MULTIPLICATION FORMULAS FOR PRODUCTS OF BERNOULLI AND EULER POLYNOMIALS

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1. Put

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\ \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\ . \tag{1.1}$$

The following multiplication formulas are familiar [5, pp. 18, 24]:

$$B_m(kx) = k^{m-1}\sum_{r=0}^{k-1} B_m\left(x + \frac{k}{r}\right)\ , \tag{1.2}$$

$$E_m(kx) = k^m \sum_{r=0}^{k-1} (-1)^r E_m\left(x + \frac{k}{r}\right) \qquad (k \text{ odd})\ . \tag{1.3}$$

Let $\overline{B}_m(x)$, $\overline{E}_m(x)$ denote, respectively, the Bernoulli and Euler functions defined by

$$\overline{B}_m(x) = B_m(x)(0 \le x < 1),\ \overline{B}_m(x + 1) = \overline{B}_m(x)\ ,$$
$$\overline{E}_m(x) = E_m(x)(0 \le x < 1),\ \overline{E}_m(x + 1) = -\overline{E}_m(x)\ , \quad (m \ge 1)\ .$$

Then $\overline{B}_m(x)$ and $\overline{E}_m(x)$ also satisfy the multiplication formulas (1.2), (1.3).

In this note we obtain some generalizations of (1.2) and (1.3) suggested by a recent result of Mordell [4]. In extending some results of Mikolás [3], Mordell proves the following theorem. Let $f_1(x), \cdots, f_n(x)$ denote functions of x of period 1 that satisfy the relations

$$\sum_{r=0}^{k-1} f_i\left(r + \frac{r}{k}\right) = C_i^{(k)} f_i(kx) \qquad (i = 1, \cdots, n)\ , \tag{1.4}$$

where $C_i^{(k)}$ is independent of x. Let $a_1, \cdots, a_n$ be positive integers that are relatively prime in pairs. Then if the integrals exist and $A = a_1 a_2 \cdots a_n$,

$$\begin{aligned} &\int_0^A f_1\left(\frac{x}{a_1}\right) f_2\left(\frac{x}{a_2}\right) \cdots f_n\left(\frac{x}{a_n}\right) dx \\ &= A\int_0^1 f_1\left(\frac{Ax}{a_1}\right) f_2\left(\frac{Ax}{a_2}\right) \cdots f_n\left(\frac{Ax}{a_n}\right) dx \\ &= C_1^{(a_1)} C_2^{(a_2)} \cdots C_n^{(a_n)} \int_0^1 f_1(x) f_2(x) \cdots f_n(x) dx\ . \end{aligned} \tag{1.5}$$

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2. We first prove

THEOREM 1. *Let* $n \geq 1$; $m_1, \cdots, m_n \geq 1$; $a_1, a_2, \cdots, a_n$ *positive integers that are relative prime in pairs*; $A = a_1, a_2, \cdots, a_n$. *Then*

$$\sum_{r=0}^{kA-1} \overline{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right)\overline{B}_{m_2}\left(x_2 + \frac{r}{a_2 k}\right) \cdots \overline{B}_{m_n}\left(x_n + \frac{r}{a_n k}\right) \tag{2.1}$$
$$= C \sum_{r=0}^{k-1} \overline{B}_{m_1}\left(a_1 x_1 + \frac{r}{k}\right)\overline{B}_{m_2}\left(a_2 x_2 + \frac{r}{k}\right) \cdots \overline{B}_{m_n}\left(a_n x_n + \frac{r}{k}\right),$$

where

$$C = a_1^{1-m_1} a_2^{1-m_2} \cdots a_n^{1-m_n} . \tag{2.2}$$

In the first place for $n = 1$ it follows from (1.2) for arbitrary $a \geq 1$ that

$$\sum_{r=0}^{ka-1} \overline{B}_m\left(x + \frac{r}{ak}\right) = \sum_{r=0}^{k-1} \sum_{s=0}^{a-1} \overline{B}_m\left(r + \frac{s}{a} + \frac{r}{ak}\right)$$
$$= \sum_{r=0}^{k-1} \overline{B}_m\left(ax + \frac{r}{k}\right),$$

which agrees with (2.1).

For the general case, let S denote the left member of (2.1). Put

$$A_s = a_1 a_2 \cdots a_s \qquad (1 \leq s \leq n)$$

and replace r by $skA_{n-1} + r$. Then

$$S = \sum_{r=0}^{kA_{n-1}-1} \overline{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \overline{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right)$$
$$\cdot \sum_{s=0}^{a_n-1} \overline{B}_{m_n}\left(x_n + \frac{A_{n-1}s}{a_n} + \frac{r}{a_n k}\right)$$
$$= \sum_{r=0}^{kA_{n-1}-1} \overline{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \overline{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right)$$
$$\cdot \sum_{s=0}^{a_n-1} \overline{B}_{m_n}\left(x_n + \frac{s}{a_n} + \frac{r}{a_n k}\right)$$
$$= a_n^{1-m_n} \sum_{r=0}^{kA_{n-1}-1} \overline{B}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right) \cdots \overline{B}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1} k}\right)$$
$$\cdot \overline{B}_{m_n}\left(a_n x_n + \frac{r}{k}\right).$$

