1)^{k-1})x_{i_2}\cdots x_{i_k}$$
$$= x_{i_2}\cdots x_{i_k} \text{ if } k - \text{is odd};$$
$$\frac{1}{2}[x_{i_1}, x_{i_2}\cdots x_{i_k}] = \frac{1}{2}(1-(-1)^{k-1})x_{i_1}\cdots x_{i_k}$$
$$= x_{i_1}\cdots x_{i_k} \text{ if } k \text{ is even.}$$

Thus every $x_{i_1}\cdots x_{i_k} \in [J, M]$.

Now let V have add dimension $r(\geqq 3)$, and choose a basis $x_1, \cdots, x_r$ for V satisfying the same conditions as above, so $\{x_1, \cdots, x_r, u\}$ is an orthonormal basis for W. By the above calculations $[J, M]$ contains $x_{i_1}\cdots x_{i_k}$ if $k < r$ and also contains u and $x_{i_1}\cdots x_{i_k}u$ for all k. The only term left is $x_1\cdots x_r$, but this is clearly in $Z(M)$. Thus $M = Z(M) + [J, M]$.

Next we consider the simple Jordan algebra $H(F_n)$, $n \geqq 3$, of $n \times n$ symmetric matrices over the base field F. U_s is F_n, and M may be taken as F_n also ([7], Prop. 19.2). Since F_n is the enveloping algebra of $H(F_n)$, the set of elements in M that commute with those of J is just the center $F1$ of F_n, and we have to show $F_n = F\cdot 1 + [J, F_n]$ under the assumption that the characteristic p does not divide n. Let e_{ij} denote the matrix units in F_n. Then $[e_{ii}, e_{ij}] = e_{ij} \in [J, F_n]$ for $i \neq j$. Also $[e_{ij} + e_{ji}, \alpha e_{ji}] = \alpha(e_{ii} - e_{jj})$ for $\alpha \in F$. Now let $x = \sum \alpha_{ij}e_{ij}$ be a matrix of trace zero: $\sum_i \alpha_{ii} = 0$; then $x = \sum_{i\neq j}\alpha_{ij}e_{ij} + \sum_{i\neq 1}\alpha_{ii}(e_{ii} - e_{11})$ since $\alpha_{11} = -\sum_{i\neq 1}\alpha_{ii}$ so $x \in [J, M]$. Finally, for arbitrary $y = \sum \beta_{ij}e_{ij}$, if $\tau =$ trace of $y = \sum_i \beta_{ii}$, let $x = y - \frac{1}{n}\tau 1$ (1 being the identity matrix). Then x has trace zero, so $\varepsilon[J, M]$, and $y = \frac{1}{n}\tau 1 + x \in F1 + [J, M]$. We note that if p divides n, then every element in $F1 + [J, M]$ has trace zero so $M \neq F1 + [J, M]$.

Now let E be the two dimensional algebra over F and $J = H(E_n)$. Then $U_s = E_n$ and M may be taken as Q_n (Q being the quaternion algebra): this is shown in the proof of [7], Prop. 19.4. Since $E_n = E \otimes F_n$, $Q_n = Q \otimes F_n$, and the centralizer of E in Q is E, the centralizer of E_n (and also of J) in Q_n is $E\cdot 1 = \{\alpha 1/\alpha \in E\}$. For any $q \in Q$, $qe_{ij} = [e_{ii}, qe_{ij}] \in [J, M]$, and $qe_{ii} - qe_{jj} = [e_{ij} + e_{ji}, qe_{ji}] \in [J, M]$. Let $r \in E, q \in Q$, then $[re_{ij} + \bar{r}e_{ji}, qe_{ji}] = rqe_{ii} - qre_{jj} = rq(e_{ii} - e_{jj}) + [r, q]e_{jj}$ for $i \neq j$. Since $rq(e_{ii} - e_{jj}) \in [J, M]$, $[r, q]e_{jj} \in [J, M]$ also, so $[E, Q]1 \in [J, M]$. From these calculations it follows that if $x = \sum q_{ij}e_{ij}$ and $\tau = \sum_i q_{ii}$ then $\frac{1}{n}\tau 1 \in [J, M]$. Also $Q = E + [E, Q]$ and $[E, Q]1 \in [J, M]$ so $\frac{1}{n}\tau = c + d$, $c \in [J, M]$, $d \in E$, and $x \in [J, M] + E1$, which we had to show.

