A CLASS OF RESIDUE SYSTEMS (mod r) AND RELATED ARITHMETICAL FUNCTIONS, II. HIGHER DIMENSIONAL ANALOGUES

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1. Introduction. In an earlier paper [3] with a similar name (to be referred to as I) we introduced the idea of a direct factor set (P-set) and the residue system (mod n) associated with such a set. We first review briefly these concepts. Two non-vacuous subsets P, Q of the positive integers Z are said to form a conjugate pair of direct factor sets provided the following two conditions are satisfied:

(i) an integer $n > 0$ is in P (or Q) if and only if, for each factorization, $n = n_1 n_2$, $(n_1, n_2) = 1$, n_1 and n_2 are also in P (or Q),

(ii) every positive integer n possesses a unique factorization of the form, $n = ab$ such that $a \in P$, $b \in Q$. A set of integers a(mod n) such that $(a, n) \in P$ is said to form a P-reduced residue system (mod n), or P-system (mod n), and the number of elements in such a system is denoted by $\phi_P(n)$. The fundamental result of I was a generalization of the Möbius inversion formula to conjugate pairs of direct factor sets. This result is reformulated in § 2 of the present paper.

In this paper we extend the notion of a P-system (mod n) from the set of integers X to t-dimensional vectors over X (briefly, X_t-vectors), $t \geqq 1$. The one dimensional case ($t = 1$) is the case already investigated in I. Two X_t-vectors, $A = \{a_i\}$, $B = \{b_i\}$, are said to be congruent (mod t, n), written $A \equiv B(\text{mod } t, n)$, provided $a_i \equiv b_i(\text{mod } n)$, $i = 1, \cdots, t$. Moreover, we place $(a_i) = (a_1, \cdots, a_t)$, using the convention, $(0, \cdots, 0) = 0$, and define vector sums and scalar multiples in the usual way. A P-reduced residue system (mod t, n), or P-system (mod t, n), is defined to be a maximal set of mutually incongruent X_t-vectors (mod t, n), $\{a_i\}$, satisfying $((a_i), n) \in P$. The number of elements in such a system depends only on t and n, and is denoted $J_{t,P}(n)$ and called the (t, P)-totient of n. In case P is the unit set 1, $J_{t,P}(n)$ reduces to the ordinary Jordan totient, $J_{t,1}(n) = J_t(n)$. A P-system with $P = Z$ is called a complete residue system (mod t, n); clearly $J_{t,Z}(n) = n^t$.

REMARK 1.1. An X_t-vector whose components are in Z will be called a Z_t-vector, and a P-system (mod t, n) consisting of elements of Z_t alone will be called a *positive* P-system (mod t, n).

We summarize now the salient points of the paper. In § 2 an enumerative principle for X_t-vectors (Theorem 2.1) is formulated, generalizing a result proved in [3, § 3] in the case $t = 1$. This result is used,

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in conjunction with the inversion principle of I, to obtain an evaluation of $J_{t,P}(n)$. A function $\phi_{\alpha,P}(n)$, formally generalizing $J_t(n)$, is also introduced, along with a generalized divisor function $\sigma_{\alpha,P}(n)$. Certain closely related functions, $\phi^*_{\alpha,P}(n)$ and $\sigma^*_{\alpha,P}(n)$ are also defined in § 2.

In § 3 we introduce the zeta function $\zeta_P(s)$ associated with a direct factor set P. In case $P = Z$, $\zeta_P(s)$ is the ordinary ζ-function, $\zeta(s)$. Employing the generalized inversion function $\mu_P(n)$ of I we also define "reciprocal" ζ-functions $\tilde{\zeta}_P(s)$ and obtain in (3.8) a generalization ($P = 1$, $Q = Z$) of the familiar fact,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \zeta^{-1}(s) , \qquad s > 1 , \tag{1.1}$$

where $\mu(n)$ denotes the Möbius function. Broad generalizations of other basic identities involving ζ-functions are also deduced.

In § 4 we obtain mean value estimates for the functions $\phi_{\alpha,P}(n)$ and $\sigma_{\alpha,P}(n)$, valid for *arbitrary* direct factor sets P, extending basic properties of $\phi(n)$ and $\sigma(n) = \sigma_{1,Z}(n)$. For example, (4.5) reduces in case $\alpha = 1$, $P = 1$, to the celebrated result [1, Theorem 330] of Mertens for the Euler ϕ-function,

$$\sum_{n \leqq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x) . \tag{1.2}$$

Using results of § 4, we obtain in § 5 (Theorem 5.1) for $t \geqq 2$, the asymptotic density of Z_t-vectors $\{a_i\}$, such that $(a_i) \in P$. Numerous special cases are considered (Corollary 5.2). We mention that Corollary 5.3, in case $t = 2$, yields a result of Kronecker asserting that the density of the integral pairs with a fixed greatest common divisor r is $6/\pi^2 r^2$.

