SEMI-GROUPS OF CLASS (C_0) IN L_p DETERMINED BY PARABOLIC DIFFERENTIAL EQUATIONS

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1. Introduction. This paper treats mixed Cauchy problems for the parabolic partial differential equation in one space variable;

$$u_t = p(x)u_{xx} + q(x)u_x + r(x)u \ . \tag{1.1}$$

Our results are for non-singular equations, that is, the variable x is restricted to a finite interval $[a, b]$, and the function p is real-valued with $p(x) > 0$ on $[a, b]$. The functions q and r may be complex-valued. We require that p, q and r be in $L_\infty[a, b]$ and that p, p' and q be absolutely continuous with p', p'' and q' in $L_\infty[a, b]$.

We impose usual boundary conditions $\pi(u) = 0$ by

$$M_{i1}u(a) + N_{i1}u(b) + M_{i2}u'(a) + N_{i2}u'(b) = 0, i = 1, 2 \ . \tag{1.2}$$

The constants M_{ij}, N_{ij} are real or complex and the matrix $(M_{ij}; N_{ij})$ has rank two.

With Equation (1.1) is associated the ordinary differential operator

$$A = p(x)D^2 + q(x)D + r(x)I, D = \frac{d}{dx} \ . \tag{1.3}$$

With the above restrictions on the coefficients, A is defined in $L_p[a, b]$[1], $1 \leq p < \infty$, as a closed operator with dense domain, $D(A)$, given by

$$D(A) = \{u \in L_p \mid u \text{ and } u' \text{ are absolutely continuous and } u, u', u'' \in L_p\} \ . \tag{1.4}$$

The boundary conditions define restrictions A_π of A to subdomains,

$$D(A_\pi) = \{u \in L_p \mid u \text{ and } u' \text{ are absolutely continuous, } \pi[u] = 0, \text{ and } u, u', u'' \in L_p\} \ . \tag{1.5}$$

Our problem is to determine those A_π which generate *semi-groups of class* (C_0) in $L_p[a, b]$ (see Hille and Phillips [1], p. 320). Our main result is

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1 We denote by $L_p[a, b], 1 \leq p < \infty$ the complex Lebesgue space defined by Lebesgue measure on $[a, b]$. Any Lebesgue space defined by a different measure μ will be denoted by $([a, b], \mu)$.

THEOREM 4. *If π is regular*[2], *the operator A_π is the infinitesimal generator of a semi-group of class (C_0) in $L_p[a, b]$, $1 \leq p < \infty$.*

The theory of adjoint semi-groups (Hille and Phillips [10], p. 426) can be used to extend the results of Theorem 4 to the Banach space $L_\infty[a, b]$. However, these results apply only in proper closed subspaces of L_∞, and for brevity we do not include them.

In § 6 we investigate the necessity of regularity of π for the generation of a semi-group of class (C_0) by the special operators $\Omega_\pi = D^2$ in $L_p[0, 1]$. We have the partial result

THEOREM 5. *Let π and π^+ be adjoint boundary conditions relative to the operator D^2. If both Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in any $L_p[0, 1]$, $1 < p < \infty$, then π and π^+ are regular.*

We also show that for certain non-regular π the operator $\Omega_\pi = D^2$ can be defined either in $L_1([0, 1], dx^2)$ or in $L_1([0, 1], d(1-x)^2)$ as the generator of a semi-group of class (C_0). These operators can be shown to be equivalent to singular operators in $L_1[0, 1]$.

We give, what seems to be, the first application of the Feller-Phillips-Miyadera Theorem (Hille and Phillips [10], p. 360); other applications have been of its corollary, the Hille-Yosida Theorem. Probably Theorem 2, where this theorem is applied, can also be proved by an appropriate use of spectral resolutions of the operators $\Omega_\pi = D^2$ in L_1 and L_2, however, we use spectral resolutions in only one instance. In any case, the eigenfunctions of the A_π can be used to give analytic representations of the semi-groups. In essence, we simply establish in L_p a certain type of behavior near $t = 0$ of solutions to the heat equation with a variety of boundary conditions.

Extensive application of semi-group theory to parabolic differential equations have been made by W. Feller ([4], [6], [7], [8]) and E. Hille [9]. Their papers contain our results for those real differential equation and real boundary conditions which determine positivity preserving semi-groups in L_1 and in L_2.

We plan in a later paper to present a study which we have made of the hyperbolic equation

$$u_{tt} + a(x)u_t = p(x)u_{xx} + q(x)u_x + r(x)u \, . \tag{1.6}$$

2. Equivalent semi-group.

We make considerable use of the following notions. If $\{T_t\}$ is a semi-group of class (C_0) on a Banach space U and if H is a linear homeomorphism of U onto another Banach space V, then it is easily shown that $\{S_t\}$ defined by

$$S_t = HT_tH^{-1} \tag{2.1}$$

[2] See G. D. Birkhoff [1], p. 383; J. D. Tamarkin [12]; or Coddington and Levinson [2], pp. 299-305.

is a semi-group of class (C_0) on V. We say that $\{T_t\}$ and $\{S_s\}$ are *homeomorphically equivalent*.

