INVOLUTIONS ON BANACH ALGEBRAS

PAUL CIVIN AND BERTRAM YOOD

1. Introduction. We present here a systematic study of involutions (conjugate linear anti-automorphisms of period two) on a complex Banach algebra B. Particular attention is given to two types of involutions which make frequent appearance in the literature on Banach algebras-symmetric involutions (xx^* has non-negative spectrum for all x) and proper involutions ($xx^* = 0$ implies $x = 0$) where $x \to x^*$ is the involution.

In these introductory remarks we confine ourselves to B semi-simple. We show first that there exist such B, commutative and not commutative, possessing no involutions. If B is not commutative and possesses a continuous (symmetric) involution then B has non-denumerably many distinct (symmetric) involutions. This is false for B commutative. Any continuous symmetric involution is proper. The converse is not true but is shown to hold for B an annihilator algebra in the sense of [1]. Any two continuous symmetric involutions which permute must be the same. This is false for proper involutions. The conclusion is valid for proper involutions for B simple with a non-zero socle.

For B^* and H^*-algebras we can say more, for example, any B^*-algebra or H^*-algebra which is not commutative possesses symmetric involutions of arbitrarily large norm.

2. General theory. Throughout this paper we are concerned with complex Banach algebras. By an *involution* on a Banach algebra, we mean a conjugate linear anti-automorphism of period two. By a *real involution* we mean a real linear anti-automorphism of period two.

We turn our attention first to the theory of real linear involutions on a commutative Banach algebra B.

2.1 DEFINITION. Let $*$ be a real involution on the commutative Banach algebra B. Let $\mathfrak{M}$ be the space of maximal regular ideals of B. Let, for $M \in \mathfrak{M}$

$$(1) \qquad \sigma(M) = \{f^* \mid f \in M\} .$$

From algebra we see that $\sigma(M) \in \mathfrak{M}$ and that σ is a one-to-one mapping of $\mathfrak{M}$ onto $\mathfrak{M}$ which is of period two.

2.2. THEOREM. *The mapping σ is a homeomorphism of $\mathfrak{M}$ onto $\mathfrak{M}$. For each $M \in \mathfrak{M}$ either*

Received July, 29, 1958. This research was supported in part by the National Science Foundation, grant NSF-G-5865.

$$f^*(\sigma M) = f(M) \text{ for all } f \in B, \text{ or}$$
$$f^*(\sigma M) = \overline{f(M)} \text{ for all } f \in B.$$

Proof. Let j be an identity for B modulo M, $jx - x \in M$ for all $x \in B$. Then $j^*x - x \in \sigma(M)$ for all $x \in B$. Consequently $j(M) = 1$ and $j^*(\sigma(M)) = 1$. Next observe that $(ij)^2 + j \in M$, and $(ij)^{*2} + j^* \in \sigma(M)$. Thus $(ij)^*(\sigma(M)) = \pm i$.

Suppose that $f(M) = a + bi$, with a and b real. Then since $j(M) = 1$, $f - aj - bij \in M$, and $f^* - aj^* - b(ij)^* \in \sigma(M)$. Thus

$$f^*(\sigma(M)) = aj^*(\sigma(M)) + b(ij)^*(\sigma(M)) = f(M) \text{ or } \overline{f(M)} ,$$

where the choice is independent of the particular $f \in B$ that is employed. Let

$$S_1 = \{M \in \mathfrak{M} | f^*(M) = f(\sigma(M)), \text{ all } f \in B\} ,$$
$$S_2 = \{M \in \mathfrak{M} | f^*(M) = \overline{f(\sigma(M))}, \text{ all } f \in B\} .$$

The sets S_1 and S_2 are disjoint and their union is $\mathfrak{M}$.

Let $Q_1 = \{M \in \mathfrak{M} | (if)^*(M) = if^*(M), \text{ all } f \in B\}$,

$Q_2 = \{M \in \mathfrak{M} | (if)^*(M) = -if^*(M), \text{ all } f \in B\}$.

The sets Q_1 and Q_2 are disjoint. If $M \in S_1$, then $(if)^*(M) = if(\sigma(M)) = if^*(M)$, so $S_1 \subset Q_1$. Likewise $S_2 \subset Q_2$, so $S_n = Q_n$, $n = 1, 2$. Now

$$Q_1 = \bigcap_{f \in B} \{M \in M | [(if)^* - if^*](M) = 0\} .$$

Thus $Q_1 = S_1$ is closed. Likewise $Q_2 = S_2$ is closed. Thus S_1 and S_2 are open and closed.

Since $\sigma^{-1} = \sigma$, it is sufficient if we show that σ is continuous. Let $M_0 \in S_1$. Consider a basic neighborhood U of $\sigma(M_0)$,

$$U = \{M \in \mathfrak{M} || f_k(M) - f_k(\sigma(M_0))| < \varepsilon, \ \varepsilon > 0, \ k = 1, \cdots, n, f_k \in B\}$$

Let

$$V = \{M \in \mathfrak{M} || f_k(M) - f_k(M_0)| < \varepsilon, k = 1, \cdots, n\} \cap S_1 .$$

For $M \in V, f_k^*(M_0) = f_k(\sigma(M_0))$ and $f_k(\sigma(M)) = f_k^*(M)$. Since V is open and $\sigma(V) \subset U$, σ is continuous on S_1. Similarly σ is continuous on S_2.

It might be noted that $\sigma(S_j) = S_j$, $j = 1, 2$. For let $M \in S_1$, then $f(M) = f^*(\sigma(M))$ for all f. Thus $f^*(\sigma\sigma(M)) = f(\sigma M)$ for all f, so $\sigma(M) \in S_1$, that is $\sigma(S_1) \subset S_1$. But then $S_1 \subset \sigma(S_1)$ so $S_1 = \sigma S_1$.

2.3. COROLLARY. *Let B be a commutative semi-simple Banach algebra with a connected space of maximal regular ideals $\mathfrak{M}$. Then* *every real involution is either complex linear or conjugate linear.*

Proof. The connectedness of $\mathfrak{M}$ forces either $S_1 = \phi$ or $S_2 = \phi$.

2.4 COROLLARY. *Let B be a semi-simple commutative Banach algebra. Then B admits an involution if and only if there is a homeomorphism σ of period two of the maximal regular ideal space $\mathfrak{M}$ onto itself such that for each $x \in B$, there is a $y \in B$ such that $x(\sigma(M)) = \overline{y(M)}$ for each $M \in \mathfrak{M}$.*

Proof. The only if statement is immediate from Theorem 2.2. Suppose that the given condition is satisfied. By semi-simplicity the y associated with a given x is unique. The definition of $x^* = y$, is easily seen to yield an involution.

2.5 THEOREM. *Let B be a commutative regular semi-simple Banach algebra with space of maximal regular ideals $\mathfrak{M}$. A real involution $*$ on B is proper ($xx^* = 0$ implies $x = 0$) if and only if the corresponding homeomorphism σ of $\mathfrak{M}$ is the identity.*

Proof. For the notion of a regular Banach algebra see [9, p. 82]. Suppose $*$ is proper and σ is not the identity. Take $M_0 \in \mathfrak{M}$ with $\sigma(M_0) \neq M_0$. Let U be a neighborhood of M_0 such that $\sigma(M_0) \notin \bar{U}$. Then $M_0 \notin \overline{\sigma(U)}$. Let $V = U \cap \mathfrak{M} - \overline{\sigma(U)}$. Then $V \cap \sigma(V)$ is empty. Since $\sigma(V)$ is an open set containing $\sigma(M_0)$, by regularity there exists $x \in B$ such that $x(\sigma(M_0)) = 1$ and $x(M) = 0$, $M \notin \sigma(V)$. For any $M \in \mathfrak{M}$, $xx^*(M) = x(M)x(\sigma(M))$ or $xx^*(M) = x(M)\overline{x(\sigma(M))}$. Clearly $xx^*(M) = 0$. As B is semi-simple $xx^* = 0$, $x \neq 0$ and $*$ is not proper. The converse is trivial.

Thus for such B the only possible proper conjugate linear involution is conjugation.

The question naturally arises whether an algebra may have no involution or whether it may have a finite number of involutions. In the examples which follow we show that both possibilities may occur in the commutative case. We also exhibit a not commutative algebra which has no involution. However we show in Theorem 2.20 that for a semi-simple Banach algebra which is not commutative if one involution exists, there must be an uncountable number of distinct involutions.

Let D denote the compact set in the plane which consists of a two cell together with certain arcs and simple closed curves as indicated in Fig. 1.