In § 6 we generalize the so-called "second Möbius inversion formula" to conjugate sets P, Q (Theorem 6.1). Application of this extended inversion relation yields in (6.3) a generalization of broad scope of Meissel's well known identity,

$$\sum_{1 \leqq n \leqq x} \mu(n) \left[\frac{x}{n} \right] = 1 . \tag{1.3}$$

We also evaluate in § 6 a generalization to P-sets of Legendre's totient function $\phi(x, n)$, defined to be the number of integers a such that $1 \leqq a \leqq x$, $(a, n) = 1$.

REMARK 1.2. It is noted that many of the results of this paper are valid, not merely for direct factor sets, but for quite arbitrary sets of integers P. For example, this is true in the case of Corollary 5.1. Moreover, a number of the remaining results can be reformulated in such a manner as to be valid for arbitrary sets P. We shall restrict our attention, however, to direct factor sets, reserving the treatment of more general sets for a later paper, to be based on other methods. The advantage of a separate treatment of direct factor sets arises from the applicability of the generalized inversion theorem.

2. Generalized totient and divisor functions. Let P and Q denote an arbitrary conjugate pair of direct factor sets, and define, as in I,

$$\rho_P(n) = \begin{cases} 1 & (n \in P) \\ 0 & (n \notin P) , \end{cases} \tag{2.1}$$

$$\mu_P(n) = \sum_{d\delta = n} \rho_P(d)\mu(\delta) . \tag{2.2}$$

The functions $\rho_P(n)$ and $\mu_P(n)$ are termed, respectively the *characteristic* function and *inversion* function of the set P. The inversion formula of I can be restated in the form,

$$f(n) = \sum_{d\delta = n} \rho_Q(d)g(\delta) \rightleftarrows g(n) = \sum_{d\delta = n} \mu_P(d)f(\delta) . \tag{2.3}$$

This principle is a direct consequence of the relation,

$$\sum_{d\delta = n} \mu_P(d)\rho_Q(\delta) = \rho(n) , \tag{2.4}$$

where $\rho(n) = \rho_1(n)$ (that is, $\rho(n) = 1$ or 0 according as $n = 1$ or $n > 1$). Note that $\mu_P(n)$ reduces to $\mu(n)$ when $P = 1$.

In order to evaluate $J_{t,P}(n)$, we shall need the following results generalizing Theorem 4 of I to t dimensional vectors.

THEOREM 2.1. *If d ranges over the divisors of n contained in Q, and for each d, x ranges over the elements of a P-system* (mod t, δ), *$d\delta = n$, then the set dx constitutes a complete residue system* (mod t, n).

We omit the proof, which is analogous to the proof in case $t = 1$. On the basis of this result it follows immediately that

$$\sum_{d\delta = n} \rho_Q(d)J_{t,P}(\delta) = n^t . \tag{2.5}$$

Application of (2.3) to (2.5) yields

THEOREM 2.2.

$$J_{t,P}(n) = \sum_{d\delta = n} d^t\mu_P(\delta) . \tag{2.6}$$

Define now for α an arbitrary real number, the generalized totient,

$$\phi_{\alpha,P}(n) = \sum_{d\delta = n} d^\alpha\mu_P(\delta) , \tag{2.7}$$

so that $\phi_{\alpha,P} = J_{t,P}(n)$ in case $\alpha = t$ is a positive integer. We also define analogously a generalized divisor function by placing

(2.8) $$\sigma_{\alpha,P}(n) = \sum_{d\delta=n} d^\alpha \rho_P(\delta) = \sum_{\substack{d\delta=n \\ \delta\in P}} d^\alpha .$$

Corresponding to the functions $\phi_{\alpha,P}(n)$, $\sigma_{\alpha,P}(n)$ we define related functions,

(2.9) $$\phi^*_{\alpha,P}(n) = \sum_{d|n} d^\alpha \mu_P(d)$$

(2.10) $$\sigma^*_{\alpha,P}(n) = \sum_{d|n} d^\alpha \rho_P(d) = \sum_{\substack{d|n \\ d\in P}} d^\alpha .$$

The following simple relations are noted.