Lastly we have to consider $J = H(Q_n)$. (We are assuming $n \neq 3$, but everything we say is valid for $n = 3$ also provided the module M

is assumed to be a $U_s^{(2)}$ module. $H(Q_3)$ has one other type of unital module which we will consider later.) Here $U_s = Q_n = M$, the center of M is $F1$, and $M = F1 + [J, M]$ if p does not divide n: the proof is the same as above.

4. We have left the simple Jordan algebras $H(C_3)$ and $H(Q_3)$ but before considering these we have to prove Theorem 3 for the alternative algebras C and Q. In fact, to prove Theorem 3 for any separable alternative algebra over any base field we need only prove it when: (a) the base field is algebraically closed (this is easy to see because of the forms (11), (12) for inner derivations); (b) the algebra is simple and the bimodule is unital ([13]); (c) the algebra is either C or Q (since, as shown in [7], all other simple alternative algebras and bimodules are actually associative algebras and associative bimodules).

If A is any alternative algebra with identity and N a unital alternative bimodule, then on introducing the product $a \cdot b$ for $a \in A, b \in A$, or $\in N$, we obtain a special Jordan algebra $(A, \cdot)$ and a U'' module $(N, \cdot)$ for $(A, \cdot)$.

From now on, let A denote either C or Q, N a unital alternative bimodule, and assume the base field algebraically closed. $(A, \cdot)$ is then a simple Jordan algebra of degree two: in fact A has an involution $a \to \bar{a}$, and $(A, \cdot) = F1 + V$ where $V = A^- = \{a \in A \mid \bar{a} = -a\}$ and for $a, b \in A^-$ we set $(a, b) = a \cdot b = ab + ba$. $F1$ is the center of A, and the nucleus of A if $A = C$. If D is a derivation of A into N, it is also a derivation of $(A, \cdot)$ into $(N, \cdot)$, and by our results for Jordan algebras of degree two,

$$\begin{aligned} D(a) &= \textstyle\sum_i (x_i \cdot a) \cdot m_i - x_i \cdot (a \cdot m_i) \\ &= \textstyle\sum_i a[R_{x_i} + L_{x_i}, R_{m_i} + L_{m_i}] \end{aligned} \tag{13}$$

where $x_i \in A$, $m_i \in N$, and $aR_x = ax$, $aL_x = xa$.

(If $A = C$, then N is a direct sum of copies of C, and may be taken as C also, and it is possible to give an even shorter proof of (13) in this case: the derivation D maps C^- into itself since $C^- = \{(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c) \mid a, b, c \in C\}$. Also, $D(\alpha 1) = 0$ for $\alpha \in F$, so for $x, y \in C^-, D(x \cdot y) = 0 = D(x) \cdot y + x \cdot D(y)$, or $(D(x), y) + (x, D(y)) = 0$ where $(x, y) = (x \cdot y)1$. Thus D is a skew linear transformation in C^-, and so, relative to an orthonormal basis $\{x_i\}$ for C^-,

$$D(x) = \sum_{i<j} \alpha_{ij}((x, x_i)x_j - (x, x_j)x_i) = \sum_{i<j} \alpha_{ij}(x_i, x, x_j).)$$

Since D satisfies (13), it is inner in the sense that it belongs to the Lie algebra generated by the right and left multiplications. However, we want to show that D has the specific form (11).