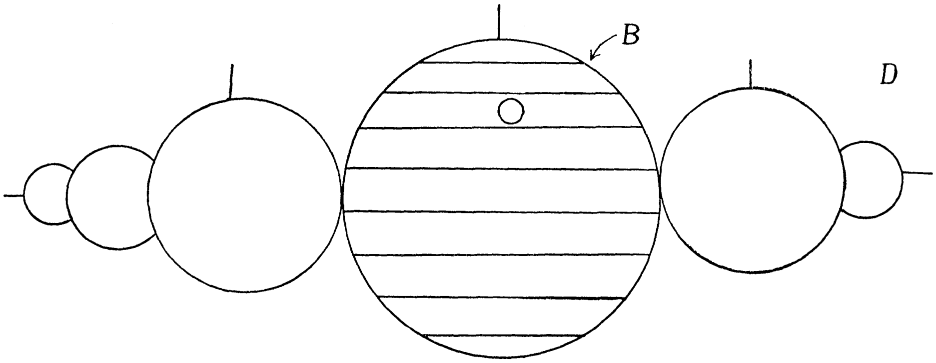

Fig. 1.

Say σ is a periodic homeomorphism of period two of D onto D. Let 0 be the open two cell, and B the boundary of 0. Then $\sigma(0) = 0$, and $\sigma(B) = B$. Now any periodic mapping of a simple closed curve has [17, p. 264] either all fixed points, just two fixed points or has no fixed points. By considering the order of the point of $D \sim B$, together with $\sigma(B) = B$, one sees that σ is pointwise fixed on $D \sim B$. Thus for the disc $0 \cup B$, $\sigma(0 \cup B) = 0 \cup B$ and $\sigma(x) = x$ for $x \in B$. It then follows from a result of Kerekjarto [8], that σ is pointwise fixed on 0. Thus D admits no homeomorphism of period at most two other than the identity mapping.

2.6 EXAMPLE. With D as above, $C(D)$, is a commutative semi-simple Banach algebra admitting exactly one involution. This follows from Theorem 2.2, since $\sigma(M) = M$ is forced for each $M \in \mathfrak{M}$.

2.7 LEMMA. *Let B be a semi-simple Banach algebra whose elements are complex valued continuous functions. Further suppose that the functions 1 and z are in B and that the maximal ideal space $\mathfrak{M}$ is a set in the complex plane. Let E be compact set in the complex plane intersecting $\mathfrak{M}$ in a point. Let A denote the algebra consisting of all continuous extensions of the elements of B to $\mathfrak{M} \cup E$. Then the maximal ideal space of A is $\mathfrak{M} \cup E$.*

Proof. Let B_0 be the subalgebra of A consisting of those elements which are constant on E. Let A_0 be the subalgebra of A consisting of the functions vanishing on $\mathfrak{M}$. Since $\mathfrak{M} \cap E$ is a point, one clearly has $A = B_0 \oplus A_0$. Let μ be a non-zero multiplicative linear functional on A, and let $\{p\} = \mathfrak{M} \cap E$.

In the decomposition $A = B_0 \oplus A_0$ we have $u_1 = u_0 + 0$, where u_1 is the unit for A and u_0 is the unit for B_0. Thus the restriction of μ to B_0 is not zero. Also since $\mathfrak{M} \cap E$ is a point, A_0 consists of all continuous functions vanishing at p. Hence there is a point $t_0 \in E$ such that for any $g \in A_0$, $\mu(g) = g(t_0)$, whether μ restricted to A_0 is zero or not.

Let $w \in A$, so $w = f + g$, $f \in B_0$, $g \in A_0$. By the remarks above there is a point z_0 in $\mathfrak{M}$, independent of f, and $t_0 \in E$ such that $\mu(f) = f(z_0)$ and $\mu(g) = g(t_0)$. Hence $\mu(w) = \mu(f) + \mu(g) = f(z_0) + g(t_0) = w(z_0) + w(t_0) - w(p)$. If one applies this formula to $w = z^2$ and makes use of the multiplicative property of μ, one obtains

$$0 = p^2 + z_0 t_0 - z_0 p - t_0 p = (p - z_0)(p - t_0) .$$

Thus $z_0 = p$, or $t_0 = p$. In the first case $\mu(w) = w(t_0)$; in the second $\mu(w) = w(z_0)$. So all the nontrivial multiplicative linear functionals are given by the points of $\mathfrak{M} \cup E$.

2.8 EXAMPLE. There exists a semi-simple commutative Banach algebra which admits no involution.

Let D be as in Figure 1. Let A be the collection of functions analytic in the interior of the cell 0 and continuous on D. Since D can be obtained from the closed two cell by adjoining successively three compact sets having one point contact with the set already available, Lemma 2.7 applies and the maximal ideal space of A is D. Clearly A is semi-simple. Since D admits no periodic homeomorphism of period at most two other than the identity, by Theorem 2.2 any involution must satisfy $f'(M) = \overline{f(M)}$. However because of the analyticity in the open cell, the latter functions are not in the algebra. Thus no involution can exist.

2.9 LEMMA. *Let A be a commutative Banach algebra with identity e_1 and no involution. Let B be a Banach algebra with identity e_2 where B is not commutative. Suppose further that 0 and e_2 are the only idempotents in the center of B. Then the direct sum $A \oplus B$ of these two algebras has no involution.*

Proof. Suppose that $A \oplus B$ has an involution $'$. Let $e_1' = u + v$, $u \in A$, $v \in B$. Since e_1' is an idempotent, so are u and v. Since e_1 is in the center of $A \oplus B$ so is e_1'. Then v is in the center of B and thus $v = 0$ or $v = e_2$. If $v = 0$ we have for $x \in A$, $x' = e_1' x' \in A$ so that $A' = A$. This is impossible as A has no involution. Therefore $v = e_2$. Now $e_1' + e_2' = e_1 + e_2$ as $e_1 + e_2$ is the identity for $A \oplus B$. Then $e_2' = e_1 - u \in A$. For $y \in B$, $y' = y' e_2' \in A$. Therefore $'$ defines an anti-isomorphism of B into A. Since A is commutative so is B. This is a contradiction.

2.10 EXAMPLE. Let A be the semi-simple commutative algebra described in Example 2.8 with no involution. Let B be the Banach algebra of algebra of all 2×2 matrices over the complex field. Then

$A \oplus B$ is a semi-simple Banach algebra which is not commutative and has no involution.

We now turn to the theory of involutions on Banach algebras without the hypothesis of commutativity, or with a hypothesis that the algebra be not commutative. It is convenient to consider certain special classes of involutions.

2.11 DEFINITIONS. Let $*$ be an involution on a Banach algebra B Let $x \in B$, and let $sp(x)$ denote the spectrum of x. Let $\rho(x)$ denote the spectral radius of x,

$$\rho(x) = \sup \{|\lambda| \mid \lambda \in sp(x)\} .$$

The spectrum and spectral radius of an element x relative to a subalgebra B_0 of B are denote by $sp(x \mid B_0)$ and $\rho(x \mid B_0)$ respectively. An element $x \in B$ is called *self-adjoint* if $x = x^*$ and H is used for the set of self adjoint elements in B. An element x is called *skew* if $x^* = -x$ and K is used for the set of skew elements in B. One has $B = H \oplus K$. Call $*$ symmetric if $sp(xx^*) \subset [0, \infty)$ for all $x \in B$. Call $*$ *Hermitian-real*, if $sp(x)$ is real for all $x \in H$. Call $*$ *regular* if $\rho(x) = 0$ and $x \in H$ imply $x = 0$. As in Theorem 2.5, we call $*$ *proper* if $xx^* = 0$ implies $x = 0$.

As shown by Kaplansky [5, p. 402], if $*$ is symmetric then $*$ is Hermitian-real.

2.12 THEOREM. *If $*$ is Hermitian-real and regular then $*$ is proper.*

Proof. Suppose $xx^* = 0$. Now $(x^*x)^2 = 0$, so that $\rho(x^*x) = 0$ and $x^*x = 0$. Let $x = h + k$, $h \in H$, $k \in K$. Then

$$0 = xx^* = h^2 - k^2 - hk + kh = x^*x = h^2 - k^2 + hk - kh .$$

Therefore $hk = kh$ and $h^2 = k^2$. Note that $sp(h^2) \subset [0, \infty)$ and, as $sp(k)$ is pure-imaginary $sp(k^2) \subset (-\infty, 0]$. Thus $sp(h^2) = (0)$. Then $h^2 = 0$. But then $\rho(h) = 0$ and $h = 0$. Similarly $k = 0$, and thus $x = 0$.