(2.11a) $$\phi^*_{-\alpha,P}(n) = \frac{\phi_{\alpha,P}(n)}{n^\alpha} ,$$

(2.11b) $$\sigma^*_{-\alpha,P}(n) = \frac{\sigma_{\alpha,P}(n)}{n^\alpha} .$$

Corresponding to the case $P = 1$, we place $\phi_{\alpha,1}(n) = \phi_\alpha(n)$, $\phi^*_{\alpha,1} = \phi^*_\alpha(n)$, and corresponding to the case $P = Z$, we write $\sigma_{\alpha,Z}(n) = \sigma_\alpha(n) = \sigma^*_{\alpha,Z}(n)$.

The following result is a generalization of [3, Theorem 8, $\alpha = 1$] and can be proved similarly.

THEOREM 2.3.

(2.12) $$\phi_{\alpha,P}(n) = \sum_{d\delta=n} \phi_\alpha(d)\rho_P(\delta) .$$

We also note, by inversion of (2.7), the following generalization of (2.5).

(2.13) $$\sum_{d\delta=n} \rho_Q(d)\phi_{\alpha,P}(\delta) = n^\alpha .$$

3. The zeta-functions of a P-set.

REMARK 3.1. In the definitions and general results of this section, s is assumed to be limited to values for which all occuring series converge absolutely.

First we define for real s,

(3.1) $$\zeta_P(s) = \sum_{n=1}^\infty \frac{\rho_P(n)}{n^s} = \sum_{\substack{n=1 \\ n\in P}}^\infty \frac{1}{n^s} .$$

The function $\zeta_P(s)$ will be called the *zeta-function* of the direct factor set P. Note that $\zeta_Z(s) = \zeta(s)$, $\zeta_1(s) = 1$. We define the *reciprocal* zeta-function of P by

(3.2) $$\tilde{\zeta}_P(s) = \sum_{n=1}^\infty \frac{\mu_P(n)}{n^s} ;$$

the function $\zeta_Q(s)$ will be designated the *conjugate* zeta-function of P.

By (1.1) it follows that $\tilde{\zeta}(s) \equiv \tilde{\zeta}_1(s) = 1/\zeta(s)$. We mention that Diricelet series of the form (3.1), (3.2) were discussed by Wintner [10, Chapter II] in case P is a semigroup generated by a set of primes.

First we prove two relations analogous to (2.4).

LEMMA 3.1.

$$\sum_{d\delta=n} \rho_P(d)\rho_Q(\delta) = 1 \,. \tag{3.3}$$

Proof. This is an immediate consequence of property (ii) of the conjugate pair P, Q.

LEMMA 3.2.

$$\sum_{d\delta=n} \mu_P(d)\mu_Q(\delta) = \mu(n) \,. \tag{3.4}$$

Proof. By the definition of $\mu_P(n)$, we have, with the left member of (3.4) denoted by $S(n)$,

$$\begin{aligned} S(n) &= \sum_{d\delta=n} \sum_{\substack{DD'=d \\ D'\in P}} \mu(D) \sum_{\substack{EE'=\delta \\ E'\in Q}} \mu(E) = \sum_{\substack{DD'EE'=n \\ D'\in P,\, E'\in Q}} \mu(D)\mu(E) \\ &= \sum_{DE\mid n} \mu(D)\mu(E) \sum_{\substack{D'E'=n/DE \\ D'\in P,\, E'\in Q}} 1 \,. \end{aligned}$$

By property (ii), it follows then that

$$S(n) = \sum_{DE\mid n} \mu(D)\mu(E) = \sum_{D\mid E} \mu(D) \sum_{E\mid (n/D)} \mu(E) \,,$$

and (3.4) results by the fundamental property of $\mu(n)$, ((2.4) with $P = 1$, $Q = Z$).

The following relations are basic.

THEOREM 3.1.

$$\zeta_P(s)\zeta_Q(s) = \zeta(s) \,, \tag{3.5}$$

$$\tilde{\zeta}_P(s)\tilde{\zeta}_Q(s) = \zeta^{-1}(s) \,, \tag{3.6}$$

$$\zeta_P(s)\tilde{\zeta}_Q(s) = 1 \,. \tag{3.7}$$

Proof. By the nature of the Dirichlet product, (3.5), (3.6), and (3.7) follow, respectively, from (3.3), (3.4), and (2.4).