If ω is a constant and α a real positive constant, and if $\{T_t\}$ is a semi-group of class (C_0), then $\{S_t\}$ defined by

$$S_t = e^{\omega t} T_{\alpha t} \tag{2.2}$$

is a semi-group of class (C_0).[1]

We make the following

DEFINITION 1. Let $\{T_t\}$ and $\{S_t\}$ be semi-groups of class (C_0) defined respectively on Banach spaces U and V. Then $\{T_t\}$ and $\{S_t\}$ are said to be *equivalent* provided there exist constants ω and α, α real and $\alpha < 0$, such that $\{T_t\}$ and $e^{\omega t} S_{\alpha t}$ are homeomorphically equivalent.

For our applications we need the following theorem, which is easily verified.[2]

THEOREM 1. *Let $\{T_t\}$ and $\{S_t\}$ be equivalent semi-groups of class (C_0) defined respectively in Banach spaces U and V,* i.e.

$$S_t = H(e^{\omega t} T_{\alpha t}) H^{-1} . \tag{2.3}$$

The infinitesimal generators A and B are related by

$$B = (\omega I + \alpha H A H^{-1}) , \quad D(B) = HD(A) . \tag{2.4}$$

The resolvents of A and B are related by

$$R(\lambda; B) = HR(\lambda - \omega; \alpha A) H^{-1} . \tag{2.5}$$

We make now

DEFINITION 2. Let A and B be closed operators defined respectively in Banach spaces U and V with dense domains. Then A and B are said to be *equivalent* provided there exists a linear homeomorphism H of U onto V such that (i) $D(B) = HD(A)$ and (ii) $B = (\omega I + \alpha H A H^{-1})$ for some constants ω and α, α real and $\alpha > 0$.

3. **Boundary conditions.** The linear forms in (1.2) define a two dimensional sub-space of a four dimensional complex vector space. It is convenient for our discussion to specify such subspaces by Grassman coordinates, which are defined by

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[1, 2] See Hille and Phillips [**10**], Theorem 12.2.2 and Theorem 13.6.1.

$$A = \begin{vmatrix} M_{11} & N_{11} \\ M_{21} & N_{21} \end{vmatrix} B = \begin{vmatrix} N_{11} & M_{12} \\ N_{21} & M_{22} \end{vmatrix} C = \begin{vmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{vmatrix}$$
(3.1)
$$D = \begin{vmatrix} M_{11} & N_{12} \\ M_{21} & N_{22} \end{vmatrix} E = \begin{vmatrix} M_{12} & N_{12} \\ M_{22} & N_{22} \end{vmatrix} F = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}$$

These coordinates satisfy the quadratic relationship

$$FC - BD = AE , \tag{3.2}$$

and they are unique to within a constant of proportionality. Also, any constants, not all zero, which satisfy (3.2) define by (3.1) a set of conditions π of rank 2 (Hodge and Pedoe [11], p. 312).

We now define, for brevity in the sequel, four types of boundary conditions by the following sets:

$$\begin{aligned} \tau_1 &= \{\pi \mid E = B + D = 0\} \\ \tau_2 &= \{\pi \mid E \neq 0, \text{ or } E = 0 \text{ and } B + D \neq 0, \text{ or } A \neq 0 \text{ and} \\ &\qquad B = C = D = E = F = 0\} , \\ \tau_3 &= \{\pi \mid F = C = 0 \text{ and one and only one of } A, B, D, E \neq 0\} , \\ \tau_4 &= \{\pi \mid A = E = 0, B = D = 1 \text{ and } FC = 1\} . \end{aligned} \tag{3.3}$$

Sets τ_1 and τ_2 have only the absorbing boundary conditions in common, i.e. $u(a) = u(b) = 0$. Sets τ_3 and τ_4 are disjoint subsets of τ_2. The set τ_3 contains only separated endpoint boundary conditions. Representatives of these types in the form of (1.2) are easily determined by imposing the defining conditions in (3.1).

It is a simple matter to check that all boundary conditions in the set τ_2 are regular in the sense of G. D. Birkhoff. With one exception, $u(a) = u(b) = 0$, all π in the set τ_1 are non-regular.

