ON RADICALS AND CONTINUITY OF HOMOMORPHISMS INTO BANACH ALGEBRAS

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1. Introduction. All Banach algebras considered are over the real field and all homomorphisms considered are algebraic (real-linear). An algebra is called semi-simple, strongly semi-simple, or strictly semi-simple, if its Jacobson radical [5], Segal radical [10], or strict radical [8], respectively, is the zero ideal; that is, if its regular maximal right ideals, its regular maximal two-sided ideals, or those of its two-sided ideals which are regular maximal right ideals, intersect in the zero ideal. Rickart [9, Corollary 6.3] proved that a semi-simple commutative Banach algebra has the property that every homomorphism of a Banach algebra into it is continuous. Call an algebra with this property an *absolute* algebra. Yood [12, Theorem 3.5] proved that every homomorphism of a Banach algebra onto a dense subset of a strongly semi-simple Banach algebra is continuous. Thus a strongly semi-simple Banach algebra is "almost" absolute. The question arose: Is a (noncommutative) semi-simple or strongly semi-simple Banach algebra necessarily absolute? A negative answer is furnished in the present note. It is shown that in order for a Banach algebra to be absolute it is sufficient that it be strictly semi-simple and necessary that it have zero as its only nilpotent element. The latter condition is shown to be sufficient for some special Banach algebras to be absolute.

2. Necessary condition for a Banach algebra to be absolute.

THEOREM 1. *An absolute Banach algebra has no nonzero nilpotent elements.*

Proof. Suppose the Banach algebra B contains a nonzero nilpotent element. Then there exists a nonzero $v \in B$ such that $v^2 = 0$. Let A be an infinite dimensional Banach algebra such that $A^2 = (0)$. Since A is an infinite dimensional complete vector space, there exists a discontinuous linear functional on A; denote it by $f(x)$. Let $\pi(x) = f(x)v$. Since $f(x)$ is linear and $v^2 = 0$, π is seen to be a homomorphism of A into B.

Let $\|y\|$ be a Banach norm for B. Then $\|\pi(x)\| = |f(x)|\ \|v\|$ since $f(x)$ is a scalar. Since $f(x)$ is discontinuous $|f(x)|$ is not bounded and

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hence π is not bounded. Thus π is discontinuous so that B is not absolute.

It is known that if a Banach algebra B is semi-simple and has a unique norm then a homomorphism of a Banach algebra *onto* B is necessarily continuous [9, Theorem 6.2]; that the resulting proposition is false if the word *into* is substituted for *onto* follows from our Theorem 1. Indeed, this theorem shows that for a Banach algebra to be absolute it is not sufficient that it have the properties of being simple, semi-simple, strongly semi-simple and having an identity and a unique norm. Thus, these properties are possessed by the algebra of all 2 by 2 matrices over the reals, under a Banach norm, and yet, since this algebra has nonzero nilpotent elements, Theorem 1 shows that it is not absolute.

3. Sufficient condition for a Banach algebra to be absolute. In [8] there was introduced the concept of a strictly semi-simple algebra. It was shown [8, Theorems 2 and 3] that a Banach algebra B is strictly semi-simple if and only if it is isomorphic to a subalgebra of $C(X, Q)$, the algebra of quaternion-valued functions continuous, and vanishing at ∞, on a locally compact Hausdorff space X.

THEOREM 2. *A strictly semi-simple Banach algebra B is absolute.*

Proof. Let A be a Banach algebra with a homomorphism T into $B \subset C(X, Q)$. Let $T_x(a) = T(a)(x)$. The kernel of T_x is closed since Q is simple, and therefore T_x is continuous, whence T_x is of bound 1. That T is continuous can now be shown by the 6-line argument of Loomis [7, p. 77]. One could also use [12, Theorem 3.5].

COROLLARY 1 (Rickart). *A semi-simple commutative Banach algebra is absolute.*

4. Concerning some special Banach algebras. For each subset S of a Banach algebra B, let $S_L(S_R)$ denote the set of all left (right) annihilators of S. B is called an annihilator algebra [3] if $B_R = 0 = B_L$ and if $I_L \neq 0$ ($I_R \neq 0$) for each proper closed right (left) ideal I, where 0 denotes the zero ideal.

Lemma 1 is due to Forsythe and McCoy [4, p. 524].

LEMMA 1. *In a ring without nonzero nilpotent elements every idempotent is in the center.*

THEOREM 3. *That a Banach algebra B have zero as its only nilpotent element is both a necessary and a sufficient condition for B to be either strictly semi-simple or absolute, provided any of the following conditions is satisfied:*

(a) *B is finite-dimensional.*
(b) *B satisfies the descending chain condition on right ideals.*
(c) *B is a semi-simple annihilator algebra.*

Proof. If B is strictly semi-simple, then $B \subset C(X, Q)$ by [8] and hence has only zero as a nilpotent element. If B is absolute, then zero is its only nilpotent element by Theorem 1. Conversely, suppose B has no nonzero nilpotent elements.

Suppose condition (a) or (b) holds. Then B has a nilpotent radical and therefore is semi-simple; also B is then a direct sum of division algebras and therefore has the property that every left (or right) ideal is two-sided [2, p. 463]. Thus B is strictly semi-simple and therefore absolute by Theorem 2.

Suppose condition (c) holds. Let M be any regular maximal right ideal in B. Bonsall and Goldie [3, pp. 155-6] show that for any semi-simple annihilator algebra B, $M_L = Be$ where e is a nonzero idempotent, B is a minimal (closed) left ideal, eB a minimal (closed) right ideal, $(eB)_L$ a maximal left ideal, and $(Be)_R = M$.

If B has no nonzero nilpotent elements, then e is in the center by Lemma 1 so that $Be = eB$ is a two-sided ideal. But the left and right annihilators of a closed two-sided ideal are identical [3, p. 159] so that $(eB)_L = (Be)_R = M$.

Since $(eB)_L$ is a left ideal, M, which was any regular maximal right ideal in B, has been shown to be a left ideal. Thus B is strictly semi-simple since it is semi-simple by hypothesis, and therefore absolute by Theorem 2.

COROLLARY 2. *An H^* algebra B is commutative if and only if any of the following properties is satisfied:*
(a) *B has no nonzero nilpotent elements.*
(b) *B is strictly semi-simple.*
(c) *B is absolute.*

Proof. An H^* algebra is the closure of the direct sum of matrix algebras M_σ [1, pp. 379–380]. If condition (a) holds, then each M_σ must have zero as its only nilpotent element and therefore must be one-dimensional. Hence each M_σ is generated by an idempotent e_σ which, by Lemma 1, is in the center. For $u, v \in \Sigma M_\sigma$, $u = \Sigma r_k e_k$, $v = \Sigma s_t e_t$, r_k, s_t scalars, $uv = vu$ so that ΣM_σ is commutative and therefore so is its closure, B. Thus condition (a) implies that B is commutative.

Suppose B is commutative. Since an H^* algebra is semi-simple, if commutative it is strictly semi-simple and $\subset C(X, Q)$ by [8], so that zero is its only nilpotent element. Hence condition (a) prevails.

The remainder of the corollary follows immediately from Theorem 3 since an H^* algebra is a semi-simple annihilator algebra [6, p. 697].

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