By Theorem 3.1 one obtains the following generalization of (1.1):

COROLLARY 3.1.

$$\tilde{\zeta}_P(s) = \frac{\zeta_P(s)}{\zeta(s)} = \frac{1}{\zeta_Q(s)} \,. \tag{3.8}$$

The equality of the first two expressions in (3.8) is equivalent to the fact [3, (4.6)],

$$\sum_{d|n} \mu_P(d) = \rho_P(n) . \tag{3.9}$$

The following identities can be verified by Dirichlet multiplication, in connection with (3.8), (2.13), and (2.11a).

THEOREM 3.2.

$$\sum_{n=1}^{\infty} \frac{\phi_{\alpha,P}(n)}{n^s} = \frac{\zeta(s-\alpha)}{\zeta_Q(s)} = \frac{\zeta(s-\alpha)\zeta_P(s)}{\zeta(s)} ; \tag{3.10}$$

$$\sum_{n=1} \frac{\phi^*_{\alpha,P}(n)}{n^s} = \frac{\zeta(s)}{\zeta_Q(s-\alpha)} = \frac{\zeta(s)\zeta_P(s-\alpha)}{\zeta(s-\alpha)} . \tag{3.11}$$

THEOREM 3.3.

$$\sum_{n=1}^{\infty} \frac{\sigma_{\alpha,P}(n)}{n^s} = \zeta(s-\alpha)\zeta_P(s) ; \tag{3.12}$$

$$\sum_{n=1}^{\infty} \frac{\sigma^*_{\alpha,P}(n)}{n^s} = \zeta(s)\zeta_P(s-\alpha) . \tag{3.13}$$

Note that in case $P = Z$, both (3.12) and (3.13) reduce to [7, Theorem 291].
It is also noted, on the basis of (3.12) and (3.8), that

COROLLARY 3.3.

$$\zeta_Q(s) \sum_{n=1}^{\infty} \frac{\sigma_{\alpha,P}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} . \tag{3.14}$$

Multiplying (3.14) by $\zeta_P(s)$ and comparing coefficients, one obtains the arithmetical relation.

COROLLARY 3.4.

$$\sigma_{\alpha,P}(n) = \sum_{d\delta=n} \sigma_\alpha(d)\mu_P(\delta) . \tag{3.15}$$

This analogue of (2.12) can also be proved arithmetically on the basis of (3.9) and the definition of $\sigma_{\alpha,P}(n)$.

In the remainder of this section, we list for later reference, explicit evaluations of $\zeta_P(s)$ for various sets P. Let k and r denote fixed positive integers and p a fixed prime. We define direct factor sets $P = A_k, B_k, C_p, D_r, E_r$ as follows: A_k (the set of kth powers), B_k (the set of k-free integers), C_p (the non-negative powers of p), D_r (the divisors of r), E_r (the complete divisors of r). A divisor d of r is said to be complete if $(d, r/d) = 1$.

We have the following representations.

$$\zeta_{A_k}(s) = \zeta(ks) \qquad (ks > 1), \tag{3.16}$$

$$\zeta_{B_k}(s) = \frac{\zeta(s)}{\zeta(ks)} \qquad (s > 1), \tag{3.17}$$

$$\zeta_{C_p}(s) = \frac{p^s}{p^s - 1} \qquad (s > 0), \tag{3.18}$$

$$\zeta_{D_r}(s) = \frac{\sigma_s(r)}{r^s} = \sigma_{-s}(r) , \tag{3.19}$$

$$\zeta_{E_r}(s) = \frac{\sigma'_s(r)}{r^s} = \sigma'_{-s}(r) , \tag{3.20}$$

where $\sigma'_s(r)$ denotes the sum of the sth powers of the complete divisors of r. For a proof of (3.17) we refer to [7, Theorem 303]; (3.18) results on summing a geometric series.