2.13. LEMMA. *Let B be a Banach algebra with an involution $x \to x^*$. Then any maximal commutative $*$-subalgebra B_0 is closed.*

Proof. The point of the lemma is that the conclusion holds even though $x \to x^*$ may be discontinuous. Let $z \in \bar{B}_0$. Since $zx = xz$, $x \in B_0$ we have $xz^* = z^*x$, $x \in B_0$. But if $z = \lim x_n$, $x_n \in B_0$, we have $x_n z^* = z^* x_n$ for all n and thus $zz^* = z^*z$. If z were not in B_0, then B_0 could not be a maximal commutative $*$-subalgebra.

2.14. LEMMA. *Let B be a Banach algebra and B_1 be the algebra obtained by adjoining an identity e to B. Let $*$ be an involution on B which is extended to an involution on B_1 by defining $(\lambda e + x)^* = \bar{\lambda}e + x^*$. If $*$ is symmetric or proper or Hermitian-real or regular on B then it has the same property on B_1.*

Proof. Consider $u = (\lambda e + x)(\lambda e + x)^* = |\lambda|^2 e + h$ where

$$h = \lambda x^* + \bar{\lambda}x + xx^* \in B .$$

Suppose $*$ is proper on B. Then if $u = 0$, $\lambda = 0$, $xx^* = 0$ and $x = 0$. Suppose that $*$ is symmetric on B. We must show that $sp(u) \subset [0, \infty)$. Clearly h is self-adjoint. Let B_0 be a maximal commutative $*$-subalgebra of B containing h. By Lemma 2.13, B_0 is closed. Let $\mathfrak{M}$ be the space of regular maximal ideals of B_0. If $y \in B_0$ then $sp(y \mid B) = sp(y \mid B_0)$ (except perhaps for the value zero). See [5, p. 402].

If B_0 is a radical algebra then $sp(h) = (0)$ so that

$$sp(u) = |\lambda|^2 + sp(h) \subset [0, \infty) .$$

Suppose that B_0 is not a radical algebra. Let $M_0 \in \mathfrak{M}$. There exists $z \in B_0$ such that $z(M_0) \neq 0$. Note that $z^*(M) = \overline{z(M)}$, $M \in \mathfrak{M}$. Consider $w = z(|\lambda|^2 e + h)z^* = |\lambda|^2 zz^* + zhz^*$. Clearly $w \in B_0$ and

$$w = z(\lambda e + x)(\lambda e + x)^* z^* = (\lambda z + zx)(\lambda z + zx)^* = |\lambda|^2 zz^* + zhz^* .$$

But $w(M_0) = |z(M_0)|^2(|\lambda|^2 + h(M_0))$. As $z(M_0) \neq 0$, $h(M_0) \in [-|\lambda|^2, \infty)$. Since M_0 is arbitrary in M, $sp(h) \subset [-|\lambda|^2, \infty)$. Therefore $sp(u) = |\lambda|^2 + sp(h) \subset [0, \infty)$.

Suppose that $*$ is Hermitian-real on B. Let $\lambda e + x$ be self-adjoint in B_1, $x \in B$. Then λ is real and x is self-adjoint in B. Since $sp(\lambda e + x) = \lambda + sp(x)$, $*$ is Hermitian real on B_1. Suppose that $*$ is regular on B and $\rho(\lambda e + x) = 0$, λ real and $x \in H$. Then if $\lambda \neq 0$ we see that x^{-1} exists in B_1 which is impossible. Thus $\rho(x) = 0$ and $x = 0$.

2.15 LEMMA. *An involution $*$ on the Banach algebra B is regular if and only if every maximal commutative $*$-subalgebra is semi-simple.*

Proof. Suppose that $*$ is regular. Let B_0 be a maximal commutative $*$-subalgebra. By Lemma 2.13, B_0 is closed. Let w be in the radical R of B_0, $w = h + k$, $h \in H$, $k \in K$. Since $*$ is an anti-automorphism of B_0, $R^* = R$. Then $w^* = h - k \in R$ so that $h \in R$, $k \in R$. Then $\rho(h \mid B_0) = 0$, so that $\rho(h) = 0$ and $h = 0$. Likewise $k = 0$.

Suppose, conversely, that the condition holds. Let $h \in H$, $\rho(h) = 0$. There exists a maximal commutative $*$-subalgebra B_0 containing h. Since

B_0 is a semi-simple Banach algebra by Lemma 2.13 and $\rho(h|B_0) = \rho(h)$, it follows that $h = 0$.

2.16 THEOREM. *Let B be a semi-simple Banach algebra with a symmetric involution $*$. Then the following statements are equivalent.* (1) *$*$ is continuous* (2) *$*$ is regular* (3) *there exists a faithful $*$-representation of B as operators on a Hilbert space.*

Proof. In view of Lemma 2.14, there is no loss of generality in assuming that B has an identity e. That (1) implies (3) has been shown by Gelfand and Neumark [10]. (They assume $\|e\| = 1$ and $\|x^*\| = \|x\|$ which is not necessary for this conclusion.) That (3) implies (1) follows from a result of Rickart [13, Lemma 5.3]. It is clear that (3) implies (2).

It is then sufficient to show that (2) implies (3). Assume (2). Let $h \in H$, $\|h\| < 1$ and let B_0 be a maximal commutative $*$-subalgebra containing e and h. By Lemmas 2.12 and 2.14 B_0 is a semi-simple Banach algebra. Hence $*$ is continuous on B_0 (see [13, Corollary 6.3]). Then the standard square root argument [10, p. 116] shows that there exists $y \in H$, $y^2 = e - h$. Let f be a positive linear functional on $B(f(yy^*) \geqq 0$ for all $y \in B)$. As in [10, p. 117] we see that

$$|f(x)| \leqq f(e)\|x\|, \quad x \in H\,. \tag{1}$$

In contrast to the Gelfand-Neumark development we do not have the right at this stage to assert that f is bounded since we did not assume $*$ to be continuous.

For any $x, y \in B$ the following inequality has been established by Kaplansky, [7, p. 55], by an algebraic computation (n is any positive integer).

$$f(y^*x^*xy) \leqq f(y^*y)^{1-2^{-n}} f[y^*(x^*x)^{2^n}y]^{2^{-n}} \tag{2}$$

Then from (1) and (2) we obtain

$$f(y^*x^*xy) \leqq f(y^*y)^{1-2^{-n}} [f(e)\|y^*\|\,\|y\|]^{2^{-n}} \|(x^*x)^{2^n}\|^{2^{-n}}$$

Let $n \to \infty$. Then

$$f(y^*x^*xy) \leqq f(y^*y)\rho(x^*x)\,. \tag{3}$$

From (3) (with $y = e$) and the Bunjakowsky-Schwarz inequality or as in [10, p. 117] we obtain $|f(x)|^2 \leqq f(e)\rho(x^*x)$, $x \in B$.

Let $I_f = \{x \in B | f(x^*x) = 0\}$. Then $\mathfrak{H}'_f = B/I_f$ is a pre-Hilbert space if we define, for two cosets ξ and η, $(\xi, \eta) = f(y^*x)$, $y \in \eta$, $x \in \xi$. Let π be the natural homomorphism of B onto B/I_f. As in [10, p. 120] we associate with $x_0 \in B$ an operator A_{x_0} on $\mathfrak{H}'_f$ defined by $A_{x_0}(\xi) = \pi(x_0 x)$. By (3),

$$||A_{x_0}(\xi)||^2 = f[(x_0 x)^*(x_0 x)] \leqq f(x^* x)\rho(x_0^* x_0) .$$

Thus $||A_{x_0}(\xi)||^2 \leqq \rho(x_0^* x_0)||\xi||^2$ so that A_{x_0} is bounded. Hence A_{x_0} may be extended to T_{x_0}, a bounded operator on the completion $\mathfrak{H}_f$ of $\mathfrak{H}'_f$. The mapping $x \to T_x$ is a $*$-representation of B. By the arguments of [10], there is a faithful $*$-representation of B as operators on the direct sum $\mathfrak{H}$ of all the $\mathfrak{H}_f(f$ running through the set F of all positive linear functionals) if the reducing ideal $\{x \in B | f(x^* x) = 0$, and $f \in F\} = (0)$.

