ON NONLINEAR POSITIVE OPERATORS

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Introduction. The purpose of the present paper is to apply the well known Schauder fixed point theorem, in its general from due to Tychonov [8], to the situation of nonlinear (or rather, not necessarily linear) maps defined on (or on a subset of) the "positive" cone in a partially ordered locally convex linear space. Throughout this paper, no use is made of possible linear properties of the maps under consideration. As far as the author is informed, there is little history to the study of such mappings; the only work done seems to be contained in papers by Krein-Rutman [2], Rothe [9] and Morgenstern [3]. In [2], the Schauder theorem is largely applied to linear maps (where it can be avoided) and a few nonlinear cases[1]. In [4], the author paid attention mainly to the case of linear compact maps in general locally convex spaces. At the end of that paper, with a somewhat sketchy proof, a general nonlinear theorem[2] is stated which however seems to need some improvement.

In this paper, the essential proposition resulting from the fixed point theorem is stated in the form of three different theorems to throw some light on potential ways of argument. While Th. 1, depending on a special convexity argument, is of a different character, Th. 2 is almost a special case of Th. 3. But as Banach spaces with normal order cones (with which Th. 2 is concerned) seem to be the most important ones in nonlinear analysis, it might be useful to have the theorem stated separately, a much simpler proof than that of Th. 3 going with it. Applications have been selected so as to furnish a non-trivial example to each of the three theorems, the one to Th. 1 showing that it is not always fruitful to restrict attention to normed topologies. It is understood that each example constitutes a new result in its respective field.

Preliminary material. In the present section, we are going to collect some theorems and definitions on which argumentation will be primarily based in the sections to follow. The main tool will be the

FIXED POINT THEOREM (Tychonov). *Let E be a locally convex linear space, M a convex compact subset of E. If T is a continuous map on M into M, then T has a fixed point $x_0 \in M$.*

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[1] Also, considerations are restricted to Banach spaces.

[2] Satz 3.1. This is restated and proved in this paper as Th. 3. The additional assumption to be made in [4] may be any one of hypotheses α,β stated with Th. 3 of the present paper.

For the proof, see [8]. To make this theorem more easily applicable to mappings that carry sets not necessarily compact into compact ones, we state the following slightly more general

FIXED POINT THEOREM (2nd form). *Let M be a complete convex set in E. If T is continuous on M into M such that $T(M)$ is relatively compact, then T has a fixed point $x_0 \in M$.*

Proof. Let M_1 be the closed convex hull of $T(M)$. As M is complete, M_1 is compact (Bourbaki [1], p. 81) and since obviously $T(M_1) \subset M_1$, Tychonov's theorem yields the desired result.

Let E be a linear space over the real scalar field. A partial ordering of E is a binary relation " $<$ " such that

1 $$x < x \text{ for all } x \in E .$$

2 $$\{x < y \,\&\, y < z\} \Rightarrow x < z .$$

3 $$\{x < y \,\&\, y < x\} \Rightarrow x = y .$$

Such an ordering is said to be compatible with the linear structure of E if in addition

4 $$\{x > 0 \,\&\, \lambda \geqq 0\} \Rightarrow \lambda x > 0 .$$

5 $$x > y \Rightarrow x + z > y + z \qquad \text{for all } z \in E .$$

The set of all $x \in E$ such that $x > 0$ is a convex cone C which contains its vertex 0, and which is proper (i.e. $C \cap -C = \{0\}$). C will be referred to as the positive cone with respect to a given partial ordering of E[3]). Conversely, each cone in E with the listed properties defines a partial ordering satisfying axioms 1 through 5, $x < y$ meaning $y - x \in C$.

Let E be a linear space, partially ordered by some such cone C. If T is a mapping defined on a subset of C, we will say T is positive whenever the range of T is in C. If E is, moreover, a topological space, T will be called strictly positive if $T(x_n) \to 0$ implies $x_n \to 0$ for any sequence $\{x_n\}$ in the domain of T.

Examples.

1. Let E be Hilbert space L_2 (0, 1) in its natural order, i.e. the positive cone C consisting of all elements $f : f(t) \geqq 0$, $t \in [0, 1]$. The positive mapping, defined on all of E,

$$T(f) = \int_0^t f^2(\tau) d\tau$$

[3] In this paper, all orderings are understood to be compatible with the linear structure of the space involved. Also, we exclude the trivial case $C = \{0\}$.

is not strictly positive. Take $f_n = \sqrt{nt^{n-1}}$, then $\|f_n\| = 1$. Now as

$$T(f_n) = t^n, \|T(f_n)\| = \frac{1}{\sqrt{2n+1}} \to 0 .$$

2. Let E be the B-space of continuous functions on the unit interval, with its natural partial order. Let $K(t, \tau)$ be $\geqq 0$ (but $\not\equiv 0$) and continuous on the unit square. If $P(z)$ is a polynomial with non-negative coefficients,

$$T(f) = \int_0^1 K(t, \tau)P[f(\tau)]d\tau$$

is strictly positive if and only if the constant term in P is > 0.