We mention the following special cases of (3.10) and (3.12), which result on the basis of (3.16) and (3.17), respectively.

$$\sum_{n=1}^{\infty} \frac{\phi_{\alpha, A_k}(n)}{n^s} = \frac{\zeta(s-\alpha)\zeta(ks)}{\zeta(s)} \qquad (s > \alpha,\ s > 1), \tag{3.12}$$

$$\sum_{n=1}^{\infty} \frac{\sigma_{\alpha, B_k}(n)}{n^s} = \frac{\zeta(s-\alpha)\zeta(s)}{\zeta(ks)} \qquad (s > \alpha,\ s > 1). \tag{3.22}$$

4. Mean values of totient and divisor functions. In this section we prove, along classical lines, some simple estimates for the functions introduced in § 2. We require no more than the following elementary facts:

$$\sum_{n \leqq x} \frac{1}{n^\alpha} = \begin{cases} O(1) & \text{if } \alpha > 1 , \\ O(\log x) & \text{if } \alpha = 1 , \\ O(x^{1-\alpha}) & \text{if } \alpha < 1 ; \end{cases} \tag{4.1}$$

$$\sum_{n \leqq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + \begin{cases} O(x^\alpha) & \text{if } \alpha \geqq 0 , \\ O(1) & \text{if } -1 < \alpha < 0 ; \end{cases} \tag{4.2}$$

$$\sum_{n > x} \frac{1}{n^\alpha} = O\left(\frac{1}{x^{\alpha-1}}\right) , \qquad \alpha > 1 . \tag{4.3}$$

LEMMA 4.1. *For P an arbitrary direct factor set, $\mu_P(n)$ is bounded; in fact, for each $n > 0$, $\mu_P(n) = 1, -1,$ or 0.*

Proof. In view of the factorability [3, Theorem 1] of $\mu_P(n)$, it suffices to prove the lemma in case $n = p^h$, p prime, $h > 0$. We have then by (2.2),

$$\mu_P(p^h) = \rho_P(p^h) - \rho_P(p^{h-1}) ,$$

so that

$$\mu_P(p^h) = \begin{cases} 1 & (p^h \in P,\ p^{h-1} \notin P) \\ -1 & (p_h \notin P,\ p^{h-1} \in P) \\ 0 & (\text{otherwise}). \end{cases} \tag{4.4}$$

The lemma is proved.

As a consequence of Lemma 4.1, one obtains

COROLLARY 4.1. *The series* (3.2) *is absolutely convergent for* $s > 1$.

In the following, x will be assumed > 1.

THEOREM 4.1. *For all* $\alpha > 0$

$$\sum_{n \leq x} \phi_{\alpha,P}(n) = \left(\frac{x^{\alpha+1}}{\alpha+1}\right)\frac{1}{\zeta_Q(\alpha+1)} + O(e_\alpha(x)) , \tag{4.5}$$

$$\sum_{n \leq x} \sigma_{\alpha,P}(n) = \left(\frac{x^{\alpha+1}}{\alpha+1}\right)\zeta_P(\alpha+1) + O(e_\alpha(x)) , \tag{4.6}$$

where

$$e_\alpha(x) = \begin{cases} x^\alpha & (\alpha > 1) \\ x \log x & (\alpha = 1) \\ x & (\alpha < 1). \end{cases}$$

Proof. We prove (4.5). By (2.7)

$$\begin{aligned} \Phi_{\alpha,P}(x) \equiv \sum_{n \leq x} \phi_{\alpha,P}(n) &= \sum_{n \leq x} \sum_{\substack{\delta \mid n \\ (d\delta = n)}} \delta^\alpha \mu_P\left(\frac{n}{\delta}\right) \\ &= \sum_{d\delta \leq x} \delta^\alpha \mu_P(d) = \sum_{d \leq x} \mu_P(d) \sum_{\delta \leq x/d} \delta^\alpha . \end{aligned} \tag{4.7}$$

Hence by (4.2) and Lemma 4.1,

$$\begin{aligned} \Phi_{\alpha,P}(x) &= \sum_{d \leq x} \mu_P(d)\left\{\frac{(x/d)^{\alpha+1}}{\alpha+1} + O\left(\left(\frac{x}{d}\right)^\alpha\right)\right\} \\ &= \frac{x^{\alpha+1}}{\alpha+1}\sum_{d \leq x}\frac{\mu_P(d)}{d^{\alpha+1}} + O\left(x^\alpha \sum_{d \leq x}\frac{1}{d^\alpha}\right) . \end{aligned}$$

By (4.1) and Corollary 4.1, one may write then

$$\Phi_{\alpha,P}(x) = \frac{x^{\alpha+1}}{\alpha+1}\left\{\tilde{\zeta}_P(\alpha+1) - \sum_{d > x}\frac{\mu_P(d)}{d^{\alpha+1}}\right\} + (e_\alpha(x)) . \tag{4.8}$$

But by Lemma 4.1 and (4.3), it follows that

$$\sum_{d > x}\frac{\mu_P(d)}{d^{\alpha+1}} = O\left(\sum_{d > x}\frac{1}{d^{\alpha+1}}\right) = O\left(\frac{1}{x^\alpha}\right) \tag{4.9}$$

for all $\alpha > 0$. By (4.8), (4.9), and (3.8) the proof of (4.5) is complete.