4. $\Omega_\pi = D^2$ in $L_1[0, 1]$ and $L_2[0, 1]$. For the special operator $\Omega_\pi = D^2$ on $[0, 1]$ we need

LEMMA 1. *Ω_π in $L_p[0, 1], 1 \leq p < \infty$, is a closed operator with dense domain. Except for those non-regular π given by*

$$\begin{aligned} &\alpha u(0) + u(1) = 0 \\ &\alpha u'(0) - u'(1) = 0 , \quad \alpha^2 = 1 , \end{aligned} \tag{4.1}$$

the resolvent $R(\lambda; \Omega_\pi)$ exists for all λ, $\Re(\lambda) > \omega_0 \geq 0$ for some ω_0, and $R(\lambda; \Omega_\pi)$ is expressed in all $L_p, 1 \leq p < \infty$, by a Green's function as

$$R(\lambda; \Omega_\pi)[u](.) = \int_0^1 G(., t, \lambda)u(t)dt . \tag{4.2}$$

The proof of Lemma 1 is easy and is omitted.[3] We, however, shall refer to the explicit expression for $G(x, t, \lambda)$ which is

$$(4.3)\quad G(x,t,\lambda) = \frac{1}{\Delta}\begin{cases} -F\sqrt{\lambda}\,sh\sqrt{\lambda}(x-t) + sh\sqrt{\lambda}t[Ash\sqrt{\lambda}(1-x) \\ \quad + D\sqrt{\lambda}\,ch\sqrt{\lambda}(1-x)] + ch\sqrt{\lambda}t[B\sqrt{\lambda}\,sh\sqrt{\lambda}(1-x) \\ \quad - E\lambda ch\sqrt{\lambda}(1-x)]\,, \\ \text{for } t \leq x, \text{ and} \\ -C\sqrt{\lambda}\,sh\sqrt{\lambda}(t-x) + (\text{above with } x \text{ and } t \text{ interchanged}), \\ \text{for } t \geq x\,. \end{cases}$$

The function $\Delta(\lambda)$ is given in terms of (3.1) by

$$(4.4)\quad \Delta(\lambda) = (F+C)\lambda + A\sqrt{\lambda}\,sh\sqrt{\lambda} + (B+D)\lambda ch\sqrt{\lambda} - E\lambda^{3/2}sh\sqrt{\lambda}$$

where the principle value of $\sqrt{\lambda}$ is chosen for $\Re(\lambda) \geq 0$.

In § 5 it will be shown that our main result, Theorem 4, follows easily from the rather difficult

THEOREM 2. *If π is regular, then $\Omega_\pi = D^2$ generates a semi-group of class (C_0) in $L_1[0, 1]$ and in $L_2[0, 1]$.*

We prove Theorem 2 by a series of lemmas. Our method of proof amounts to proving this theorem for the subsets τ_3 and τ_4 of the set τ_2 of regular π. These results are then used to define a factorization of $R(\lambda; \Omega_\pi)$ for any regular π by which we reduce estimates on $\|[R(\lambda; \Omega_\pi)]^n\|$, $n = 1, 2, \cdots$, which are needed for an application of the Feller-Phillips-Miyadera Theorem, to estimates on certain functions of the complex parameter λ.

The necessity for estimating $\|[R(\lambda; \Omega_\pi)]^n\|$ for $n > 1$ results when Ω_π generates a semi-group $\{T_t\}$ for which $\|T_t\|$ is not bounded by $e^{\omega t}$ for any ω. Whether or not $\|T_t\| \leq e^{\omega t}$ for a semi-group of class (C_0) in a Banach space.[4] In one instance, part (b) of Lemma 3, we are able to guess an equivalent norm for $L_1[0, 1]$ so that the Hille-Yosida Theorem can be applied, whereas in the L_1 norm this does not seem to be the case.

We have the easy

LEMMA 2. *For π in the set τ_3, Ω_π generates a semi-group of class (C_0) both in $L_1[0, 1]$ and in $L_2[0, 1]$.*

[3] See Coddington and Levinson [**2**], pp. 300-305.

[4] See Feller [**5**] where it is shown that if $\{T_t\}$ is a semi-group of class (C_0) in a Banach space, then an equivalent norm can always be defined *by the semi-group* so that in this norm $\|T_t\| < e^{\omega t}$.

Proof. (a) For $L_2[0, 1]$ all such Ω_π are self-adjoint with negative spectrum and a set of eigenfunctions which are a basis for $L_2[0, 1]$. It follows easily that such Ω_π generate semi-groups of contracting operators in $L_2[0, 1]$. (b) In $L_1[0, 1]$ we have by Fubini's Theorem, since $G(x, t, \lambda)$ is continuous,

$$\|R(\lambda\,; \Omega_\pi)u\| \leq \int_0^1\int_0^1 |G(x, t, \lambda)|\,|u(t)|\,dtdx$$
$$\leq \|u\|_1 \max_{0\leq t\leq 1}\int_0^1 |G(x, t, \lambda)|\,dx\ .$$

From (4.3) for these special π one gets easily

$$\|R(\lambda; \Omega_\pi)\| \leq \frac{1}{\lambda}\ . \tag{4.6}$$

By the Hille-Yosida Theorem, Ω_π generates a semi-group of contracting operators. This completes the proof.