Let $x \in H$, $sp(x) \subset (0, \infty)$. Let B_0 be a maximal commutative $*$-subalgebra of B containing e and x. For $y \in B_0$, $sp(y | B_0) = sp(y | B)$ by [10, p. 109]. As B_0 is semi-simple by Lemma 2.15 it follows that there exists $z \in H \cap B_0$, $z^2 = x$, $sp(z) \subset (0, \infty)$ (see [10, p. 159]). Let

$$P = \{x \in B | x \in H \text{ and } sp(x) \subset [0, \infty)\} .$$

The arguments of [10, p. 160] now show that P is a cone in H. Let $x = e - u$, $u \in H$, $||u|| < 1$. As noted above there exists $w \in H$, $w^2 = e - u$. Also $sp(x) = sp(w^2) \subset [0, \infty)$. Hence $e \in \operatorname{int}(P \cap H)$. Everything is now arranged for the validity of the reasoning of [10, p. 161] to show that the reducing ideal of B coincides with the radical of B.

2.17 COROLLARY. *Any symmetric continuous involutions on a semi-simple Banach algebra is proper.*

Proof. This is immediate from Theorems 2.16 and 2.12. The converse of Corollary 2.17 is false. Let B be the algebra of all complex-valued functions analytic in $|z| < 1$ and continuous in $|z| \leqq 1$. Define an involution $*$ on B by $f^*(z) = \overline{f(\bar{z})}$. Then $*$ is proper but not symmetric.

2.18 LEMMA. *Let $'$ and $*$ be two involutions on a Banach algebra B. Then $'*'$ is an involution on B which is symmetric or proper or Hermitian-real or regular if and only if $*$ has the corresponding property.*

Proof. Set $\# = '*'$. It is readily verified that $\#$ is an involution. Note that $* = '\#'$. Therefore it is sufficient to show that $\#$ inherits any of the stated properties from $*$.

Let $x = y^{*\prime}$. Then $xx^\# = (yy^*)'$. If $xx^\# = 0$ and $*$ is proper then $yy^* = 0$, $y = 0$ and $x = 0$. Also $sp(xx^\#) = \overline{sp(yy^*)}$. Thus if $*$ is symmetric so is $\#$. Suppose $*$ is Hermitian-real. Let $x = x^\#$. Then $x' = x'^*$ so that $sp(x')$ is real. Therefore $sp(x)$ is real. If $*$ is regular, $x = x^\#$ and $\rho(x) = 0$ hold, then $\rho(x') = 0$, $x' = 0$ and $x = 0$.

2.19 LEMMA. *Let B be a Banach algebra with an identity and an involution *. If $y \in H$ and y^{-1} exists then the mapping $'$ defined by $x' = y^{-1}x^*y$ is an involution. If $y = u^2$, $u \in H$ then $'$ can be expressed as $= {}^{\#*\#}$ where $^\#$ is an involution.*

Proof. Since $(y^{-1})^* = (y^*)^{-1}$ it is easy to check that $'$ is an involution on B. Let $y = u^2$, $u \in H$. Define $^\#$ by the rule $z^\# = u^{-1}z^*u$. Then $x^{\#*\#} = u^{-1}x^\# u = u^{-2}x^*u^2 = x'$, $x \in B$. Clearly if * is continuous so is $'$.

For B with an identity and an involution *, let

$$P^+ = \{x \in H \mid sp(x) \subset (0, \infty)\} .$$

It is known [**11**, p. 27)] that if * is continuous then each $x \in P^+$ can be written in the form $x = u^2$ where $u \in P^+$.

2.20 THEOREM. *Let B be a Banach algebra which is not commutative and which has a continuous involution *. Then B has non-denumerably many distinct involutions. If * is symmetric (Hermitian-real), these involutions may be chosen to be symmetric (Hermitian-real).*

Proof. Let B_1 be the algebra obtained by adjoining an identity e to B; extend * to B_1 by $(\lambda e + x)^* = \bar{\lambda} e + x^*$. Consider any two involutions on B_1 of the form $x' = y_1^{-1}x^*y_1$, $x^\# = y_2^{-1}x^*y_2$, where y_k^{-1} exists, $y_k \in H$, $k = 1, 2$. Note that $B' = B$, $B^\# = B$ and that $' = {}^\#$ if $'$ agrees with $^\#$ on B. Therefore the first statement follows if we show that there are non-denumerably many involutions on B_1 of the form $'$. Let Q be the set of invertible elements of H.

Let Z be the center of B_1. We show first that $' = {}^\#$ if and only if $(Z \cap H)y_1 = (Z \cap H)y_2$. Suppose $' = {}^\#$. Then $y_2y_1^{-1}x = xy_2y_1^{-1}$, $x \in B$ or $y_2y_1^{-1} \in Z$. Then $y_2y_1^{-1}y_1 = y_1y_2y_1^{-1}$, whence $y_1^{-1}y_2 = y_2y_1^{-1}$. Hence $y_2y_1^{-1} \in Z \cap H$. Now $Z \cap H$ is a real subalgebra of H. Thus $(Z \cap H)y_2 \subset (Z \cap H)y_1$. Hence $(Z \cap H)y_1 = (Z \cap H)y_2$. Assume this relation. Since $e \in Z \cap H$, $y_2y_1^{-1} \in Z \cap H$. Then clearly $' = {}^\#$. Since $Z \cap H$ is closed and multiplication by y_1 is a homeomorphism of B_1, $(Z \cap H)y_1$ is a closed real linear manifold in H.

Suppose that there are at most a denumerable number of involutions on B_1 of the form $'$. Then there are at most denumerable many distinct closed linear manifolds in H of the form $(Z \cap H)y$, $y \in Q$. Denote this collection by $\{E_n\}$. Now as $e \in Z \cap H$, each $y \in Q$ is contained in at least one E_n, namely $(Z \cap H)y$. Thus $Q \subset \cup E_n$. Let S be the set of real multiples of e and form, for each n, $R_n = S + E_n$. R_n is a closed linear manifold in H. Let $w \in H$. For sufficiently large real λ, $\lambda e + w \in Q$ and therefore $H = \cup R_n$. Suppose some $R_n = H$, $R_n = S + (Z \cap H)y$ where $y \in Q$. Then $(Z \cap H)y = H$. Since $e \in H$, $y^{-1} \in Z \cap H$ and thus

$y \in Z \cap H$. As $Z \cap H$ is a subalgebra of $H, H \subset Z$ and B_1 is commutative. Therefore $R_n \neq H$. By the Baire category theorem we have a contradiction as H is of the second category.

Suppose $*$ is symmetric (Hermitian-real). Consider only those involutions $'$, where $x' = y^{-1}x^*y$ with $y \in P^+$. By Lemmas 2.18 and 2.19, each $'$ is symmetric (Hermitian-real). If there were only a denumerable many such involutions the first argument above would show that there are only denumerably many closed linear manifolds of H of the form $(Z \cap H)y$; $y \in P^+$. Let $\{E_n\}$ be that collection and form $\{R_n\}$ as in the earlier argument. We now have $P^+ \subset \cup E_n$. If $w \in H$, since $*$ is Hermitian-real, $\lambda e + w \in P^+$ for sufficiently large real λ. Thus $H = \cup R_n$. The balance of the argument is exactly as given earlier.

The result is Theorem 2.20 in false if B is commutative, see Example 2.6.

2.21 THEOREM. *Let B be a Banach algebra with a continuous involution $*$ which is symmetric (Hermitian-real). If B is not commutative then there exists a continuous symmetric (Hermitian-real) involution $'$ such that $'* \neq *'$.*

Proof. Adjoin an identity e to B forming B_1 and extend $*$ to B_1 in the usual way. It is enough to show by Lemmas 2.18 and 2.19 that there exists $y \in P^+$, $x' = y^{-1}x^*y$, $x \in B$, where $'* \neq *'$. Suppose $'* = *'$ for all such y. A simple computation shows $y^2 \in Z$, (the center of B_1) for all $y \in P^+$. Since every $u \in P^+$ can be written as $u = v^2$, $v \in P^+$, then $P^+ \subset Z$. Let $w \in H$, $\|e - w\| < 1$. Then w^{-1} exists and $w = h^2$ for some $h \in H$. Since $sp(h)$ is real, $w \in P^+$. Hence Z contains a ball of H. Consequently, as Z is a linear space, $Z \cap H = H$. This shows that B_1 is commutative which is a contradiction.

This result is false if B is commutative. See Example 2.6. We can improve Theorem 2.21 for B semi-simple.

2.22 THEOREM. *Let B be a semi-simple Banach algebra with a continuous symmetric involution $*$. Let $'$ be any symmetric involution on B such that $'* = *'$. Then $' = *$. If B is not commutative there exists non-denumerably many symmetric involutions which do not permute with $*$.*

Proof. We show first that $' = *$ if and only if $x' = -x^*$ implies $x = 0$. Given any $z \in B$ consider $y = z - z'^*$. Then $y^* = -y'$ so that $y = 0$ and $z' = z^*$.