The proof of (4.6) is similar and the details will be omitted; likewise for the following result.

THEOREM 4.2. *For all* $\alpha > 0$

$$\sum_{n\leqq x}\phi^*_{-\alpha,P}(n) = \frac{x}{\zeta_Q(\alpha+1)} + O(e^*_\alpha(x)) , \tag{4.10}$$

$$\sum_{n\leqq x}\sigma^*_{-\alpha,P}(n) = x\zeta_P(\alpha+1) + O(e^*_\alpha(x)) , \tag{4.11}$$

were $e^*_\alpha(x) = x^{-\alpha}e_\alpha(x)$ *and* $e_\alpha(x)$ *is defined as in Theorem* 4.1.

5. Asymptotic density of vector sets. We shall refer to the greatest common divisor (a_i) of the components of a Z_t-vector $\{a_i\}$ as the *index factor* of the vector. Let S be a set of positive integers and let $N_t(x, S)$ denote the number of Z_t-vectors with components $a_i \leqq x$ $(i = 1, \cdots, t)$ and with index factor in S. Then place

$$\delta_t(S) = \lim_{x\to\infty}\frac{N_t(x, S)}{x^t} ,$$

(if this limit exists) and call $\delta_t(S)$ the asymptotic density of the set of Z_t-vectors with index factor in S. We now prove the principal result of this section.

THEOREM 5.1. *If t is an integer* $\geqq 2$, *then*

$$N_t(x, P) = \frac{x^t}{\zeta_Q(t)} + \begin{cases} O(x\log x) & \textit{if } t = 2, \\ O(x^{t-1}) & \textit{if } t > 2. \end{cases} \tag{5.1}$$

Proof. For positive integral r, $x \geqq 1$, place

$$\Phi_{r,P}(x) = \sum_{n\leqq x} J_{r,P}(n) = \sum_{n\leqq x}\phi_{r,P}(n) , \qquad \Phi_{0,P}(x) = 1 .$$

Let j be a fixed integer, $1 \leqq j \leqq t$, and let $i_1, \cdots, i_j$ be a set of distinct integers satisfying $1 \leqq i < \cdots < i_j \leqq t$. Consider all Z_t-vectors such that the components in the positions $i_1, \cdots, i_j$ have the same value n, the components in the remaining positions are $\leqq n$, and the index factor is in P. Denote by S_j the set of all such vectors, including repetitions, obtained by letting n range over the set, $1 \leqq n \leqq x$, and by choosing the set, $i_1, \cdots, i_j$, in every possible way. Then if $N(S_j)$ denotes the number of elements in S_j, it follows that

$$N(S_j) = \binom{t}{j}\Phi_{t-j,P}(x) . \tag{5.2}$$

Consider now a fixed Z_t-vector, $\beta_k \in S_k$, $1 \leqq k \leqq t$, with exactly k of its components equal to n and the remaining components $<n$. Then β_k appears $\binom{k}{j}$ times in S_j, it being understood that $\binom{k}{j} = 0$ if $j > k$. In view of the fact,

$$\sum_{j=1}^{t} (-1)^{j+1}\binom{k}{j} = 1 ,$$

it follows that β_k is contained exactly once in the set

$$\sum_{j=1}^{t} (-1)^{j+1} S_j .$$

Consequently

$$N_t(x, P) = \sum_{j=1}^{t} (-1)^{j+1} N(S_j) ;$$

hence by (5.2),

$$N_t(x, P) = \sum_{j=1}^{t} (-1)^{j+1}\binom{t}{j}\Phi_{t-j,P}(x) .$$

The theorem follows by (4.5) on taking limits.

As a corollary of Theorem 5.1 one obtains by (3.8),

COROLLARY 5.1 (cf. [2, p. 8]). *If* $t \geqq 2$, *then* $\delta_t(P)$ *exists and is given by*

$$\delta_t(P) = \frac{1}{\zeta_Q(t)} = \frac{\zeta_P(t)}{\zeta(t)} . \tag{5.3}$$

As in § 3 let r and k denote positive integers and p a positive prime. On the basis of the evaluations (3.16)—(3.20), we obtain the following special cases of Corollary 5.1.