Continuing in this way we get

$$S = a_{n-1}^{1-m_{n-1}} a_n^{1-m_n} \sum_{r=0}^{kA_{n-2}-1} \overline{B}_{m_1}\Big(x_1 + \frac{r}{a_1 k}\Big) \cdots \overline{B}_{m_{n-2}}\Big(x_{n-2} + \frac{r}{a_{n-2}k}\Big)$$

$$\cdot \overline{B}_{m_{n-1}}\Big(a_{n-1}x_{n-1} + \frac{r}{k}\Big)\overline{B}_{m_n}\Big(a_n x_n + \frac{r}{k}\Big)$$

$$= a_1^{1-m_1} \cdots a_n^{1-m_n} \sum_{r=0}^{k-1} \overline{B}_{m_1}\Big(a_1 x_1 + \frac{r}{k}\Big)\overline{B}_2\Big(a_2 x_2 + \frac{r}{k}\Big)$$

$$\cdots \overline{B}_{m_n}\Big(a_n x_n + \frac{r}{k}\Big) .$$

For $k = 1$, (2,1) reduces to

$$\sum_{r=0}^{A-1} \overline{B}_{m_1}\Big(x_1 + \frac{r}{a_1}\Big)\overline{B}_2\Big(x_2 + \frac{r}{a_2}\Big) \cdots \overline{B}_n\Big(x_n + \frac{r}{a_n}\Big) \tag{2.3}$$

$$= C \cdot \overline{B}_{m_1}(a_1x_1)\overline{B}_{m_2}(a_2x_2) \cdots \overline{B}_{m_n}(a_nx_n) ,$$

where C is defined by (2.2); (2.3) may be considered a direct generalization of (1.2).

We remark that a formula like (2.1) holds for any set of functions satisfying (1.4).

We note also that the formula (2.2) can be proved by means of the Chinese remainder theorem. This remarks applies also to formulas (3.4) and (4.8) below.

3. In the next place we have

THEOREM 2. *Let n be odd and ≥ 1; $m_1, \cdots, m_n \geq 1$; $a_1, a_2, \cdots, a_n$ positive odd integers that are relatively prime in pairs; $A = a_1 a_2 \cdots a_n$; k odd ≥ 1. Then*

$$\sum_{r=0}^{kA-1} (-1)^r \overline{E}_{m_1}\Big(x_1 + \frac{r}{a_1 k}\Big) \cdots \overline{E}_{m_n}\Big(x_n + \frac{r}{a_n k}\Big) \tag{3.1}$$

$$= C' \sum_{r=0}^{k-1} (-1)^r \overline{E}_{m_1}\Big(a_1 x_1 + \frac{r}{k}\Big) \cdots \overline{E}_{m_n}\Big(a_n x_n + \frac{r}{k}\Big) ,$$

where

$$C' = a_1^{-m_1} a_2^{-m_2} \cdots a_n^{-m_n} . \tag{3.2}$$

The proof is similar to that of Theorem 1, but makes use of (1.3) in place of (1.2); also the formula

$$\overline{E}_m(x + r) = (-1)^r \overline{E}_m(x) \qquad (m \geq 1) \tag{3.3}$$

is needed.

For $n = 1$ and a odd, we have

$$\sum_{r=0}^{ka-1}(-1)^r \bar{E}_{m_1}\left(x + \frac{r}{ak}\right) = \sum_{r=0}^{k-1}(-1)^{sk}\bar{E}_m\left(x + \frac{s}{a} + \frac{r}{ak}\right)$$
$$= a^{-m}\sum_{r=0}^{k-1}(-1)^r \bar{E}_m\left(ax + \frac{r}{k}\right),$$

which agrees with (3.1). For the general case let S' denote the left member of (3.1). Then

$$S' = \sum_{r=0}^{kA_{n-1}-1}\sum_{s=0}^{a_n-1}(-1)^{r+s}\bar{E}_{m_1}\left(x_1 + \frac{sA_{n-1}}{a_1} + \frac{r}{a_1 k}\right)\cdots$$
$$\cdot \bar{E}_{m_{n-1}}\left(x_{n-1} + \frac{sA_{n-1}}{a_{n-1}} + \frac{r}{a_{n-1}k}\right)$$
$$\cdot \bar{E}_{m_n}\left(x_n + \frac{sA_{n-1}}{a_n} + \frac{r}{a_n k}\right).$$

If we put

$$sA_{n-1} = qa_n + t \qquad (0 \leq t < a_n),$$

then $s \equiv q + t \pmod 2$, so that

$$\bar{E}_{m_n}\left(x_n + \frac{sA_{n-1}}{a_n} + \frac{r}{a_n k}\right) = (-1)^q \bar{E}_{m_n}\left(x_n + \frac{t}{a_n} + \frac{r}{a_n k}\right).$$