The proof is not so easy for

LEMMA 3. *For π in the set τ_4, Ω_π generates in $L_1[0, 1]$ and in $L_2[0, 1]$ a semi-group of class (C_0).*

Proof. Any π in the set τ_4 is given by

$$\begin{aligned} au(0) + u(1) &= 0 \\ au'(0) + u'(1) &= 0 \end{aligned} \qquad a \neq 0\ . \tag{4.7}$$

We note that if the complex constant a in (4.7) is such that $|a| = 1$, then the conditions π are self-adjoint relative to the operator D^2.

(a) We set $\sigma = \log|a|$ and define a linear homeomorphism H of $L_2[0, 1]$ onto $L_2[0, 1]$ by

$$H[u](x) = e^{-\sigma x}u(x)\ . \tag{4.8}$$

The operator $\tilde{\Omega}_\pi$ equivalent to Ω_π is

$$\tilde{\Omega}_\pi = D^2 + 2\sigma D + \sigma^2 I\ . \tag{4.9}$$

Now $\tilde{\Omega}_\pi$ is a perturbation by the unbounded operator

$$B = 2\sigma D + \sigma^2 I \tag{4.10}$$

of the operator $\Omega_{\tilde{\pi}}$, where $\tilde{\pi}$ is given by

$$\begin{aligned} \alpha u(0) + u(1) &= 0 \\ \alpha u'(0) + u'(1) &= 0\ , \quad \alpha = \frac{a}{|a|} = e^{i\theta}\ . \end{aligned} \tag{4.11}$$

The domain $D(B)$ of B is the same as $D(\tilde{\Omega}_\pi) = D(\Omega_{\tilde{\pi}})$.

Now $\Omega_{\tilde{\pi}}$ is self-adjoint in $L_2[0, 1]$ with eigenvalues $\lambda_n = -(\theta + (2n+1)\pi)^2$, $n=0, \pm 1, \cdots$, and eigenfunctions $\phi_n(x) = \exp[i(\theta+(2n+1)\pi)x]$, which are a basis for $L_2[0, 1]$. Then $\Omega_{\tilde{\pi}}$ generates a contraction semi-group given by

$$T_t[u] = \sum_{n=-\infty}^{\infty} a_n e^{\lambda_n t}\phi_n(x),\ a_n = (u, \phi_n)\ . \tag{4.12}$$

We want to establish that B is in the perturbing class $\mathfrak{P}(\Omega_{\tilde{\pi}})$ of $\Omega_{\tilde{\pi}}$ (Hille and Phillips [10], p. 394). Since $D(B) = D(\Omega_{\tilde{\pi}})$ we must establish that

(4.13)
(i) $BR(\lambda; \Omega_{\tilde{\pi}})$ is bounded for some λ,
(ii) BT_t on $D(\Omega_{\tilde{\pi}})$ is bounded for all $t > 0$, and therefore extensible to $\overline{BT_t}$ on $L_2[0, 1]$, and
(iii) $\int_0^1 \| \overline{BT_t} \| dt < \infty$.

Now (i) of (4.13) follows immediately from (4.2). For (ii) of (4.13) we compute for $u \in D(\Omega_{\tilde{\pi}})$,

$$\begin{aligned} \frac{1}{2}\| BT_t(u)\|_2^2 &\le 4\sigma^2(DT_t(u), DT_t(u)) + \sigma^4\| T_t(u)\|_2^2 \\ &= 4\sigma^2 T_t(u)DT_t(u)\,|_0^1 - 4\sigma^2(T_t(u), D^2T_t(u)) \\ &\qquad + \sigma^4\| T_t(u)\|_2^2\ . \end{aligned} \tag{4.14}$$

Using the facts that $\pi(T_t(u)) = 0$, $\| T_t(u)\|_2 \le \|u\|_2$, and $\lambda_n \le 0$, we get

$$\frac{1}{2}\| BT_t(u)\|_2^2 \le \sigma^4\|u\|_2^2 + 4\sigma^2\|u\|_2^2\{\max_{-\infty\le n\le\infty} - \lambda_n e^{2\lambda_n t}\}\ . \tag{4.15}$$

Therefore, since $\lambda e^{-\lambda t}$ has on $[0, \infty)$ the maximum $1/2et$,

$$\| BT_t(u)\|_2 \le 2\sigma\left(\sigma^2 + \frac{2}{et}\right)^{1/2}\|u\|_2\ . \tag{4.16}$$

This proves (ii) in (4.13) as well as (iii)

Since $B \in \mathfrak{P}(\Omega_{\tilde{\pi}})$, the operator $\tilde{\Omega}_\pi$ generates a semi-group of class (C_0) (Hille and Phillips [10], p. 400). Since $\tilde{\Omega}_\pi$ is equivalent to Ω_π, this proves our lemma for $L_2(0, 1)$.