3. Denote by $\{E_\alpha\}$ a collection of topological linear spaces, each E_α being partially ordered by some positive cone C_α. Then the product space $E = \prod_\alpha E_\alpha$ is ordered by $C = \prod_\alpha C_\alpha$. Let A_α be a positive map on E into E_α, and consider the map

$$A(x) = (\cdots, A_\alpha(x), \cdots)$$

on E into E. Then A is strictly positive on C if and only if to each α, there is a $\beta(\alpha)$ such that $A_\beta(x) \to 0$ (in E_β) implies $x_\alpha \to 0$ (in E_α). In particular, if $A_\alpha(x) = A_\alpha(x_\alpha)$, then A is strictly positive if and only if each A_α is.

I. Morgenstern's theorem. If E is the Banach space L_1, partially ordered by the positive cone $C = \{f: f(t) \geqq 0\}$, it turns out that the intersection of C with the unit sphere $S = \{f: \|f\| = 1\}$ is convex. This is true for any abstract L-space or, more generally, for any normed space in which the norm is additive on C. To this situation Morgenstern [3] applied Schauder's fixed point theorem. He obtained the following

THEOREM (Morgenstern)[4]. *Let E be a Banach space, partially ordered by a positive cone C which is closed and on which the norm is additive. Then if T is continuous and strictly positive on $C \cap \{\|x\| = c\}$, $c > 0$, mapping this set into a compact one, there is some $\lambda > 0$ and $x \in C$ such that $\lambda x = T(x)$, $\|x\| = c$.*

The proof is readily obtained by applying the fixed point theorem (2nd form) to the map $cT(x)/\|T(x)\|$ on the set $C \cap \{\|x\| = c\}$. However, it may be so arranged as to yield a much more general proposition.

[4] The theorem is stated in our terminology and a slightly more general form.

THEOREM 1. *Let E be a locally convex space, partially ordered by a complete cone C, and let T be a continuous, strictly positive transformation on C, mapping bounded sets into compact ones. Assume $H: f(x)=1$ is a closed hyperplane meeting C in a nonvoid, bounded set. Then to each $c > 0$, there is an $x \in C$ and $\lambda > 0$ with*

$$\lambda x = T(x) , \quad f(x) = c .$$

Proof. It follows from our assumptions that the continuous linear form $f(x)$ is > 0 at every non-zero point of C. For assume there is an $x_0 \in C, x_0 \neq 0$, with $f(x_0) = 0$. Then if $y_0 \in H \cap C$, we would have $f(y_0 + \mu x_0) = 1$ for all $\mu \geqq 0$ which contradicts the hypothesis that $H \cap C$ be bounded. It is now also clear that f cannot be < 0 on C. Applying the fixed point theorem (2nd form) to the map $cT(x)/f[T(x)]$ on the set $H \cap C$, we get the desired result letting $\lambda = f[T(x)]. c^{-1}$.

REMARK. We should point out the relation between Morgenstern's theorem and Th. 1. If, under the assumptions of the former, the norm coincides on C with a continuous linear form, then Morgenstern's theorem is a corollary of Th. 1. (This is the case in L_1, e.g.). Assume then, still under the assumptions of Morgenstern's theorem, that there is no such linear form. Now $C \cap \{\|x\| = c\}$ is convex ($c > 0$), so there is a closed hyperplane H separating this set from a convex open neighborhood of 0. Obviously $H \cap C$ is bounded and Th. 1 can be applied provided T is compact, continuous and strictly positive on C.

II. Banach spaces with normal positive cones. We will now extend Morgenstern's theorem to ordered Banach spaces in which the norm is not necessarily additive on the positive cone C. This assumption will be replaced by the weaker hypothesis that C is normal. A convex cone of vertex 0 in a normed space E is normal [5] if the topology of E is generated by a norm which is monotone (with respect to the order induced in E by C) on C. In terms of the given norm on $E, x \to \|x\|$, this amounts to saying there is a constant $\gamma > 0$ such that

$$\|x + y\| \geqq \gamma \|y\| \qquad \text{for all } x \in C, y \in C .$$

It can easily be checked that for all classical Banach spaces, the positive cones pertaining to their natural partial orders are normal ([5], p. 130).