Let $D_{x,z} = [R_x, R_z] + [L_x, R_z] + [L_x, L_z]$ where all the operators act on the right of elements. Then $a \to aD_{x,z}$ is a derivation for all x, z, and the following identities are valid for characteristic not 2 [(12], formulas (6), (12), (19)):

$$[L_x, R_z] = [R_x, L_z] \tag{14}$$

$$D_{x,z} = R_{[x,z]} - L_{[x,z]} - 3[L_x, R_z] \tag{15}$$

$$\begin{aligned}[R_x + L_x, R_z + L_z] &= D_{x,z} + [L_x, R_z] \\ &= R_{[x,z]} - L_{[x,z]} - 2[L_x, R_z]\end{aligned} \tag{16}$$

$$[x, y_1y_2, z] = y_1[x, y_2, z] + [x, y_1, z]y_2 + [([x, z]), y_1, y_2] \ . \tag{17}$$

Now take $A = C$, and $N = C$ (which we may do, as remarked above). Let D be a derivation satisfying (13). Since $C = F1 + C^-$, we may assume the x_i and m_i are in C^-.

Assume first that the characteristic is not 3. For y in C,

$$\begin{aligned}D(y) &= \textstyle\sum_i y[R_{x_i} + L_{x_i}, R_{m_i} + L_{m_i}] \\ &= \textstyle\sum_i y(D_{x_i,m_i} + [L_{x_i}, R_{m_i}]) \\ &= \frac{1}{3}\textstyle\sum_i y(R_{[x_i,m_i]} - L_{[x_i,m_i]} + 2D_{x_i,m_i}) \ .\end{aligned} \tag{18}$$

Since $\frac{2}{3}\sum_i D_{x_i,m_i}$ is a derivation, $\sum_i R_{[x_i,m_i]} - L_{[x_i,m_i]}$ is also a derivation, and by Lemma 1 of [12], $\sum_i [x_i, m_i]$ is in the nucleus of C. However, since the x_i and m_i are in C^-, $\sum_i [x_i, m_i]$ is also in C^-, and so $\sum [x_i, m_i] = 0$. Thus by (15),

$$\begin{aligned}D(y) &= \frac{2}{3} y \textstyle\sum_i D_{x_i,m_i} = -2\textstyle\sum_i y[L_{x_i}, R_{m_i}] \\ &= \textstyle\sum_i [-2x_i, y, m_i]\end{aligned} \tag{19}$$

and so D has form (11) with $g = 0$.

Now let the characteristic be 3, and D a derivation in C satisfying (13). By (15), for all x, z, $D_{x,z} = R_{[x,z]} - L_{[x,z]}$. Since

$$D = \textstyle\sum_i [R_{x_i} + L_{x_i}, R_{m_i} + L_{m_i}] = \sum_i D_{x_i,m_i} + [L_{x_i}, R_{m_i}]$$

is a derivation, $\sum_i [L_{x_i}, R_{m_i}]$ is also a derivation, that is, $\sum_i [x_i, yz, m_i] = y\sum_i [x_i, z, m_i] + \sum_i [x_i, y, m_i]z$ for all y, z in C. By (17), $[y, z, (\sum_i [x_i m_i])] = 0$ for all y, z, so $\sum [x_i, m_i]$ is in the nucleus of C. Since it is also in C^-, $\sum_i [x_i, m_i] = 0$. Thus $\sum_i D_{x_i,m_i} = 0$ and $D = \sum_i [L_{x_i}, R_{m_i}]$, that is, $D(y) = \sum_i [x_i, y, m_i]$.

Next we have to consider the derivations of Q. The irreducible

unital alternative bimodules for Q are Q itself, and a certain submodule of Qv, where $C = Q + Qv$: from (10) it follows that if I is any irreducible (therefore two-dimensional) right ideal in Q, then Iv is a unital irreducible bimodule for Q (called a Cayley bimodule in [7]). Thus we have only to consider derivations of Q into Iv, but since Qv is a direct sum of two submodules isomorphic to Iv, we may take the module to be Qv. The proof is almost the same as for C. If D satisfies (13), we may take the x_i in $Q^- = Q \cap C^-$; the m_i are automatically in C^-. The nucleus of Qv is zero, as shown by (10). As for C, we conclude that $\sum_i [x_i, m_i] = 0$, and so D satisfies (11), and (12) for characteristic not 3. This concludes the proof of Theorem 3.