Suppose that $x' = -x^*$. Then $xx' = -xx^*$. By symmetry, $sp(xx^*) = (0)$. By Theorem 2.16, $xx^* = 0$. Since $*$ is proper by Corollary 2.17, $x = 0$. Therefore $' = *$.

If B is not commutative Theorem 2.20 guarantees the existence of non-denumerably many symmetric involutions $^\#$ each different from *. By the above, $^{\#*} \neq {}^{*\#}$ for each such $^\#$.

3. Involutions on special algebras. For B^*-algebras, H^*-algebras and semi-simple annihilator algebras we obtain more detailed properties of involutions. We start with the B^* and the H^* cases.

Any involution $'$ on a Banach algebra B is a real-linear operator on B. If $'$ is so considered we denote its norm as an operator by $\|(')\|$.

Consider a B^*-algebra B. The defining involution * is symmetric. (See [11, p. 281].) Also the defining involution in an H^*-algebra is symmetric, [5, p. 404], (or see Theorem 3.8 below).

3.1. LEMMA. *Let B be a B^*-algebra and $'$ be any involution on B. Then $\|('^*{}')\| = \|(')\|^2$ where * is the defining involution for B.*

Proof. Clearly $\|('^*{}')\| \leqq \|(')\|^2$. Take any $x \in B$ and set $x = y^{*\prime}$, $y = x'^*$. Then

$$\|x\|^2\|('^*{}')\| \geqq \|xx'^{*\prime}\| = \|(yy^*)'\| \geqq \rho[(yy^*)']$$
$$= \rho(yy^*) = \|y\|^2 = \|x'^*\|^2 = \|x'\|^2 .$$

From this we see that $\|('^*{}')\| \geqq \|(')\|^2$.

3.1. LEMMA. *In a B^*-abgebra B, an involution is an isometry if and only if it permutes with the defining involution *.*

Proof. Let the involution $'$ permute with *. Then $'^*{}' = {}^*$ so that by Lemma 3.2, $\|(')\| = 1$. Then $\|x'\| = \|x\|$ for all $x \in B$.

Let $'$ be an isometric involution. Suppose first that B has an identity. Since $'^*$ is a linear isometric isomorphism, by [3, Lemma 8] $'^*$ permutes with *. From this it follows that $'^* = {}^*{}'$. Suppose that B has no identity. Let B_1 be the algebra obtained by adjoining an identity e to B. For $\lambda e + x$, λ scalar and $x \in B$ define

$$\|\lambda e + x\| = \sup_{\substack{\|y\| = 1 \\ y \in B}} \|\lambda y + xy\|$$

Then [11, p. 207], B_1 is a B^*-algebra with $(\lambda e + x)^* = \bar{\lambda} e + x^*$. We can also extend $'$ to B_1 by $(\lambda e + x)' = \bar{\lambda} e + x'$. Then $'$ is an involution on B_1. Also, since $'$ is an isometry on B,

$$\|(\lambda e + x)'\| = \sup_{\|y\| = 1} \|\bar{\lambda} y + x'y\| = \sup_{\|y\| = 1} \|\lambda y + yx\|$$

$$= \sup_{\|y\| = 1} \|\bar{\lambda} y + x^* y\| = \|\lambda e + x\|$$

Thus $'$ is an isometry on B_1 so that, by the above, $'^* = {}^*{}'$.

3.3. THEOREM. *Let B be a B^*-algebra or an H^*-algebra which is not commutative. Then B possesses symmetric involutions of arbitrarily large norm.*

Proof. Let B be a B^*-algebra. By Theorem 2.22, there exists a symmetric involution $'$ which does not permute with *. Then by Lemma 3.2, $\|(')\| > 1$. Set $U_1 = {}'{}^*{}'$ and for each $k > 1$ define U_k inductively by $U_k = (U_{k-1})(^*)(U_{k-1})$. Each U_k is easily seen to be an involution. Also, by Lemma 3.1, $\|U_k\| = \|U_{k-1}\|^2$ for $k > 1$, whereas $\|U_1\| = \|(')\|^2 > 1$. By Lemma 2.18, U_k is a symmetric involution.

Let B be an H^*-algebra $E(B)$ be the B^*-algebra of all bounded linear operators on B. The mapping $L: x \to L_x$ of B into $E(B)$ defined by $L_x(y) = xy$, $y \in B$ is a faithful *-representation of B. If $'$ is an involution on B it induces an involution $'$ on $L(B)$ by the rule $(L_a)' = L_{a'}$. Denote the norm of this involution on $L(B)$ by $|||(')|||$ (and the norm of $'$ as an involution on B by $\|(')\|$ as above). Since $L_{a*} = (L_a),^*$ $\|L_{a*}\| = \|L_a\|$. By [13, Corollary 5.5], B has the uniqueness or norm property. Since $\|x\|_1 = \|x'\|$ defines a complete norm on B([12, p. 1068], $\|(')\| = k < \infty$. Let $S_k(S_1)$ be the ball in B, center at the origin and radius $k(1)$. Then $S_k' \supset S_1$. Also, using the fact that * is an isometry on B, we have

$$\begin{aligned}\|L_{a'}\| &= \sup_{x \in S_1} \|a'x\| \leqq \sup_{x \in S_k} \|a'x'\| \leqq k \sup_{x \in S_k} \|xa\| \\ &= k \sup_{x \in S_k} \|a^* x^*\| = k^2 \|L_{a*}\| = k^2 \|L_a\| .\end{aligned}$$

Therefore

$$|||(')||| \leqq \|(')\|^2 . \qquad (1)$$

In particular $'$ is a continuous involution on $L(B)$. Let A be the closure of $L(B)$ in $E(B)$. The mapping $'$ of $L(B)$ onto $L(B)$ may be extended to an involution also denoted by $'$ of A onto A with the same norm and furthermore A is a B^*-algebra.

Now by Theorem 2.22 we can select an involution $'$ on B which does not permute with *. Then by Lemma 3.2 applied to * and $'$ on A, $|||(')||| > 1$. Starting with $'$ and * we form the sequence $\{U_k\}$ of involutions on B as above. Each U_k is symmetric. Since $|||(U_k)||| \to \infty$, $\|(U_k)\| \to \infty$ by (1).

The argument employed shows that if $^\#$ is any isometric involution on an H^*-algebra B then ${}^{\#*} = {}^{*\#}$. For by equation (1), $\|(^\#)\| = 1$ implies $|||(^\#)||| = 1$ whence we may apply Lemma 3.2 to $^\#$ and * on A. The converse is false. Let B be the set of all couples (x, y) of complex numbers with multiplication and addition coordinatewise. Define an inner product (α, β) for B when $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$ by $(\alpha, \beta) = x_1\bar{x}_2 + 2y_1\bar{y}_2$ and an involution on B by $(x, y)^* = (\bar{x}, \bar{y})$. This makes B an H^*-algebra in terms of the involution *. Define a new involution $'$ by $(x, y)' = (\bar{y}, \bar{x})$. It is easy to see that ${}'^* = {}^{*\prime}$ and that $'$ is not an isometry.

Then next result is an improvement in the B^*-case of Theorem 2.22 inasmuch as $'$ may be a proper involution.

3.4. THEOREM. *Let B be a B^*-algebra and $'$ any proper involution on B such that ${}'^* = {}^{*\prime}$ where * is the defining involution on B. Then $' = {}^*$.*

Proof. As in the proof of Theorem 2.22, it is sufficient to show that $x' = -x^*$ implies $x = 0$. Let $x' = -x^*$. Write $x = h + k$, $h \in H$, $k \in K$. Then $x' = h' + k'$ and $x^* = h - k$. Also $h'^* = h^{*\prime} = h'$ so $h' \in H$. Likewise $k' \in K$. We have the decomposition

$$0 = x' + x^* = (h + h') + (k' - k)$$

so that $h' = -h$ and $k' = k$.

Consider the closed subalgebra R generated by h. R is a commutative B^*-algebra. Since $'$ is an isometry on B (Lemma 3.2) and $h' = -h$ we see that $R' = R$. It follows from Theorem 2.5 that $' = {}^*$ on R. Thus $h' = h$ and $h = 0$. By considering the closed subalgebra generated by k and arguing in a like manner we see that $k = 0$. Therefore $x=0$.

Theorem 3.4 holds for H^*-algebras. We do not prove this here as the fact is a consequence of Theorem 2.2 and Theorem 3.8.

We turn to some results for algebras with minimal ideals.

We shall have occasion to extend (in our context) the following result due to Rickart [14, p. 29].