THEOREM 2. *Let E be a normed space, partially ordered by a complete normal cone C. Let T be a strictly positive transformation, which is continuous and maps bounded subsets of C into compact ones. Then to each $c > 0$, there is $x \in C$ and $\lambda > 0$ with*

$$\lambda x = T(x) , \quad \|x\| = c .$$

Proof. Consider the mapping

$$x \to S_c(x) = T(x) + |c - \|x\|\,|y$$

for fixed $0 \neq y \in C$ and $c > 0$. This is a continuous map carrying bounded subsets of C into compact ones. T being strictly positive, we have $\inf\left\{\|T(x)\|: x \in C \,\&\, \|x\| \geqq \frac{1}{2}c\right\} = \varepsilon > 0$. Hence, C being a normal cone, we obtain

$$\inf_{x \in C} \|S_c(x)\| \geqq \gamma \cdot \inf_{x \in C} \sup(\|T(x)\|, |c - \|x\|\,|\,\|y\|) \geqq \gamma \sup\left(\varepsilon, \frac{1}{2}c\|y\|\right) > 0 .$$

Thus $x \to R_c(x) = cS_c(x)\|S_c(x)\|^{-1}$ maps $C \cap \{\|x\| \leqq c\}$ into a compact subset. So by the fixed point theorem (2nd form) there is an x in this subset with $x = R_c(x)$. Clearly $\|x\| = c$, and letting $\lambda = c^{-1}\|S_c(x)\|$ we have $\lambda x = T(x)$. Since $c > 0$ is arbitrary, the proof is complete.

III. A third theorem. The theorem presented in this section weakens the assumption in Th. 2 that E be normed and removes the hypothesis that C be a normal cone. Instead, we require either one of conditions α, β of hypothesis H (s. below) to hold. As the conclusion is only established for *some* continuous semi-norm $x \to p(x)$ on E (which, however, may be assumed to generate the topology of E if E is normed), Th. 3 is not a generalization of Th. 1 or 2. We start out with a

LEMMA. *If E is a locally convex space, C a closed proper convex cone in E of vertex 0, then there exists a continuous linear form on E, non-negative on C and > 0 at a given non-zero element of C.*

Proof. Let $0 \neq y \in C$. Since C is proper and closed, there is a convex open neighborhood U of $-y$ such that C and $\bigcup_{\lambda>0}\lambda U$ do not intersect. Hence there is a closed hyperplane H separating C and $\bigcup_{\lambda>0}\lambda U$. Obviously H contains 0, so has an equation $f(x) = 0$. After a potential change of sign, f will meet the requirement.

Now let E be any locally convex space, partially ordered by a complete positive cone. A mapping T, defined on a neighborhood of 0 in C into C, will be called of type "P" if it satisfies:

1. T is continuous and strictly positive.
2. There is a neighborhood U of 0 such that the image under T of $U \cap C$ is relatively compact.

Consider

HYPOTHESIS H. We will say that hypothesis H is satisfied if one of

the two following statements is true:

α. To each compact subset of C, there exists a continuous semi-norm which is > 0 at each non-zero point of that set.

β. T is positive-homogeneous of some degree $\sigma > 0$, i.e. $T(\lambda x) = \lambda^\sigma T(x)$ for $x \in C$ and $\lambda > 0$.

For instance, condition α is automatically fulfilled if there exists a continuous norm on E (or even on C)[5]. Condition β is of course satisfied if T is a linear map.

THEOREM 3. *Assume hypothesis* H *holds and* T *is a mapping of type "* P *". Then there exists a continuous semi-norm* p *such that for each* $0 < c \leqq 1$, *there are an* $x \in C$ *and* $\lambda > 0$ *satisfying*

$$\lambda x = T(x) \ , \quad p(x) = c.$$

Proof. Let $U = \{x : q_1(x) \leqq 1\}$ be a closed neighborhood of 0 such that $T(U \cap C)$ is relatively compact. Second, let q_2 be selected, according to which one of conditions α, β in H is satisfied, as follows:

Case α. Let q_2 be a continuous semi-norm strictly positive on $T(U \cap C)$.

Case β. Let $q_2 = q_1$.

Third, by the lemma, we may choose an $y \in C$ and a continuous linear form f such that $f \geqq 0$ on C while $f(y) > 0$. We may further suppose that $\sup\{q_1(y), q_2(y), f(y)\} = 1$.