Since n is odd we therefore get

$$S' = \sum_{r=0}^{kA_{n-1}-1}(-1)^r \bar{E}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right)\cdots \bar{E}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1}k}\right)$$
$$\cdot \sum_{t=0}^{a_n-1}(-1)^t \bar{E}_{m_n}\left(x_n + \frac{t}{a_n} + \frac{r}{a_n k}\right)$$
$$= \sum_{r=0}^{kA_{n-1}-1}(-1)^r \bar{E}_{m_1}\left(x_1 + \frac{r}{a_1 k}\right)\cdots \bar{E}_{m_{n-1}}\left(x_{n-1} + \frac{r}{a_{n-1}k}\right)$$
$$\cdot a_n^{-m_n}\bar{E}_{m_n}\left(a_n x_n + \frac{r}{k}\right).$$

Continuing in this way we ultimately reach (3.1).

For $k = 1$, (3.1) becomes

$$\sum_{r=0}^{A-1}(-1)^r \bar{E}_{m_1}\left(x_1 + \frac{r}{a_1}\right)\cdots \bar{E}_{m_n}\left(x_n + \frac{r}{a_n}\right) \tag{3.4}$$
$$= C' E_{m_1}(a_1 x_1)\cdots E_{m_n}(a_n x_n),$$

subject to the conditions of the theorem.

4. **Theorem 2 can be extended further** by introducing the "Eulerian" polynomial [2] $\phi_m(x, \rho)$ defined by

$$\frac{1-\rho}{1-\rho e^t} e^{xt} = \sum_{m=0}^{\infty} \phi_m(x, \rho) \frac{t^m}{m!} \qquad (\rho \neq 1) . \tag{4.1}$$

In particular $\phi_m(x, -1) = E_m(x)$.

We shall assume that the parameter ρ is an fth root of unity. It follows easily from (4.1) that

$$\phi_{m-1}(kx, \rho) = \frac{(\rho - 1)f^{m-1}}{m} \sum_{r=0}^{f-1} \rho^r B_m\left(x + \frac{r}{f}\right) . \tag{4.2}$$

We accordingly define the function $\overline{\phi}_n(x, \rho)$ *by means of*

$$\overline{\phi}_{m-1}(kx, \rho) = \frac{(\rho - 1)e^{m-1}}{m} \sum_{r=0}^{f-1} \rho^r \overline{B}_m\left(x + \frac{r}{f}\right) . \tag{4.3}$$

It follows from (4.3) that

$$\overline{\phi}_n(x + 1, \rho) = \rho^{-1}\overline{\phi}_n(x, \rho) , \tag{4.4}$$

so that if ρ is a primitive fth root of unity, $\overline{\phi}_n(x, \rho)$ has period f. Also by means of (4.1) we readily obtain the multiplication theorem [1] *valid for* $k \equiv 1 \pmod f$

$$\sum_{r=0}^{k-1} \rho^r \phi_m\left(x + \frac{r}{k}, \rho\right) = k^{-m}\phi_m(kx, \rho) \tag{4.5}$$

and consequently

$$\sum_{r=0}^{k-1} \rho^r \overline{\phi}_m\left(x + \frac{r}{k}, \rho\right) = k^{-m}\overline{\phi}_m(kx, \rho) . \tag{4.6}$$

We may now state

THEOREM 3. *Let* $f > 1, n \equiv 1 \pmod f$; $m_1, \cdots, m_n \geq 1, a_1, a_2, \cdots, a_n$ *positive integers that are relatively prime in pairs and such that* $a_i \equiv 1 \pmod f$ *for* $i = 1, \cdots, n$; *also let* $k \equiv 1 \pmod f$. *Then if* $A = a_1 a_2 \cdots a_n$, *we have*

$$\sum_{r=0}^{kA-1} \rho^r \overline{\phi}_{m_1}\left(x_1 + \frac{r}{a_1 k}, \rho\right) \cdots \overline{\phi}_{m_n}\left(x_n + \frac{r}{a_n k}, \rho\right) \tag{4.7}$$

$$= C' \sum_{r=0}^{k-1} \rho^r \overline{\phi}_{m_1}\left(a_1 x_1 + \frac{r}{k}, \rho\right) \cdots \overline{\phi}_{m_n}\left(a_n x_n + \frac{r}{k}, \rho\right),$$

where C' *is defined by* (3.2).

The proof is very much like that of Theorem 2 and will be omitted. We remark that for $k = 1$, (4.7) becomes

(4.8)
$$\sum_{r=0}^{A-1} \rho^r \overline{\phi}_{m_1}\left(x_1 + \frac{r}{a_1}, \rho\right) \cdots \overline{\phi}_{m_n}\left(x_n + \frac{r}{a_n}, \rho\right)$$
$$= C' \overline{\phi}_{m_1}(a_1 x_1, \rho) \cdots \overline{\phi}_{m_n}(a_n x_n, \rho) .$$

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