5. We can now apply the results of 4 to $H(C_3)$ and $H(Q_3)$, and we start with $H(C_3)$.

If $J = H(C_3)$, the only irreducible unital module for J is J itself. Let $J = H(C_3)$, M be isomorphic to $H(C_3)$, and $E = J + M$, the split null extension. E may be regarded as a subspace closed under $x \cdot y = xy + yx$ of the nonassociative algebra $B = A \oplus N$, where A and N are both isomorphic to C_3, and N is an A-bimodule in the sense that $an, na \in N$ for $a \in A, n \in N$. J, M, and E are the self-adjoint elements of A, N, B, respectively (see § 10 of [7]).

Let $K = H(F_3)$ be the subalgebra of J of matrices with coefficients in F, and $L = F_3$ the subalgebra of A of matrices with coefficients in F, so $K = L \cap J$. K, J, L are all contained in A. L is in the nucleus of B (elements of L associate with all elements of B), and B and N are associative bimodules for L.

Let D be a derivation of J into M. Let $h(x) = x + D(x)$ for $x \in J$. Then $h(x \cdot y) = h(x) \cdot h(y)$ for $x, y \in J$. We want to show first that h can be extended from $K = J \cap L$ to be a homomorphism of L into B (that is, $h(a_1 a_2) = h(a_1)h(a_2)$ for $a_i \in L$), and this will follow from Theorem 7.3 of [7] if we show that $h(K)$ generates an associative subalgebra in B. For $x \in K, h(x) = x + D(x) \in L + N$, and $L + N$ is an associative subalgebra of B which contains the subalgebra generated by $h(K)$.

Thus h can be extended from K to a homomorphism of L into B. Since K generates L and $h(x) = x + D(x)$ for $x \in K, h(a) = a + d(a)$ for a in L, where d is a derivation of L into N such that $d(x) = D(x)$ for x in K. Since N is an associative bimodule for L, $d(a) = [a, n]$ for a skew element n of N. However, every skew element of C_3 is a sum of commutators of hermitian elements (by Prop. 7.3 of [7], as $C^- = [C^-, C^-]$), so $n = \sum_i [z_i, m_i]$, $z_i \in M = H(N)$, and so $D(x) = \sum_i [x, [z_i\, m_i]] = \sum (z_i, x, m_i)$ for $x \in K$.

For any y in J, the map $D_1(y) = D(y) - \sum_i (z_i, y, m_i)$ is a derivation which annihilates K, and it suffices to show D_1 is inner. Thus we

may assume D is zero on K, and so $h(x) = x$ for $x \in K$, where $h(x) = x + D(x)$ for $x \in D$. Thus $h(K) = K \subseteqq L$, so $h(K)$ is in the nucleus of B, and, using Th. 7.3 of [7] again, h can be extended from J to a homomorphism of $A = C_3$ into B. Since J generates $A, h(a) = a + d(a)$ for $a \in A$, and d is a derivation of A into N. Since $d(x) = D(x)$ for $x \in J$, we will write D for d.

Let $a = \sum_{i,j} \alpha_{ij} e_{ij}$ be an element of A, α_{ij} being in C. If I denotes the identity matrix, the elements αI, $\alpha \in C$, form a subalgebra of A isomorphic to C, and we will denote this subalgebra by CI, so $a = \sum_{i,j} (\alpha_{ij} I) e_{ij}, \alpha_{ij} I \in CI$.

Since $D(e_{ij}) = 0, D(a) = \sum_{i,j} D(\alpha_{ij} I) e_{ij}$. For $\alpha \in C, [\alpha I, e_{ij}] = 0$ for all i, j, so $[D(\alpha I), e_{ij}] = 0$ and $D(\alpha I) = \alpha' I, \alpha' \in C$. The map $\alpha I \to \alpha' I$ is clearly a derivation of CI into the bimodule $N_0 = \{n \in N \mid [e_{ij}, n] = 0$ for all $i, j\}$ which is clearly isomorphic to CI itself as CI bimodule. Thus $\alpha' = \sum_i [\sigma_i, \alpha, \tau_i]$ for elements $\sigma_i, \tau_i \in C^-$ such that $\sum [\sigma_i, \tau_i] = 0$.

If $a = \sum \alpha_{ij} e_{ij} \in J, D(a) = \sum \alpha'_{ij} e_{ij}$. We will show $D(a)$ satisfies (1) by proving this for elements $a = \alpha e_{12} + \bar{\alpha} e_{21}$ and for the elements of K and noting that J is generated by such elements.