Put $p = \sup\{q_1, q_2, |f|\}$ and consider the set $U_1 = \{x \in C : p(x) \leqq c\}$, c being any fixed real number between 0 and 1 (1 included). For any positive integer n, form the mappings

$$T_n(x) = T(x) + |1 - p(Tx)| \cdot n^{-1}y$$

and $S_n(x) = cT_n(x)/p[T_n(x)]$. Obviously, S_n is a transformation of type "P", mapping U_1 into itself, provided the denominator $p[T_n(x)]$ has a positive lower bound. To show that this is true, consider first all $x \in U_1$ such that $p(T(x)) \leqq \frac{3}{4}$. Then

$$p(T_n(x)) \geqq f[T_n(x)] \geqq \frac{1}{4n} f(y) \neq 0 \ .$$

For the remaining elements $x \in U_1$ we have $p(T(x)) > \frac{3}{4}$, hence

$$p(T_n(x)) \geqq p(T(x)) - |1 - p(T(x))| n^{-1}p(y) > \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \ .$$

[5] Condition α can be weakened so as to require the existence and continuity of the semi-norms involved only on C.

Applying the fixed point theorem (2nd form) to S_n, we are sure there is an $x_n \in U_1$ satisfying $x_n = S_n(x_n)$, $p(x_n) = c$. Letting $\lambda_n = p(T_n(x_n)) \cdot c^{-1}$, by definition of S_n we obtain

$$(*) \qquad \lambda_n x_n = T(x_n) + |1 - p(T(x_n))| n^{-1} y \ .$$

$\{T(x_n)\}$ being relatively compact, it follows that $\{\lambda_n\}$ is a bounded sequence. Assume, for the moment, that $\{\lambda_n\}$ has a positive lower bound. Then as the right-hand side of our last equation is relatively compact, so is $\{x_n\}$. (Here we may remark that for a convergent subsequence of $\{x_n\}$, the corresponding subsequence of λ's converges automatically to some $\lambda > 0$.) Hence for each limiting point of a subsequence of $\{\lambda_n x_n\}$, such that $\lambda_{n_k} \to \lambda$, $\lambda x = T(x)$ and, by continuity, $p(x) = c$.

All that remains to prove is that $\lambda_n > \eta > 0$ for all n. Suppose there were a subsequence $\{\lambda_k\}$ tending to zero. From this it would follow that $p(T(x_k)) \to 0$ which, by definition of p, in turn would imply $q_2(T(x_k)) \to 0$. On the other hand, T being strictly positive, 0 is no limiting point to the sequence $T(x_n)$ because of $p(x_n) = c$. Thus if α of H is satisfied, we arrive at a contradiction. Now assume H holds by virtue of condition β. Letting $z_n = \lambda_n x_n$, $\{z_n\}$ has a limiting point z, say. Because T is strictly positive, we must have $z \neq 0$. Multiplying equation (*) by λ_n^σ, we get

$$\lambda_n^\sigma z_n = T(z_n) + o\left(\frac{1}{n}\right) y \ .$$

Now if there were any subsequence $\{\lambda_k\}$ of $\{\lambda_n\}$ such that $\lambda_k \to 0$, we would obtain (as $\lambda_k^\sigma \to 0$) $T(z) = 0$ for some $z \in C$, $z \neq 0$. This again contradicts the hypothesis that T be strictly positive, and the proof is complete.

REMARK. Hypothesis H was needed to prove that $\{\lambda_n\}$ does not have 0 as a limiting point. The proof of Satz 3.1 in [4] is essentially the same as the one presented here, but is incorrect at the point where it says "$\lambda_0 > 0$" (l.c., p. 329, line 3 f.b.).

Applications. The remainder of this paper is concerned with a number of applications to the preceding theorems.

1. Consider the linear space ω of all real sequences $x = (x_1, x_2, \cdots)$, partially ordered by the positive cone $C = \{x : x_i \geqq 0, i = 1, 2, \cdots\}$. In the product topology (i.e. considering ω the product of countably many real lines) ω is locally convex. Let $r > 0$ be a fixed integer and let $k = (k_1, k_2, \cdots)$ denote any sequence of non-negative integers such that

$\sum_{i=1}^{\infty} k_i = r$.[6] Since the set $\{k\}$ of all such k is countable, we may arrange it into a sequence; hence consider $\{k\}$ as ordered by the natural order of subscripts.

Further, denote by x^k the product $\prod_{k_i \neq 0} x_i^{k_i}$. Now if to each k and each positive integer i there corresponds a real number $a_{ik} \geqq 0$ such that

$$\sum_{(k)} a_{ik} < C ,$$

where C is independent of i, the equations

$$y_i = \sum_{(k)} a_{ik} x^k$$

define a mapping $y = A(x)$ on the subspace of all bounded sequences into itself such that each y_i is a homogeneous form of degree r in the variables $x_1, x_2, \cdots$ (If $r = 1$, then A defines a bounded linear map on the B-space (m)).

Consider the properties

α. There are n rows in A (the first n rows, say) such that

$$\sum_{i=1}^{n} a_{ik} \geqq a_{jk} \qquad \text{for all } k \text{ and all } j > n .$$

β. $\sum_{i=1}^{n} y_i \to 0$ implies $\sum_{i=1}^{n} x_i \to 0$[7].