COROLLARY 5.2. *The asymptotic density of the* Z_t*-vectors,*, $t \geqq 2$,

(i) *with index factor a* kth power is

$$\delta_t(A_k) = \frac{\zeta(kt)}{\zeta(t)} ; \tag{5.4}$$

(ii) *with* k*-free index factor is*

$$\delta_t(B_k) = \frac{1}{\zeta(kt)} ; \tag{5.5}$$

(iii) *with index factor a non-negative power of* p *is*

$$\delta_t(C_p) = \left(\frac{p^t}{p^t-1}\right)\frac{1}{\zeta(t)} ; \tag{5.6}$$

(iv) *with index factor a divisor of r is*

$$\delta_t(D_r) = \frac{\sigma_t(r)}{r^t\zeta(t)}\,; \tag{5.7}$$

(v) *with index factor a complete divisor of r is*

$$\delta_t(E_r) = \frac{\sigma'_t(r)}{r^t\zeta(t)} = \frac{\sigma'_{-t}(r)}{\zeta(t)}\,. \tag{5.8}$$

The results contained in (5.4) and (5.5) are due originally to Gegenbauer [5]. In case $k = 1$, (5.5) becomes $\delta_t(B_1) = 1/\zeta(t)$, $t \geqq 2$ [9, p. 156]. Further specialization of (5.5) to the case $k = 1$, $t = 2$ yields the classical result [7, Theorem 332] asserting that the probability that a pair of integers be relatively prime is $6/\pi^2$. By (5.4), with $k = 2$, $t = 2$, it follows that the density of the integral pairs whose greatest common divisor is a perfect square is $\pi^2/15$. The case $p = 2$, $t = 2$ in (5.6) shows that the density of the integral pairs with greatest common divisor a power of 2 is $8/\pi^2$. By (5.7) with $r = 8$, $t = 2$, it follows that the density of the pairs of integers whose greatest common divisor is a factor of 8 is $255/32\pi^2$.

COROLLARY 5.3. *If $t \geqq 2$ and r is a positive integer, then the asymptotic density of the Z_t-vectors with index factor r is*

$$\delta_t(r) = \frac{1}{r^t\zeta(t)}\,. \tag{5.9}$$

Sketch of proof. The corollary is true in case $r = 1$, as noted above on the basis of (5.5), or alternatively by (5.7) with $r = 1$. The proof can be completed for arbitrary r by induction on the number of distinct prime factors of r and application of (5.8). The details are omitted.

The preceding corollary is due to Kronecker in case $t = 2$ [8, p. 311]. It was proved in the general case by Cesàro [1, p. 293]; a further generalization was given by G. Daniloff [4, p. 587].

6. Generalization of the second Möbius inversion formula. In case $P = 1$, $Q = Z$, the following inversion relation reduces to a familiar analogue [7, Theorem 268] of the Möbius inversion formula.

THEOREM 6.1. *Let x denote a positive real variable; then*

$$f(x) = \sum_{n \leqq x} \rho_Q(n) g\left(\frac{x}{n}\right) \rightleftarrows g(x) = \sum_{n \leqq x} \mu_P(n) f\left(\frac{x}{n}\right)\,. \tag{6.1}$$

Proof. Let $g(x)$ be defined as on the right of (6.1). Then

$$\sum_{n\leq x}\rho_Q(n)g\Big(\frac{x}{n}\Big)=\sum_{n\leq x}\rho_Q(n)\sum_{\substack{d\leq x/n\\(l=nd)}}\mu_P(d)f\Big(\frac{x/n}{d}\Big)$$
$$=\sum_{l\leq x}f\Big(\frac{x}{l}\Big)\sum_{l=dn}\mu_P(d)\rho_Q(n)=f(x)\ ,$$

on the basis of (2.4). The converse is proved similarly.

We define $[x]_P$ to be the number of positive integers $\leqq x$ belonging to P. It is evident, by property (ii) of the conjugate pair P, Q, that

$$[x]=[x]_Z=\sum_{\substack{n\leq x\\n\in Q}}\Big[\frac{x}{n}\Big]_P=\sum_{n\leq x}\Big[\frac{x}{n}\Big]_P\rho_Q(n)\ . \tag{6.2}$$

Applying the above inversion theorem to (6.), one obtains

THEOREM 6.2.