From now on we identify J with M. We note that the derivation $\alpha \to \alpha'$ in C satisfies $(\alpha)' = (\bar{\alpha}')$ since this holds for $\alpha \in C^-$ and for $\alpha \in F1$. Let $b_i = \sigma_i e_{12} + \bar{\sigma}_i e_{21}, c_i = \tau_i e_{12} + \bar{\tau}_i e_{21}$. Then

$$\begin{aligned}(b_i, a, c_i) &= (b_i \cdot a) \cdot c_i - b_i \cdot (a \cdot c_i) \\ &= (\sigma_i e_{12} + \bar{\sigma}_i e_{21}, \alpha e_{12} + \bar{\alpha} e_{21}, \tau_i e_{12} + \bar{\tau}_i e_{21}) \\ &= z_i e_{12} + \bar{z}_i e_{21}\end{aligned}$$

where $z_i = (\sigma_i \bar{\alpha}) \tau_i + (\alpha \bar{\sigma}_i) \tau_i + \tau_i (\bar{\sigma}_i \alpha) + \tau_i (\bar{\alpha} \sigma_i) - (\tau_i \bar{\alpha}) \sigma_i - (\alpha \bar{\tau}_i) \sigma_i - \sigma_i (\bar{\tau}_i \alpha) - \sigma_i (\bar{\alpha} \tau_i)$.

But $\bar{\sigma}_i = -\sigma_i, \bar{\tau}_i = -\tau_i$, so

$$\begin{aligned}z_i &= (\sigma_i \bar{\alpha}) \tau_i - (\alpha \sigma_i) \tau_i - \tau_i (\sigma_i \alpha) + \tau_i (\bar{\alpha} \sigma_i) \\ &\quad - (\tau_i \bar{\alpha}) \sigma_i + (\alpha \tau_i) \sigma_i + \sigma_i (\tau_i \alpha) - \sigma_i (\bar{\alpha} \tau_i) \\ &= [\sigma_i, \bar{\alpha}, \tau_i] - [\tau_i, \bar{\alpha}, \sigma_i] - \alpha(\sigma_i \tau_i) - [\alpha, \sigma_i, \tau_i] - (\tau_i \sigma_i) \alpha \\ &\quad + [\tau_i, \sigma_i, \alpha] + \alpha(\tau_i \sigma_i) + [\alpha, \tau_i, \sigma_i] + (\sigma_i \tau_i) \alpha - [\sigma_i, \tau_i, \alpha] \\ &= 2[\sigma_i, \bar{\alpha}, \tau_i] + 4[\sigma_i, \alpha, \tau_i] + [\alpha, [\tau_i, \sigma_i]] .\end{aligned}$$

For any β in $C, \beta + \bar{\beta}$ is in the center and nucleus, so for every $\gamma, \delta, [\gamma, \beta + \bar{\beta}, \delta] = 0$ or $[\gamma, \beta, \delta] = -[\gamma, \bar{\beta}, \delta]$; also $[\bar{\gamma}, \bar{\beta}, \bar{\delta}] = -[\bar{\delta}, \bar{\beta}, \bar{\gamma}] = [\bar{\gamma}, \bar{\beta}, \bar{\delta}] = [\gamma, \bar{\beta}, \delta]$. Thus $z_i = 2[\sigma_i, \alpha, \tau_i] + [\alpha, [\tau_i, \sigma_i]]$. By assumption $\sum_i [\sigma_i, \tau_i] = 0$, so, finally, $\sum_i (b_i, a, c_i) = 2(\sum_i [\sigma_i, \alpha, \tau_i] e_{12} + [\sigma_i, \bar{\alpha}, \tau_i] e_{21})$ and $D(a) = \frac{1}{2} \sum_i (b_i, a, c_i)$ for $a = \alpha e_{12} + \bar{\alpha} e_{21}$. For $x \in H(F_3)$, $\sum_i (b_i, x, c_i) = \sum_i [x, [b_i, c_i]]$ since x is in the nucleus of C_3, and