We prove the following theorem:

If a mapping A of the above mentioned type satisfies α and β, there are a $\lambda > 0$ and an $x > 0$ for which

$$\lambda x = A(x) .$$

REMARK. If $r = 1$, then the point spectrum of the bounded map A on (m) contains a positive real number.

Proof. Consider in ω the cone $C_1 = C \cap \{x : \sum_{i=1}^{n} x_i \geqq x_j, j > n\}$. Owing to α, $A(x)$ is defined on the cone C_1 into itself. Since ω is complete and C_1 closed, C_1 is a complete cone in ω. Next we show that A, which is in general not defined but on a dense subset of ω, is continuous on C_1. Let $x_n \to x$ in C_1. It follows from the definition of C_1 that all coordinates of all the x_n are uniformly bounded, say by some

[6] A more general theorem results if we admit all k such that $\sum_{i=1}^{\infty} k_i \leqq r$.

[7] Cf. Example 3 in the preliminary section.

constant $M \geqq 1$. Given $\varepsilon > 0$ and any fixed subscript i, we can find a k_0 such that

$$\sum_{k>k_0} a_{ik} < \frac{\varepsilon}{3M^r} .$$

Then if $y_n = A(x_n)$ and $y = A(x)$, we obtain

$$|(y_n - y)_i| \leqq \sum_{k \leqq k_0} |a_{ik}x_n^k - a_{ik}x^k| + \sum_{k>k_0} a_{ik}x_n^k + \sum_{k>k_0} a_{ik}x^k .$$

The last two righthand terms are, by the choice of k_0, each less than $\frac{1}{3}\varepsilon$. The first righthand term will be less than $\frac{1}{3}\varepsilon$ for $n > n_0$ if n_0 is large enough, since there are only finitely many coordinates of both x_n and x involved. Thus we have $|(y_n - y)_i| < \varepsilon$ if $n > n_0$ and continuity is established.

Now $f(x) = \sum_{i=1}^{n} x_i$ is a continuous linear form on ω. The intersection of the hyperplane $f(x) = 1$ with C_1 is certainly bounded as $(0 \leqq)\ x_i \leqq 1, i = 1, 2, \cdots$, in that intersection. Moreover, a set $\{x\}$ is bounded in ω if and only if $|x_i| < M_i$ uniformly on $\{x\}$. If M_i can be chosen independently of i, then the set is relatively compact by the well known Tychonov theorem. Thus on C_1 closed bounded sets coincide with compact sets, and A transforms bounded sets into compact ones on C_1.

By hypothesis β, A is strictly positive on C_1. (Conditions more explicit than β may be obtained easily by applying the reasoning of Example 3, preliminary section.) Hence A meets all the requirements of Th. 1 and the proof is complete.

2. In a recent paper [7], Schmeidler proved the existence of an eigenvalue to the homogeneous algebraic integral equation of order n

$$(*) \qquad \mu^n y^n(s) - \sum_{\beta=0}^{n-1} \mu^\beta y^\beta(s) a_\beta(s, y) = 0 , \qquad (0 \leqq s \leqq 1)$$

where n is an odd integer > 0 and

$$a_\beta(s, y) = \int_0^1 \cdots \int_0^1 K_\beta(s, t_1, \cdots, t_\nu) y^{\beta+1}(t_1) \cdots y^{\beta+1}(t_\nu) dt_1 \cdots dt_\nu$$

are homogeneous integral forms with continuous kernels $K_\beta(s_1 t_1 \cdots t_\nu)$ such that $(\beta + 1)(\nu + 1) = n + 1$ and the K's are symmetric with respect to all their arguments. Schmeidler shows (*) to be the natural generalization of a linear Fredholm equation with continuous symmetric kernel. In an earlier paper [6], a theorem was stated by Schmeidler that generalizes the well known Jentzsch theorem on linear Fredholm integral equations with positive kernel. The proof of that theorem of

Schmeidler's, however, appears to be incorrect[8]. We are going to show that the theorem yet is correct and holds under weaker conditions than the ones stated in [6]. Let us call (*) an *algebraic integral equation with non-negative coefficients* if n is *any* positive integer and

$$a_\beta(s, y) = \sum_{\alpha_1+\cdots+\alpha_\nu=n-\beta} \int_0^1 \cdots \int_0^1 K_{\beta\alpha_1} \cdots \alpha_\nu(s, t_1, \cdots, t_\nu) y(t_1)^{\alpha_1} \cdots y(t_\nu)^{\alpha_\nu} dt_1 \cdots dt_\nu$$

are homogeneous integral forms of order $n-\beta$ with continuous kernels $K_{\beta\alpha_1\cdots\alpha_\nu}(s, t_1, \cdots, t_\nu) \geqq 0$. We will prove this theorem:

If some α_β contains a term $\int K(s, t_1, \cdots, t_\nu) y(t_1)^\alpha \cdots y(t_r)^\alpha dt_1 \cdots dt_\nu$ such that α is the highest power occurring in any α_β, and if

$$\int_0^1 K(s, t_1, \cdots, t_\nu) ds \geqq \delta > 0 \qquad \textit{for } (t_1, \cdots, t_\nu) \in [0, 1]^\nu;$$

then (*) has an eigenvalue $\mu_0 > 0$ with eigenfunction $y_0(s) \geqq 0$.