(b) In $L_1[0, 1]$ we do not use a perturbation argument as in $L_2[0, 1]$ because of the difficulty in proving (ii) of (4.13) without using orthogonality relations.

Again let $\sigma = \log|a|$ and introduce in $L_1[0, 1]$ an equivalent norm by

$$\|f\|_0 = \int_0^1 |f(x)|e^{-\sigma x}dx\ . \tag{4.17}$$

The identity mapping of $L_1[0, 1]$ under these two norms is a linear homeomorphism and Ω_π is equivalent to itself.

We get by Fubini's Theorem

$$\|R(\lambda; \Omega_\pi)u\|_0 \leq \int_0^1 |u(t)| \int_0^1 |G(x, t, \lambda)| e^{-\sigma x} dx dt \ . \tag{4.18}$$

The Grassman coordinates for (4.7) are $A = E = 0, B = D = a, C = 1$, and $F = a^2$, and from (4.3) for real $\lambda, \lambda > \sigma^2(\sigma = \log|a|)$,

$$|G(x, t, \lambda)| \leq \frac{\begin{cases} |a|^2 sh\sqrt{\lambda}(x - t) + |a| sh\sqrt{\lambda}(1 + t - x), & t \leq x \\ sh\sqrt{\lambda}(t - x) + |a| sh\sqrt{\lambda}(1 + x - t), & t \geq x \end{cases}}{\lambda(-1 - |a|^2 + 2|a| ch\sqrt{\lambda})} \tag{4.19}$$

We recognize the right-hand side of (4.19) as the Green's function, $G_1(x, t, \lambda)$, for d^2/dx^2 and the real boundary conditions π_1 given by

$$\begin{aligned} -|a|u(0) + u(1) &= 0 \\ -|a|u'(0) + u'(1) &= 0 \end{aligned} \tag{4.20}$$

for which $A = E = 0, B = D = |a|, C = -1$, and $F = -|a|^2$.

Now the function $e^{-\sigma x}$ is an eigenfunction of the operator $\Omega_{\pi_1^+}$ for the eigenvalue σ^2, where π_1^+ is the adjoint of π_1, which is represented by (4.20) if $|a|$ is replaced by $|a|^{-1}$. Since these are real boundary conditions, $G_1(x, t, \lambda)$, for real λ, defines the Green's function for $\Omega_{\pi_1^+}$ if integration is done with respect to the variable x. Therefore for (4.18) we have with λ real

$$\begin{aligned} \|R(\lambda; \Omega_\pi)u\|_0 &\leq \int_0^1 \frac{|u(t)| e^{-\sigma t}}{\lambda - \sigma^2} dt \\ &\leq \frac{\|u\|_0}{\lambda - \sigma^2}, \ \lambda > \sigma^2 \ . \end{aligned} \tag{4.21}$$

This proves that Ω_π generates a semi-group of class (C_0) in L_1 normed by $\|u\|_0$, and therefore in L_1 with the usual norm. This completes the proof of our lemma.

The extension to all π in the set τ_2 is based on

LEMMA 4. *Let π be in the set τ_2. Then*

$$R(\lambda; \Omega_\pi) = \sum_{i=1}^{6} f_i(\lambda) R(\lambda; \Omega_{\pi_i}) \ , \tag{4.22}$$

where π_1 and π_2 are in the set τ_4 and $\pi_3, \cdots, \pi_6$ are in the set τ_3. The functions $f_i(\lambda)$ are given by

(42.3) $$f_i(\lambda) = \alpha_i \frac{\Delta_i(\lambda)}{\Delta(\lambda)}, \qquad i = 1, 2, \cdots, 6,$$

where the α_i *are constants and* $\Delta(\lambda)$ *for* π *and* $\Delta_i(\lambda)$ *for* π_i *are defined by* (4.4).