3.5. THEOREM. (Rickart). *Let R be a ring and $x \to x^*$ be a mapping of R onto R of period two with $(xy)^* = y^*x^*$ and $xx^* = 0$ implying $x = 0$. Let I be a minimal right (left) ideal of R. Then there exists a unique idempotent e, $e = e^*$, such that $I = eR (I = Re)$.*

3.6. THEOREM. *Let B be a Banach algebra. Let $'$ and * be two proper involutions on B such that ${}'^* = {}^{*\prime}$ and let I be a minimal right ideal. Then there exists a unique idempotent e, $e = e^* = e'$ such that $I = eB$.*

Proof. By Theorem 3.5 there exists a unique idempotent e, $e = e^*$

such that $I = eB$. We have to show $e' = e$. By the Gelfand-Mazur theorem eBe consists of all scalar multiples of e. We may then write $ee'e = \lambda e$, where λ is a scalar. Since $'^* = {}^{*}{}'$, $(ee'e)^* = ee'e = \bar{\lambda}e$ whence λ is real. Let a be real and set $w = ae + ee'$. Simple computations give $ww^* = (a^2 + 2a\lambda + \lambda)e$ and $ww' = (a^2 + 2a + \lambda)ee'$. Note also that $w = 0$ implies $(a + 1)ee' = 0$, as e' is an idempotent. Thus $w = 0$ implies $a = -1$, as $'$ is proper. Suppose $\lambda < 1$. The choice $a = -1 + (1-\lambda)^{1/2}$ makes $ww' = 0$ and thus $w = 0$. Then $a = -1$ which is impossible. Hence $\lambda \geqq 1$. Suppose $\lambda > 1$. The choice $a = -\lambda + (\lambda^2 - \lambda)^{1/2}$ makes $ww^* = 0$, $w = 0$. Then $(\lambda - 1)^2 = \lambda^2 - \lambda$ or $\lambda = 1$. This contradiction shows that $\lambda = 1$.

Therefore $ee'e = e$. Then $(e - ee')(e - ee')' = 0$ so that $e = ee'$. Applying $'$ to this relation we have $e' = ee'$ and $e = e'$.

3.7. THEOREM. *Let B be a semi-simple Banach algebra with a Hermitian-real involution $*$. Let I be a minimal right (left) ideal. Then there exists a unique self-adjoint idempotent e such that $I = eB(I = Be)$.*

Proof. We show first that for any idempotent j, $jj^* = 0$ implies $j = 0$. For $j - j^* \in K$ so that $j - j^*$ has a quasi-inverse y,

$$j - j^* + y - (j - j^*)y = 0 .$$

If $jj^* = 0$, left multiplication by j shows that $j = 0$.

Let I be a minimal right ideal. Then there exists an idempotent j such that $I = jB$. Now $jj^* \neq 0$ and $jj^*j = \lambda j$ for some scalar λ. Then $jj^*jj^* = \lambda jj^*$ and, by taking $*$ of both sides we see that λ is real. As above there exists y, $j - j^* + y - (j - j^*)y = 0$. Multiplication on the left and right by j yields $(1 - \lambda)j + jj^*yj = 0$. If $\lambda = 0$ then $j^*jj^* = 0$ so that multiplication on the left by j^* yields $j^*j = 0$. This is impossible. Then $e = \lambda^{-1}jj^*$ is a self-adjoint idempotent generator for I. The uniqueness of e follows as in [14, p. 30].

For an algebra B and a subset S let $L(S)(R(S))$ denote the left (right) annihilator of S in B. Following Bonsall and Goldie [1]. We call a Banach algebra B an *annihilator algebra* if B has no absolute left or right divisors of zero and if $L(I) \neq (0)$ $(R(I) \neq (0))$ for each proper closed right (left) ideal. By [4, p. 697] every H^*-algebra is an annihilator algebra.

3.8. THEOREM. *Let B be a semi-simple annihilator algebra with an involution $*$. Then the following are equivalent.* (a) *$*$ is symmetric.* (b) *$*$ is Hermitian-real and* (c) *$*$ is proper.*

Proof. If $*$ is symmetric then $*$ is Hermitian-real by [5, p. 402]. Let (b) hold. Suppose that $x^*x = 0$ for some $x \in B$. If $x \neq 0$ then xB is a proper right ideal which contains a minimal right ideal I by [1, p. 158]. For some idempotent $e, e = e^*$, $I = eB$ by Theorem 3.7. There exists $y \in B$ such that $e = xy$. Then $e = e^*e = y^*x^*xy = 0$, which is impossible. Therefore (b) implies (c).

Suppose that $*$ is proper. If $*$ is not symmetric there exists $x \in B$ where $-x^*x$ has no quasi-inverse and $I = \{-x^*xy - y \mid y \in B\}$ is a proper regular right ideal of B. Now I is contained in some regular maximal right ideal M. By hypothese $L(M)$ is a non-zero left ideal and therefore, by [1, p. 158] and Theorem 3.5, contains a self-adjoint idempotent e. Then $e(-x^*xy - y) = 0$ for all y. Also $(-ex^*x - e)y = 0$ for all y. Therefore $e = -ex^*xe = -ex^*(ex^*)^*$. The idempotent e can be chosen as a generator of a minimal right ideal so that we can write $exe = \alpha e$ where α is a scalar. Let $\alpha = a + bi$ where a, b are real and set $c = a + (a^2 + 1)^{1/2}$. Then $(ex^* - ce)(ex^* - ce)^* = (-1-2ca + c^2)e = 0$. Hence $ex^* = ce$, $xe = (ex)^* = ce$ and $-e = ex^*xe = c^2e$. Thus $c^2 = -1$ which is a contradiction. Therefore $*$ is symmetric.

3.9. EXAMPLE. Let B be the semi-simple Banach algebra whose elements f are functions of two complex variables x_i, $j = 1, 2$, such that each $f \in B$ is analytic for $|x_i| < 1$ and continuous for $|x_i| \leqq 1$. Define f^* by $f^*(x_1, x_2) = \overline{f(\bar{x}_1, \bar{x}_2)}$ and f' by and f' by $f'(x_1, x_2) = \overline{f(\bar{x}_2, \bar{x}_1)}$. Then it is easily verified that $*$ and $'$ are proper involutions, that $'* = *'$ but $' \neq *$.

We call a Banach algebra *simple* if it is semi-simple and has no proper closed two sided ideals. By the *socle* of a semi-simple algebra A with minimal one sided ideals we mean the algebraic sum of its minimal left (right) ideals. For properties of the socle see [2, Chapter 4].

Let $I_j, j = 1, 2$ be distinct minimal right ideals in a simple Banach algebra B, $I_j = e_jB$, with $e_j = e_j^2 \neq 0$, $j = 1, 2$. A slight variation of the argument used by Kaplansky in the case $e_1e_2 = e_2e_1 = 0$ [4, p. 693] shows that e_1Be_2 is one-dimensional. (See also [11, p. 293].)

3.10. THEOREM. *Let $'$ and $*$ be two permuting proper involutions on a simple Banach algebra with non-zero socle. Then $' = *$.*

Proof. As in the proof of Theorem 2.22, we must show that if $x' = -x^*$ then $x = 0$. Take such an element x. Let I be any minimal right ideal. By Theorem 3.6 there exists an idempotent e, $I = eB$, $e = e' = e^*$. Consider $exe = \lambda e$ where λ is a scalar. Then $0 = e(x' + x^*)e = 2\bar{\lambda}e$. Therefore $\lambda = 0$ and $exe = 0$. Let I_1 be any other minimal right ideal, $I_1 = e_1B$, $e_1^2 = e_1 = e_1' = e_1^*$. We shall show that $exe_1 = 0$. Note that the socle of B is dense in B.

Suppose that $exe_1 \neq 0$. Now since B is simple, eBe_1 is one dimensional. Let w be any non-zero element of eBe_1. Write $exe_1 = \lambda w$, $\lambda \neq 0$. Then $0 = e_1(x' + x^*)e = \bar{\lambda}(w' + w^*)$. Thus $w' + w^* = 0$. It follows that $e_1(y' + y^*)e = 0$ for all $y \in B$. In particular $y = e_1$ shows $e_1 e = 0 = ee_1$.

Write $x = h + k$, $h \in H$, $k \in K$. As in the proof of Theorem 3.4, $h' = -h^*$, $k' = k = -k^*$. Since $exe_1 \neq 0$ then either $ehe_1 \neq 0$ or $eke_1 \neq 0$. Suppose that $eke_1 \neq 0$.