Proof. We first state a

LEMMA. *Consider the mapping $\varphi: (a_0, \cdots, a_{n-1}) \to z_0$ where z_0 is the greatest real root of*

$$z^n - \sum_{\beta=0}^{n-1} a_\beta z^\beta = 0 . \tag{1}$$

Then φ is defined and continuous on the set $\{a_\beta \geqq 0; 0 \leqq \beta \leqq n-1\} \subset E^n$.

It is clear that $z_0 = 0$ if and only if $a_0 = a_1 = \cdots = a_{n-1} = 0$, and φ is continuous at that point. At any other point, however, z_0 is a simple root which implies continuity of φ.

Recalling that α is the highest power of y in any a_β, we observe that $a_\beta(s, y)$ $(0 \leqq \beta \leqq n-1)$ exist for all $y(s) \in L_\alpha(0, 1)$. Moreover, each $a_\beta(s, y)$ is a continuous map on $[0, 1] \times L_\alpha$ into the space $C(0, 1)$ of continuous functions on $[0, 1]$. For

$$|a_\beta(s, y) - a_\beta(t, \tilde{y})| \leqq |a_\beta(s, y) - a_\beta(s, y)| + |a_\beta(s, \tilde{y}) - a_\beta(t, \tilde{y})| \tag{2}$$

where the first righthand term can be estimated by expressions of the form

$$I = \int K[y(t_1) - \tilde{y}(t_1)](y(t_1)^{\alpha_1-1} + \cdots \tilde{y}(t_1)^{\alpha_1-1}) y(t_2)^{\alpha_2} \cdots y(t_\nu)^{\alpha_\nu} dt_1 \cdots dt_\nu .$$

Using Hölder's inequality we arrive at an estimate

$$|I| \leqq \text{const} \cdot \|y - \tilde{y}\| P(\|y\|, \|\tilde{y}\|)$$

where $P(u, v)$ is a homogeneous polynomial of order $n-\beta-1$ in u, v

[8] The treacherous point is that the mapping $\eta \to y$, [**6**] p. 252 above, is not continuous.

and $\| \ \|$ denotes the norm in L_α. Thus, $a_\beta(s, y) - a_\beta(s, \tilde{y}) \to 0$ uniformly in s as $\tilde{y} \to y$ in L_α. The second righthand term in (2) can be estimated by terms

$$II = \int [K(s, t_1 \cdots t_\nu) - K(t, t_1 \cdots t_\nu)]\tilde{y}(t_1)^{\alpha_1} \cdots \tilde{y}(t_\nu)^{\alpha_\nu} dt_1 \cdots dt_\nu;$$

so if $|s - t| < \delta(\varepsilon)$, the K's being continuous, we obtain

$$|II| \leqq \varepsilon \int \tilde{y}(t_1)^{\alpha_1} dt_1 \cdots \int \tilde{y}(t_\nu)^{\alpha_\nu} dt_\nu \leqq \varepsilon \|\tilde{y}\|^{n-\beta}$$

remembering that $\alpha_1 + \cdots + \alpha_\nu = n - \beta$ and $\int_0^1 y^\gamma dt \leqq \left(\int_0^1 y^\alpha dt\right)^\alpha$ if $\gamma \leqq \alpha$. This proves continuity of a_β on $[0, 1] \times L_\alpha$[9]. But from the above reasoning it is obvious that $\{a_\beta(s, y)\}$ is an equicontinuous, bounded set of functions if y runs through any bounded set of L_α. Hence, by the lemma, we have established:

The mapping $y(s) \to z_0(s)$, z_0 being defined as the greatest real root of (1) *for each $s \in [0, 1]$, maps any bounded subset of the positive cone in L_α onto a set of equicontinuous, non-negative and uniformly bounded functions over* $[0, 1]$.