Proof. We use the Grassmann coordinates to define the π_i as follows. By adding and subtracting constants we write π as $\sum_{i=1}^{6} \alpha_i \pi_i$ where

(4.24)
$$\begin{aligned}
\pi&: \ (A, B, C, D, E, F), \\
\pi_1&; \ (0, 1, C - X, 1, 0, F - \bar{X}), \\
\pi_2&: \ \left(0, 1, \frac{X}{|X|}, 1, 0, \frac{\bar{X}}{|X|}\right), \\
\pi_3&: \ (1, 0, 0, 0, 0, 0), \\
\pi_4&: \ (0, 1, 0, 0, 0, 0), \\
\pi_5&: \ (0, 0, 0, 1, 0, 0), \\
\pi_6&: \ (0, 0, 0, 0, 1, 0),
\end{aligned}$$

$\alpha_1 = 1, \alpha_2 = |X|, \alpha_3 = A, \alpha_4 = B - 1 - |X|, \alpha_5 = D - 1 - |X|$ and $\alpha_6 = E$. Now X has to be chosen so that the coordinates of π_1 satisfy (3.2). X is given by

(4.25)
$$X = C - \rho e^{i\theta}, \ \theta = \arg(C - \bar{F}) \text{ and}$$
$$\rho = \frac{|C - \bar{F}| + \sqrt{|C - \bar{F}|^2 + 2}}{2}$$

Using the linearity of the numerator of the Green's function (4.3) in the constants A, B, C, D, E, and F, we get the expression (4.23).

We shall apply to the functions $f_i(\lambda)$ of Lemma 6 the following:

THEOREM 3. *Let* $f(\lambda)$ *be analytic in a half plane* $\Re(\lambda) > \alpha$. *Let* $f(\lambda)$ *satisfy either of the following conditions*:

(i) $f(\lambda)$ *is real for real* λ *and* $(-1)^k f^{(k)}(\lambda) \geq 0$ (*or* ≤ 0) *for all real* λ, $\lambda > \alpha$, $k = 0, 1, \cdots$, i.e., f *is completely monotonic in* $(\alpha, +\infty)$.

(ii) (a) $\int_{-\infty}^{\infty} |f(\sigma + i\tau)| d\tau < M < +\infty$, $\sigma > \alpha$, M *independent of* σ.

(b) $\lim_{|\tau| \to \infty} f(\sigma + i\tau) = 0$ *uniformly in every closed subinterval of* $\alpha < \sigma < +\infty$.

Then there exist real numbers $K > 0$ *and* ω *such that*

(4.26) $$\sum_{k=0}^{n} \frac{|f^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k < K, \text{ for } n = 0, 1, \cdots,$$

and λ *real*, $\lambda > \omega$.

Proof. Suppose that (i) holds and that $f(\lambda) > 0$ for real λ (otherwise replace f by $-f$). Then $|f_i^{(k)}(\lambda)| = (-1)^k f_i^k(\lambda)$ and with $\omega = \alpha + 1$

$$(4.27) \qquad \sum_{k=0}^{\infty} \frac{|f_i^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k = \sum_{k=0}^{\infty} \frac{f_i^k(\lambda)}{k!} (\omega - \lambda)^k = f(\omega),\ \lambda \geq \omega ,$$

since f is analytic in the region $\Re(\lambda) > \alpha$. Then (4.26) follows with $K = |f(\alpha + 1)|$ and $\omega = \alpha + 1$.

Suppose that condition (ii) holds. Then f is the Laplace transform (Widder [13], p. 265) of a function $\phi(t)$ for which $\phi(t) = 0, t < 0$ and $|\phi(t)| \leq Me^{\sigma - t}, \sigma > \alpha$. We have (Widder [13], p. 57)

$$(4.28) \qquad f^{(k)}(\lambda) = \int_0^{\infty} (-t)^k e^{-\lambda t} \phi(t) dt \qquad \Re(\lambda) > \alpha .$$

So with $\omega = \alpha + 2$ and real $\lambda, \lambda > \omega$,

$$(4.39) \qquad \sum_{k=0}^{n} \frac{|f^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k \leq \int_0^{\infty} e^{-\omega t} |\phi(t)| dt \leq M .$$

Therefore (4.26) follows with $K = M$ and $\omega = \alpha + 2$.

We finally come to

Proof of Theorem 2. We shall establish the existence of real constants M and $\omega > 0$ such that in both L_1 and L_2 for real λ

$$(4.30) \qquad \|[R(\lambda; \Omega_\pi)]^{n+1}\| \leq \frac{M}{(\lambda - \omega)^n} , \quad \lambda > \omega, n = 1, 2, \cdots .$$

By the Feller-Phillips-Miyadera Theorem this will prove our theorem.

In the representation (4.22) for $R(\lambda; \Omega_\pi)$, each Ω_{π_i} generates a semigroup of class (C_0) in L_1 and in L_2, either by Lemma 2 or by Lemma 3. Then for each $R(\lambda; \Omega_{\pi_i}), i = 1, 2, \cdots, 6$ (4.30) holds in $L_p, p = 1, 2,$ and M and $\omega > 0$ can be chosen independently of i and p.