Set $u = eke_1$. Then $u' = e_1 ke$. We have $uu' = \alpha e$, $u'u = \beta e_1$ where α and β are non-zero scalars. Since uu' is self-adjoint under $'$, α and β are real. Clearly $\alpha u = uu'u = \beta u$. Then $\alpha = \beta$. Suppose $\alpha = -\gamma^2 < 0$. Then $(u + \gamma e)(u + \gamma e)' = 0$ as $ee_1 = 0$. This implies that $u = -\gamma e$ which is impossible. Set $v = \alpha^{-1/2} u$. Then $vv' = e$ and $v'v = e_1$. Consider the matrix units e_{ij} for the algebra M_2 of all 2×2 matrices over the complex field. If we make e correspond with e_{11}, v with e_{12}, v' with e_{21} and e_1 with e_{22}, we see that the subalgebra A generated by e_1, v, v' and e_1 is a copy of M_2. Also $A' = A^* = A$. By Theorem 3.8, $'$ and * are symmetric on A so that $' = {}^*$ on A by Theorem 2.22. But $0 \neq u' = -u^*$. Therefore $eke_1 = 0$.

If $ehe_1 \neq 0$ set $u = ehe_1$, $u^* = e_1 he$ and proceed in the same way using * as $'$ was employed above. Therefore $exe_1 = 0$.

It follows that $exQ = 0$ where Q is the socle of B. Consequently $exB = 0$ and $ex = 0$. Since e is an idempotent generator for an arbitrary minimal right (or left) ideal, $Qx = 0$ and $x = 0$. This completes the proof.

4. Real involutions on commutative Banach algebras. In this section B will denote a commutative Banach algebra over the complex field. The space of maximal regular ideals of B is denoted as earlier by $\mathfrak{M}$. With respect to a real involution $'$, we denote

$$\{x \in B \mid x' = x\} \text{ by } H, \text{ and } \{x \in B \mid x' = -x\} \text{ by } K.$$

The item that is not available for real involutions as it is for involutions is that $K = iH$. Our object in this section is to relate the real involution structure in B to certain properties of $\mathfrak{M}$.

4.1. LEMMA. *A commutative semi-simple Banach algebra is infinite dimensional if and only $\mathfrak{M}$ is infinite.*

Proof. By [9, p. 59] there is no loss in assuming that B has an identity. Suppose B is infinite dimensional. By a result of Kaplansky [6, p. 379] there exists an element of B with infinite spectrum. Thus $\mathfrak{M}$ is infinite.

Suppose $\mathfrak{M}$ is infinite. By arguments of Šilov [15, p. 37], there exists an element $w \in B$ with infinite spectrum. Then B is infinite dimensional for otherwise each element in B satisfies a polynomial equation and thus has finite spectrum.

4.2. THEOREM. *Let B be an infinite dimensional commutative semi-simple Banach algebra with a real involution $'$. Then H is infinite dimensional.*

Proof. Suppose H is finite dimensional. Now powers of elements in H are also in H. Thus each $x \in H$ satisfies a polynomial equation with real coefficients. Let $f \in B$, $f = h + k$, $h \in H$, $k \in K$. Since $(f - h)^2 \in H$, we see that $f - h$ satisfies a real polynomial equation, as does h. Standard arguments show that f also satisfies a polynomial equation and hence has finite spectrum. Since f was arbitrary, the result of Kaplansky [6, p. 376] cited above implies B finite dimensional, consequently H must be infinite dimensional.

4.3. COROLLARY. *If B is a commutative Banach algebra with a real involution $'$, and $\mathfrak{M}$ is infinite, then H is infinite dimensional.*

Proof. Consider B/R where R is the radical of B. Since $R' = R,'$ defines a new real involution on B/R, for if $a - b \in R_1$ the $a' - b' \in R$. Let H_0 be the set of self adjoint elements of B/R. By Theorem 4.2 H_0 is infinite dimensional. If π is the natural mapping of B onto B/R, we have $\pi H = H_0$. The inequality in one direction is immediate. On the other hand suppose $a + R \in H_0$, with $a = h + k$, $h \in H$, $k \in K$. Then $a' + r_1 = a + r_2$, with $r_i \in R$, and $h - k + r_1 = h + k + r_2$. Thus $k \in R$ and $h \in a + R$, so $\pi h = a + R$. Thus H is infinite dimensional.

4.4. LEMMA. *Let A be a semi-simple algebra over the reals and I a finite-dimensional two-sided ideal of A. Then $A = I \oplus L(I)$ where $L(I) = R(I)$ is a two-sided ideal.*

Proof. I is semi-simple and finite-dimensional so I has an identity e. Now $L(I) = R(I)$ by algebra [1, p. 159].

Now clearly $I = eA = Ae$ and $e^2 = e$. By the Peirce decomposition

$$A = eA \oplus (1 - e)A = Ae \oplus A(1 - e)$$

where $(1 - e)A = R(I) = L(I) = A(1 - e)$.

4.5. THEOREM. *Let A be a semi-simple algebra over the reals. Then there exists an automorphism on A with period two and K finite-dimensional if and only if A possesses a finite-dimensional ideal I on which there is an automorphism of period two.*

Proof. Suppose an automorphism of A of period two exists with K finite-dimensional. Denote it by *. Let $f_1, \cdots, f_n$ be a basis for K. Let I be the two sided ideal generated by K. We show that I is finite-dimensional.

Let $x \in A$, $x = h + k$, $h \in H$, $k \in K$. Let $\sum a_i f_i = y \in K$. Clearly $hy \in K$. Then if $k = \sum b_i f_i$,

$$xy = hy + ky = hy + \sum b_j f_j \sum a_i f_i \, .$$

This shows that xy lies in the finite-dimensional subspace of A generated by $f_1, \cdots, f_n$ and the $f_i f_j$, $i, j = 1, \cdots, n$. Likewise yx lies in this subspace. Hence I is finite-dimensional. In fact, clearly I equals the linear space generated by $f_1, \cdots, f_n$ and the $f_i f_j$. Clearly $I^* = I$.

Suppose conversely that A has a finite-dimensional ideal I and there exists an automorphism $x \to x'$ of period 2 on I. By Lemma 4.4 we can write $A = I_1 \oplus I$ where I_1 is an ideal. Define for $x = u + v$, $u \in I_1$, $v \in I$

$$x^* = u + v' \, .$$

Then $x \to x^*$ is an automorphism of period two. For this we need only check $(xy)^* = x^* y^*$. Note if $x = u + v$, $y = r + s$ in the decomposition that $us \in I_1 \cap I = (0)$ and likewise $0 = vr = us' = v'r$,

$$(xy)^* = (ur + vs)^* = ur + (vs)' = ur + v's', \text{ and}$$
$$(x^* y^*) = (u + v')(r + s') = ur + v's' \, .$$

Also $K \subset I$, for if $(u + v) = -(u + v)^* = -u - v'$ then, since we have a direct sum, $u = -u$, $v = -v'$. Thus $u = 0$ and $K \subset I$.

4.6. THEOREM. *Let B be a commutative semi-simple Banach algebra. Then the following are equivalent:*

(1) *There exists a real involution with K finite-dimensional.*

(2) *There exists a finite-dimensional ideal I of B.*

(3) *$\mathfrak{M}$ has isolated points.*

Proof. By the preceding theorem (1) implies (2). We show that (2) implies (1). By Theorem 4.5, it is sufficient to show that I has a real involution. But I is a semi-simple finite-dimensional commutative Banach algebra with identity. Let $\mathfrak{M}_1$ be the space of maximal ideals of I. But Lemma 4.1, $\mathfrak{M}_1$ is finite. Then I is isomorphic to $C(\mathfrak{M}_1)$ and thus there is a natural involution on I. Thus (2) implies (1).

We next show that (1) implies (3). For consider $f \in K$. Since $f^3, f^5, \cdots$ are in K and K is finite dimensional, f satisfies a polynomial equation with real coefficients. Thus $f(M)$ takes on only a finite number of values.

Let $e_1, \cdots, e_n$ be generators for K. Let $M_0 \in \mathfrak{M}$ where $e_1(M_0) \neq 0$.

We show M_0 is an isolated point of $\mathfrak{M}$. Let $E = \{M \in \mathfrak{M} \mid e_k(M) = e_k(M_0),\ k = 1, \cdots, n\}$. It is sufficient to show that $E = \{M_0\}$. For suppose this has been established. For each k let $c_{k,1}, \cdots, c_{k,n(k)}$ be the distinct values of $e_k(M)$, $c_{k,1} = e_k(M_0)$. Let $\varepsilon_k = \max_j |c_{k,j} - c_{k\,1}|/2$ or if $r(k) = 1$ set $\varepsilon_k = |c_{k,1}|/2$. Let $U = \{M \in \mathfrak{M} \mid |e_k(M) - e_k(M_0)| < \varepsilon,\ k = 1, \cdots, n\}$ where $\varepsilon = \min \varepsilon_k$. This neighborhood contains only M_0.