Thus the map $y \to z_0$ satisfies the assumptions of Th. 2 if we can show that it is strictly positive. For that end, let $\|z_0\| \to 0$. If K is the kernel mentioned in our present theorem, we get by (1)

$$z_0(s)^n \geqq z_0(s)^\beta \int_0^1 \cdots \int_0^1 K(s, t_1, \cdots, t_\nu) y(t_1)^\alpha \cdots y(t_\nu)^\alpha dt_1 \cdots dt_\nu$$

where $\nu\alpha + \beta = n$. Since z_0 is the greatest real root of (1), there follows

$$z_0(s)^{n-\beta} \geqq \int_0^1 \cdots \int_0^1 K(s, t_1, \cdots, t_\nu) y(t_1)^\alpha \cdots y(t_\nu)^\alpha dt_1 \cdots dt_\nu .$$

Integrating this last equation, we obtain

$$(3) \qquad \int_0^1 z_0(s)^{\nu\alpha} ds \geqq \delta\left[\int_0^1 y(t)^\alpha dt\right]^\nu = \delta \|y\|^{\nu\alpha} .$$

Now assume first that y is bounded in L_α as $z_0 \to 0$. Then z_0 runs through a uniformly bounded set of continuous functions and hence, as it converges to 0 in measure, $\int_9^1 z_0(s)^{\nu\alpha} ds \to 0$. By (3) this implies $\|y\| \to 0$. This excludes that $\|y\| \to \infty$ as $z_0 \to 0$, for division of z and y by $\|y\|$ (remember that (*) is homogeneous) would lead to $\|y\| = 1$ while $z_0 \to 0$.

[9] In general, α_β are not continuous for the weak topology on L_α.

Thus $y \to z_0$ is strictly positive and the application of Th. 2 ends the proof.

3. Let Ω be a compact region in Euclidean n-space, $C(\Omega)$ the B-space of continuous functions on Ω, and $D(\Omega)$ the space of continuously differentiable functions on Ω in the topology of uniform convergence of the function and its first order derivatives. Assume a kernel $K(s, t) \geqq 0$ is given on $\Omega \times \Omega$ and a real function $f(s; u, p_i)$ of $2n+1$ arguments (letting $s = (x_1, \cdots, x_n)$) such that these conditions are satified:

1°. $K(\psi) = \int_\Omega K(s, t)\psi(t)dt$ is a compact linear transformation on $C(\Omega)$ into $D(\Omega)$ which has an eigenvalue $\lambda_1 > 0$ with an adjoint eigenfunction $\varphi(s)$ that is $\geqq 0$ and bounded except on an Ω-subset of (Lebesgue) measure 0.

2°. f is a continuous real valued function, defined for $s \in \Omega$, $u \geqq 0$, $|p_i| < \infty$ $(i = 1, \cdots, n)$ and such that

$$f(s; u, p_i) \geqq \begin{cases} \alpha u & \text{if } 0 \leqq u \leqq \delta\ , \\ K & \text{if } \delta < u\ , \end{cases}$$

where α, δ, K are three suitably chosen positive constants. Then the following theorem holds:

Under conditions 1°., 2°. *the nonlinear integro-differential equation*

$$\lambda u(s) = \int_\Omega K(s, t) f\left(t; u(t), \frac{\partial u}{\partial x_i}(t)\right)dt$$

has for each $c > 0$ *at least one solution* $u \geqq 0$ *with* $\lambda = \lambda(u) > 0$ *and* $\|u\| = c$, *where* $\| \ \|$ *denotes a suitably chosen norm of* $D(\Omega)$. *Moreover,* $\lambda(u)$ *satisfies the inequality* $\lambda \geqq \lambda_1 \cdot \inf(\alpha, Kc^{-1})$.

Proof. $D(\Omega)$ is partially ordered by the positive cone $C = \{u : u \geqq 0 \text{ on } \Omega\}$ but we note that C is not a normal cone (cf. sec. II). For any $\varepsilon > 0$, the transformation[10]

$$T_\varepsilon(u) = \int_\Omega K(s, t)\left[f\left(t; u, \frac{\partial u}{\partial x_i}\right) + \varepsilon\right]dt$$

is, due to the continuity of f and condition 1°., compact and continuous on C into C. For all $u \in C$ we have

$$\int_\Omega T_\varepsilon(u)ds \geqq \iint K(s, t)\varepsilon dtds \geqq \varepsilon \iint K(s, t)\varphi(s)dtds = \varepsilon\lambda_1 \int \varphi(s)ds > 0$$

[10] The following proof shows how cases may be handled where strict positiveness of the map involved cannot be verified. (T_ε is not necessarily strictly positive by Example 2 of the preliminary section if $\varepsilon = 0$.)