Iterates of a resolvent can be computed by

$$(4.31) \qquad [R(\lambda; \Omega_\pi)]^{n+1} = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \lambda^n} R(\lambda; \Omega_\pi) ,$$

(Hille and Phillips [10], p. 184). Making use of (4.22), (4.31) and (4.30) for each $R(\lambda; \Omega_{\pi_i})$, we get

$$(4.32) \qquad \|[R(\lambda; \Omega_\pi)]^{n+1}\| \leq \frac{M}{(\lambda - \omega)^{n+1}} \sum_{i=1}^{6} \sum_{k=0}^{n} \frac{|f_i^{(k)}(\lambda)|}{k!} (\lambda - \omega)^k ,$$

real $\lambda, \lambda > \omega$, and $n = 0, 1, \cdots$.

We suppose now that π is such that either $E \neq 0$ or $B + D \neq 0$.

The only other regular π is in the set τ_3, and has been dealt with in Lemma 2. With this assumption, each of the functions $f_i(\lambda)$ of Lemma 4 can be written as

$$f_i(\lambda) = J_i + \frac{K_i}{\sqrt{\lambda}} + \frac{L_i}{\lambda} + F_i(\lambda)\ , \qquad i = 1, 2, \cdots, 6 \tag{4.33}$$

for uniquely determined constants and a unique analytic function $F_i(\lambda)$.

For $\Re(\lambda) > 0$ we have chosen a branch of $\lambda^{1/2}$, so that the first three functions in (4.33) are analytic and satisfy condition (i) of Theorem 3. The functions $F_i(\lambda)$ are analytic and can be shown to satisfy conditions (ii) of Theorem 3. Then (4.32) and (4.33) together with Theorem 3 give our desired result (4.30). This proves our theorem.

5. A_π in $L_p[a, b]$, $1 \leq p < \infty$. With the tedious work done in § 4, we now come to our main result

THEOREM 4. *If π is regular, the operator A_π is the infinitesimal generator of a semi-group of class (C_0) in $L_p[a, b]$, $1 \leq p < \infty$.*

Proof. The assumptions on the coefficients of A in (1.3) are such that standard changes of independent and dependent variables[5] can be made to show that A_π in $L_p[a, b]$ is equivalent in the sense of Definition 2 to $\tilde{A}_\pi$ in $L_p[0, 1]$, where

$$\tilde{A}_\pi = \Omega_{\tilde{\pi}} + r_1 I\ . \tag{5.1}$$

The conditions $\tilde{\pi}$ are as in (1.2) and can readily be shown to be regular if and only if conditions π are regular.

The function r_1 in (5.1) is in $L_\infty[0, 1]$, and therefore $r_1 I$ is a bounded operator in any L_p. So $\tilde{A}_\pi$ is obtained by perturbing $\Omega_{\tilde{\pi}}$ by a bounded operator. Perturbation theory shows that $\tilde{A}_\pi$ generates a semi-group of class (C_0) if and only if $\Omega_{\tilde{\pi}}$ does (see Hille and Phillips [10], Theorem 13.2.1).

This reduces our proof to that of showing that for regular π the operators $\Omega_\pi = D^2$ generate semi-groups of class (C_0) in any $L_p[0, 1]$, $1 \leq p < \infty$. This extension of Theorem 2 we shall now give.

Let π^+ denote the boundary conditions adjoint to π relative to the operator D^2 (Coddington and Levinson [2], pp. 288–293). It is readily checked that the Grassmann coordinates (A', B', C', D', E', F') of π^+ are obtained from those of π by interchanging F and C and taking complex conjugates. From (3.3) it follows that π^+ is in the set τ_2 if and only if π is.

[5] See Courant and Hilbert [3], p. 250.

Let π, and therefore π^+, be regular boundary conditions. Then by Lemma 1 the resolvent $R(\lambda; \Omega_\pi)$ exists for $\Re(\lambda)$ greater than some ω_0, and it is expressed by (4.2).

We denote the norm of a bounded linear operator T in L_p by $N_p\{T\}$. Then by Theorem 2 and the Feller-Phillips-Miyadera Theorem (Hille and Phillips [10], p. 360), we have

$$N_p\{[R(\lambda; \Omega_\pi)]^n\} \leq M_p(\lambda - \omega_0)^{-n}, \Re(\lambda) > \omega_0 , \tag{5.2}$$

$p = 1, 2$ and $n = 1, 2, \cdots$

Now $R(\lambda; \Omega_\pi)$ is defined by (4.2) on the space of continuous functions, which is dense in $L_p[0, 1], 1 \leq p < \infty$. If we let $M = \max(M_1, M_2)$ and apply the Riesz Convexity Theorem (Zygmund [14], p. 198), we obtain (5.2) for $1 \leq p \leq 2$. By the Feller-Phillips-Miyadera Theorem, this is sufficient for Ω_π to generate a semi-group of class (C_0) in $L_p, 1 \leq p \leq 2$.