Suppose E contains $M_1 \neq M_0$. If $g \in K$, $g(M_1) = g(M_0)$ since $e_1, \cdots, e_n$ generate K. Let $h \in H$. Then $he_1 \in K$ and $he_1(M_0) = he_1(M_1)$. Since $e_1(M_0) = e_1(M_1) \neq 0$, $h(M_0) = h(M_1)$. Thus $f(M_0) = f(M_1)$ for all $f \in B$. This is impossible.

Lastly we show that (3) implies (2). For consider B_1 the algebra with 1 adjoined to B. Since $\mathfrak{M}$ has as isolated point M_0 so does the maximal ideal space $\mathfrak{M}_1$ of B_1. Then by a result of Šilov [16], B_1 contains the characteristic function ϕ of M_0. Since $\phi(B) = 0$, $\phi \in B$. It is easy to see that ϕ generates a 1-dimensional ideal of B.

4.7. THEOREM. *Let B be a complex commutative semi-simple Banach algebra with an identity. Let $x \to x^*$ be a real involution. Then we can write $B = I_1 \oplus I_2$, with I_j an ideal such that $I_j^* = I_j$, $j = 1, 2$ and with * complex linear on I_1 and conjugate linear on I_2.*

Proof. In the notation of Theorem 2.2, $\mathfrak{M} = S_1 \cup S_2$, where S_j, $j = 1, 2$ are open and closed. By a theorem of Šilov [16], there exist $e_j \in B$, $j = 1, 2$, such that $e_j(S_j) = 1$ while $e_1(S_2) = e_2(S_1) = 0$.

Let $I_j = e_j B$. Clearly $B = I_1 \oplus I_2$. Let $x \in I_1$. For $M \in S_2$,

$$x^*(M) = \overline{x(\sigma(M))}\ .$$

But $\sigma(M) \in S_2$ by the remarks at the end of the proof of Theorem 2.2, and thus $x^*(M) = 0$. Then $x^* = e_1 x^*$, for $x^* \in I_1$ and $I_1^* = I_1$. Likewise $I_2^* = I_2$.

For $x \in I_1$,

$$x^*(M) = 0 = x(M),\ M \in S_2\ ,$$
$$x^*(M) = x(\sigma(M)),\ M \in S_1\ ,$$

so clearly $x \to x^*$ is complex linear on I_1.
For $x \in I_2$,

$$x^*(M) = x(M) = 0,\ M \in S_1\ ,$$
$$x^*(M) = \overline{x(\sigma(M))},\ M \in S_2\ ,$$

Thus $x \to x^*$ is conjugate linear on I_2.

We call an algebra A *decomposable* if $A = I_1 \oplus I_2$ with $I_j \neq (0)$ an ideal, $j = 1, 2$. Otherwise we call A *indecomposable*.

4.8. THEOREM. *Let A be a semi-simple algebra over the complexes with an identity e and suppose that A has a conjugate linear automorphism $*$ of period two. A necessary and sufficient condition that A is indecomposable is that* (1) *every real linear automorphism of period two on A is either complex linear or conjugate linear and* (2) *every central idempotent of A is self adjoint under every real linear automorphism on A of period two.*

Proof. Say A is decomposable, so $A = I_1 \oplus I_2$ wtih I_j an ideal, $j = 1, 2$. Let $e = e_1 + e_2$ with $e_j \in I_j$, $j = 1, 2$. Then e_j is a central idempotent so from (2) $e'_j = e_j$ for any real automorphism $'$ of period two on A. Let $x \in I_j$, where $x' = x_1 + x_2$ with $x_k \in I_k$, $k = 1, 2$. Thus $e_j x' = e_j x_j = x_j$, and $x'_j = x e_j = x$ since $x \in I_j$. Whence $x' = x_j$, and $I'_j = I_j$, $j = 1, 2$ for any $'$.

Let $x = x_1 + x_2$, $x_j \in I_j$, $j = 1, 2$. Define $x' = x_1 + x_2^*$. Clearly $'$ has period two and is a real automorphism on A. Also for λ complex, $(\lambda x_1)' = \lambda x'_1$ while $(\lambda x_2)' = (\lambda x_2)^* = \bar{\lambda} x'_2$. Thus condition (1) is violated and we have a contradiction.

Suppose now that A is indecomposable. The only central idempotents of A are 0 and e. For if f is a central idempotent $A = fA \oplus (e - f)A$ is a decomposition of A. Clearly both 0 and e satisfy the condition in (2). Let $'$ denote a real automorphism of period two on A. From $e + (ie)^2 = 0$, we have $e + ((ie)')^2 = 0$. Let $u = 2^{-1}[(ie)' - (ie)]$, and $v = 2^{-1}[(ie)' + (ie)]$. Thus u and v are central. One easily verifies that

$$u^2 = -\frac{e + (ie)'(ie)}{2}, \qquad v^2 = -\frac{e - (ie)'(ie)}{2}$$

$$u^4 = \frac{e + (ie)'(ie)}{2}, \qquad v^4 = \frac{e - (ie)'(ie)}{2}.$$

Thus $-u^2$ is a central idempotent so either $-u^2 = e$ or $u^2 = 0$. If $-u^2 = e$, then $v^2 = 0$. Since A is semi-simple and v is central $(vA)^2 = (0)$ and $v = 0$. Likewise if $u^2 = 0$, $u = 0$. Thus $(ie)' = \pm ie$. Since e is the unit for A, condition (1) is satisfied.

4.9. COROLLARY. *Let B be a semi-simple complex commutative Banach algebra with identity e, and suppose that B has a conjugate linear automorphism of period* 2. *Necessary and sufficient conditions that $\mathfrak{M}$ be connected are that* (1) *any idempotent of B is self adjoint under each real linear automorphism of period two, and* (2) *each real linear automorphism of period two is complex linear or conjugate linear.*

Proof. Say $\mathfrak{M}$ is connected. Then by the result of Šilov [16], B is indecomposable. Hence the two conditions above hold.

Suppose $\mathfrak{M}$ is not connected. Then B is decomposable by Šilov's theorem. This contradicts the conditions of the Theorem 4.8.

Added in Proof. The use of Theorem 3.1 in a paper by R. Arens, *The maximal ideals of certain function algebras*, Pacific. J. Math. 8 (1958), 641-648 permits a simpler figure than that of Fig. 1 to be employed in Example 2.8. The paper of Arens appeared after the present paper had been accepted for publication.

References

1. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc., **14** (1954), 154-167.
2. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. **37**, (Providence, R. I. (1956).
3. R. V. Kadison, *Isometries of operator algebras*, Ann. of Math (2) **54** (1951), 325-338.
4. I. Kaplansky, *Dual rings*, Ann. Math. (2) **49** (1948), 689-701.
5. ———, *Normed algebras*, Duke Math J., **16** (1949), 339-417.
6. ———, *Ring isomorphisms on Banach algebras*, Can. J. Math. **6** (1954), 374-381.
7. ———, *Topological algebra*, Department of Mathematics, University of Chicago, 1952, mimeographed notes.
8. B. V. Kerékjártó, *Uber die periodischen Transformationen der Kreisscheibe und der Kugelflache*, Math. Ann. **80** (1919), 36-38.
9. L. H. Loomis, *An introduction to abstract harmonic analysis*, D. Van Nostrand Co., New York, 1953.
10. M. A. Naimark, *Involutive Algebren*, Sowjetische Arbeiten zur Funktionalanalysis, Berlin (1954), 89-196.
11. ———, *Normed rings*, Moscow, 1956. (Russian).
12. C. E. Rickart. *The singular elements of a Banach algebra*, Duke Math. J. **14** (1947), 1066-1077.
13. ———, *The uniqueness of norm problems in Banach algebras*, Ann. of Math. **51** (1950), 615-628.
14. ———, *Representation of certain Banach algebras on Hilbert space*, Duke Math. J. **18** (1951), 27-39.
15. G. E. Šilov, *On normed rings with one generator*, Mat. Sbornik 21 (63) (1947), 25-47 (Russian)
16. ———, *On the decomposition of a commutiative normed ring in a direct sum of ideals*, Mat. Sbornik 32 (74) (1953), 353-364, (Russian)
17. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ. 28, New York 1942.

UNIVERSITY OF OREGON