if sup ess $\varphi(s) = 1$. Hence T_ε is strictly positive and we may apply Th. 3, hypothesis H being satisfied through condition α. Following the notation of the proof to Th. 3, choose for q_1 any norm generating the topology of $D(\Omega)$, let $q_2 = \max|u|$ and write $p = \|\ \|$. (Obviously p is a norm generating the topology of $D(\Omega)$; it is also evident that, in the present case, the statement of Th. 3 holds for all $c > 0$ instead of $0 < c \leqq 1$.) Thus, c being fixed, to each $\varepsilon > 0$ there is a $u_\varepsilon \geqq 0$ and $\lambda_\varepsilon > 0$ such that

$$\lambda_\varepsilon u_\varepsilon = T_\varepsilon(u_\varepsilon)\,,\ \|u_\varepsilon\| = c\,. \tag{1}$$

We are going to show that λ_ε has a positive lower bound for $\varepsilon > 0$. Multiplying (1) by φ and integrating, we obtain

$$\begin{aligned}\lambda_\varepsilon\int_\Omega u_\varepsilon\varphi dt &= \iint K(s,t)\varphi(s)[f(u_\varepsilon(t)) + \varepsilon]dsdt \\ &= \lambda_1\int \varphi(t)[f(u_\varepsilon(t)) + \varepsilon]dt > \lambda_1\int \varphi(t)f[u_\varepsilon(t)]dt\,.\end{aligned} \tag{2}$$

Now Ω is the union of two measurable subsets Ω_1 and Ω_2 such that $u_\varepsilon \leqq \delta$ in Ω_1 whereas $u_\varepsilon > \delta$ in Ω_2. On account of condition 2°. we have

$$\int_{\Omega_1}\varphi f[u_\varepsilon]dt \geqq \alpha\int_{\Omega_1}\varphi u_\varepsilon dt$$

and

$$\int_{\Omega_2}\varphi f[u_\varepsilon]dt \geqq K\int_{\Omega_2}\varphi dt \geqq \frac{K}{c\gamma}\int_{\Omega_2}\varphi u_\varepsilon dt\,,$$

where $\gamma \leqq 1$ is a constant such that $\max|u| \leqq \gamma\|u\|$ for all $u \in D(\Omega)$[11]. Hence for the last integral in (2),

$$\int_\Omega \varphi f[u_\varepsilon]dt \geqq \alpha\int_{\Omega_1}\varphi u_\varepsilon dt + \frac{K}{c\gamma}\int_{\Omega_2}\varphi u_\varepsilon dt \geqq \inf\left(\alpha_1\frac{K}{c\gamma}\right)\int_\Omega \varphi u_\varepsilon dt$$

and we finally obtain

$$\lambda_\varepsilon \geqq \inf\left(\alpha, \frac{K}{c\gamma}\right)\cdot\lambda_1 \qquad \text{for } \varepsilon > 0\,. \tag{3}$$

Now let $\varepsilon \to 0$ in (1). As $\|u_\varepsilon\| = c$ independently of ε, the righthand side of (1) is a relatively compact sequence and so is the corresponding sequence of u_ε by (3). Thus for a common convergent sequence of λ_ε and u_ε, the limit function u satisfies $\lambda u = T_0(u)$ and the proof is complete.

[11] $p = \sup\{q_1, q_2, |f|\}$ (Th. 3) implies $\gamma \leqq 1$. If f does not depend on any p_i, the proof may be carried via Th. 2 and we will have $\gamma = 1$, $\|u\| = \max|u|$.

REMARK. If $f(s; u, p_i)$ is such that δ may be chosen arbitrarily large (e.g., if $f = \alpha u + g(s; u, p_i)$ with $g \geqq 0$), then we will have Ω_2 void and λ will satisfy the inequality

$$\lambda(u) \geqq \alpha\lambda_1 .$$

We apply the preceding theorem to the following problem.

Let Ω be a compact region in 3-space such that Green's function for the first boundary problem of potential theory exists. It is then well known that this kernel $G(s, t)$ satisfies condition 1°. of our theorem. It $f(s; u, p_i)$ is Hölder continuous with respect to all variables, then the equation

$$\lambda u(s) = \int_\Omega G(s, t) f\left(t; u, \frac{\partial u}{\partial x_i}\right) dt$$

is equivalent to the boundary problem

$$(*) \qquad \Delta u + \lambda^{-1} f\left(s; u, \frac{\partial u}{\partial x_i}\right) = 0, \; u = 0 \text{ on } \partial\Omega .$$

Hence, if f satisfies condition 2°., we have:

The nonlinear boundary problem (*) *has, for suitable values* $\lambda > 0$, *solutions* $u \geqq 0$ *such that* max u *attains any positive real number. If, moreover,* $f = u + g(s; u, p_i)$ $(g \geqq 0)$, *then for each such* λ

$$\lambda(u) \geqq \lambda_1 ,$$

where λ_1 *is the largest eigenvalue of the corresponding linear problem.*

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