$$[x]_P=\sum_{n\leq x}\mu_P(n)\Big[\frac{x}{n}\Big]\ . \tag{6.3}$$

We deduce two special cases of (6.3). Let A_k, B_k be the P-sets defined in § 3 and place (as in I), $\lambda_k(n)=\mu_{A_k}(n)$, $\mu_k(n)=\mu_{B_k}(n)$. Putting $[x]_k=[x]_{B_k}$ and nothing that $[\sqrt[k]{x}]=[x]_{A_k}$, one obtains

COROLLARY 6.1.

$$[x]_k=\sum_{n\leq x}\mu_k(n)\Big[\frac{x}{d^k}\Big]=\sum_{d^k\leq x}\mu(d)\Big[\frac{x}{d^k}\Big]\ , \tag{6.4}$$

$$[\sqrt[k]{x}]=\sum_{n\leq x}\lambda_k(n)\Big[\frac{x}{n}\Big]\ . \tag{6.5}$$

These formulas are classical [6], [9, p. 35]. Note that (6.4) and (6.5) reduce to (1.3) in the cases $k=1$ and $k=0$, respectively.

It can be shown easily, on the basis of (6.4), that $\delta_1(B_k)=1/\zeta(k)$, $k>1$ (cf. [7, Theorem 333] in case $k=2$). In words, this states that the asymptotic density of the k-free integers ($k\geqq 2$) is $1/\zeta(k)$; in conjunction with (5.5) it therefore follows that

COROLLARY 6.2. *If $kt\geqq 2$, then the asymptotic density of the Z_t-vectors with k-free index factor is $1/\zeta(kt)$.*

Finally, we consider the function $\phi_P(x,n)$ defined to be the number of positive integers $a\leqq x$ such that $(a,n)\in P$. In case $P=1$, $\phi_P(x,n)$ becomes Legendre's function $\phi(x,n)$. To deal with $\phi_P(x,n)$ we have the following extension of [3, Theorem 4] which can be proved in much the same way.

LEMMA 6.1. *Let d range over the divisors of n, $d\in Q$, and for* *each such d, let y range over the positive integers $a \leqq x/d$ such that $(a, n/d) \in P$. Then the set dy consists of the positive integers $\leqq x$*

An immediate consequence of this lemma is

THEOREM 6.3.

$$\sum_{d|n} \phi_P\left(\frac{x}{d}, \frac{n}{d}\right)\rho_Q(d) = [x] . \tag{6.6}$$

THEOREM 6.4.

$$\phi_P(x, n) = \sum_{d|n} \mu_P(d)\left[\frac{x}{d}\right] . \tag{6.7}$$

Theorem 6.4 can be deduced from (6.6) by a direct application of the following easily proved extension of (2.3).

THEOREM 6.5. *If $f(x, n)$ and $g(x, n)$ are functions of the real variable x and the positive integral variable n, then*

$$g(x, n) = \sum_{d|n} \rho_Q(d) f\left(\frac{x}{d}, \frac{n}{d}\right) \rightleftarrows f(x, n) = \sum_{d|n} \mu_P(d) g\left(\frac{x}{d}, \frac{n}{d}\right) . \tag{6.8}$$

The proof is omitted.

BIBLIOGRAPHY

1. Ernest Cesàro, *Sur le plus grand commun diviseur de plusieurs nombres*, Annali di Matematica, **13** (1885), II, 291-294.

2. ———, *Sur l'étude des événements arithmétiques*, Mémoires de l'académie royale des sciences de Belgique, **47** (1886), 13 pages.

3. Eckford Cohen, *A class of residue systems* (mod r) *and related arithmetical functions, I. A generalization of Möbius inversion*, Pacific J. Math., **9** (1959), 13-23.

4. Georges Daniloff, *Contribution à la théorie des fonctions arithmétiques*, Recueil de l'Académie des Sciences Bulgare, **35** (1941), 479-590. (Bulgarian, French summary).

5. Leopold Gegenbauer, *Einige asymptotische Gesetze der Zahlentheorie*, Sitzungsberichte der Akademie der Wissenschaften (Vienna), **92** (1885), 1290-1306.

6. ———, *Über die Anzahl der Primzahlen*, Sitzungsberichte der Akademie der Wissenschaften (Vienna), **95** (1887), 94-96.

7. G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* (3rd edition), Oxford, 1954.

8. Leopold Kronecker, *Vorlesungen über Zahlentheorie*, Leipzig, 1901.

9. I. M. Vinogradov, *Elements of Number Theory*, New York, 1954, (English translation).

10. Aurel Wintner, *The Theory of Measure in Arithmetical Semigroups*, Baltimore, 1944.

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