$$\sum_i [b_i, c_i] = \sum_i (\sigma_i\bar{\tau}_i - \tau_i\bar{\sigma}_i)e_{11} + (\bar{\sigma}_i\tau_i - \bar{\tau}_i\sigma_i)e_{22} = \sum_i [\tau_i, \sigma_i](e_{11} + {}_{22}) = 0$$

since $\sum_i [\sigma_i, \tau_i] = 0$. Thus $D(x) = \frac{1}{2}\sum (b_i, x, c_i) = 0$ for $x \in H(F_3)$. It is easy to see that $H(F_3)$ and elements a of the form $\alpha e_{12} + \bar{\alpha} e_{21}$ generate J, and so $D(z) = \frac{1}{2}\sum_i (b_i, z, c_i)$ for all $z \in J$.

Now we have to consider derivations of $H(Q_3)$ into the module $H((Qv)_3)$ consisting of hermitian matrices with elements in the subspace Qv of C. Since $H(C_3)$ is the direct sum, as $H(Q_3)$ module, of the submodules $H(Q_3)$ and $H((Qv)_3)$ it suffices to show that a derivation of $H(Q_3)$ into $H(C_3)$ has the form (1), and this can be done exactly as for derivation of $H(C_3)$ into itself; we shall therefore omit the proof. (The result is valid even in characteristic 3, but in characteristic 3 there exist other unital modules for $H(Q_3)$ (for example, Q_3) for which not all derivations satisfy (1); we will show this below.)

(b) We have shown that if J is any semi-simple Jordan algebra over an algebraically closed field and none of the special simple ideals of J has degree divisible by the characteristic of the field, then every derivation of J is of the form (1). Suppose now that J is separable over an arbitrary base field, and that the special simple ideals of J satisfy the same condition on degrees (i.e. when their *centers* are extended to their algebraic closures, the maximum number of orthogonal idempotents with sum 1 is not divisible by the characteristic). Then, when the base field is extended to its algebraic closure, J will still be semi-simple and satisfy the degree condition, for if the base field of a simple separable Jordan algebra is extended to its algebraic closure, it becomes a direct sum of isomorphic simple algebras which are also isomorphic to the algebra obtained by extending the center of the simple algebra. Extension of the base field also preserves the property of being special. A derivation of J over the original base field into a module extends by linearity to a derivation of the algebra obtained by extending the base field into the module obtained by extending the base field, and it is easy to see that the latter derivation is of the form (1) if and only if the former is.

(c) To complete the proof of the theorem, we have to show that if J is separable but does not satisfy the condition on degrees, there exist derivations not of the form (1). Let J_1 be an ideal of J which is simple, separable and special and of degree divisible by the characteristic; it suffices to find a module and a derivation for J_1 which do not satisfy (1), i.e. we may assume J is J_1, since the module and derivation for J_1 may be considered as module and derivation for J in the obvious way. Let J_E denote $J \otimes_F E$ for an extension field E of F. Then $U_s(J_E) = U_s(J)_E$. We take U_s as the module. If there is an element $m \in U_s$ such that $m \notin$ Center of $U_s + [J, U_s]$ then the derivation $x \to [x, m]$ will do.

In particular, it will suffice to find $m \notin$ Center of $U_s + [U_s, U_s]$. However, (Center of $U_s)_E =$ Center of $(U_s)_E$, and $([U_s, U_s])_E = [(U_s)_E, (U_s)_E]$, so it sufficies to show that $U_s(J_E) \neq$ Center of $U_s(J_E) + [U_s(J_E), U_s(J_E)]$ for E the algebraic closure of F.

J_E is a direct sum of isomorphic simple algebras $(J_E)_i$ and $U_s(J_E)$ is the direct sum of the algebras $U_s((J_E)_i)$, so finally it suffices to show that $U_s \neq$ Center $+ [U_s, U_s]$ if J is simple over an algebraically closed field and of degree n divisible by the characteristic. But then $U_s = D_n$ where $D = F, E,$ or Q, and in each of these matrix algebras the matrix $e_{11} \notin$ Center $+ [U_s, U_s]$ if p divides n.

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