Also by Theorem 2 and the above argument, Ω_{π^+} generates a semi-group of class (C_0) in any $L_p\,[0, 1], 1 \leq p \leq 2$. It is readily shown that Ω_{π^+} in L_q and $\Omega_\pi +$ in $L_p, 1/p + 1/q = 1, 1 < p \leq 2$, are adjoints of each other. The theory of adjoint semi-groups (Hille and Phillips [10], Chapter IV) shows that Ω_π in L_q generates a semi-group of class (C_0), since Ω_{π^+} does in L_p. This completes the proof of our theorem.

6. **Non-regular π.**

One result relating to the necessity of regularity of π for A_π to generate a semi-group of class (C_0) in $L_p[a, b]$ is given in

LEMMA 5. *If A_π generates a semi-group of class (C_0) in $L_2[a, b]$, then π is regular.*

Proof. As we saw in the proof of Theorem 4, it is sufficient to prove this result for $\Omega_\pi = D^2$ in $L_2[0, 1]$.

Let π be a set of non-regular boundary conditions. It is simply a matter of computation to show that for the function $u(x) = 1, 0 \leq x \leq \frac{1}{2}$, and $u(x) = 0, \frac{1}{2} < x \leq 1$ we get in (4.2)

$$\| R(\lambda; \Omega_\pi)u \|_2 > C\lambda^{-3/4} \tag{6.1}$$

for all real λ sufficiently large and $C > 0$. Thus, by the Feller-Phillips-Miyadera Theorem, Ω_π does not generate a semi-group of class (C_0) in $L_2[0, 1]$.[6] This proves our result.

We now have[7]

[6] Indeed, this proves that Ω_π does not generate a semi-group of the more general class (A) in $L_2[0, 1]$ since it is not true that $\lambda R(\lambda; \Omega_\pi)u \to u$ as $\lambda \to +\infty$ (Hille and Phillips [10], p. 322).

[7] By a more careful analysis, the complete result can probably be proven that regularity of π is necessary for A_π to generate a semi-group of class (C_0) in $L_p[a, b]$.

THEOREM 5. *Let π and π^+ be adjoint boundary conditions relative to the operator D^2. If both Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in any $L_p[0, 1], 1 < p < \infty$, then π and π^+ are regular.*

Proof. Suppose that Ω_π and Ω_{π^+} generate semi-groups of class (C_0) in some $L_p[0, 1]$. Then Ω_π generates a semi-group of class (C_0) in $L_q[0, 1], 1/p + 1/q = 1$. An application of the Riesz Convexity Theorem, as in Theorem 4, shows that Ω_π generates a semi-group of class (C_0) in $L_2[0, 1]$. By Lemma 5, π is regular, and therefore also π^+. This completes the proof.

For certain of the non-regular π, other Lebesgue spaces can be chosen in which operators Ω_π are defined and generate semigroups of class (C_0). The construction of these spaces is suggested by the method of proof used in part (b) of Lemma 3.

Suppose that conditions π are given by

$$\begin{aligned} u(0) &= au'(1) \\ u(1) &= 0 \end{aligned} \qquad |a| \geq 1 . \tag{6.2}$$

Then, if $G(x, \tau, \lambda)$ is the Green's function of Ω_π, it can be shown that $G_1(x, \tau, \lambda) \equiv |G(\tau, x, \lambda)|$ is the Green's function for Ω_{π_1}, where conditions π_1 are given by

$$\begin{aligned} u(0) &= 0 \\ u(1) &= |a| u'(0) . \end{aligned} \tag{6.3}$$

Also Ω_{π_1} has the real, non-negative eigenfunction $\phi(x) = \sigma^{-1} sh\sigma x$ where σ is the largest real root of $sh\sigma = |a|\sigma$. In a manner similar to that in part (b) of Lemma 3, one can show that Ω_π can be defined in the Lebesgue space $L_1([0, 1], \phi(x)dx)$ as the generator of a semi-group of class (C_0). This space is also norm equivalent to the space $L_1([0, 1], dx^2)$.

The linear homeomorphism of $L_1([0, 1], dx^2)$ onto $L_1([0, 1], d(1-x)^2)$ defined by $u(x) \to u(1-x)$, shows that $\Omega_{\tilde{\pi}}$ generates a semi-group of class (C_0) in $L_1([0, 1], d(1-x)^2)$ where the conditions $\tilde{\pi}$ are given by

$$\begin{aligned} u(0) &= 0 \\ u(1) &= -au'(0) . \end{aligned} \tag{6.4}$$

In each of these spaces, $L_1[0, 1]$ can be shown to be a dense subspace. The operators Ω_π and $\Omega_{\tilde{\pi}}$ can be shown to be equivalent to singular operators in $L_1[0, 1]$.

We do not know whether similar results hold for other non-regular π.

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