VIBRATION OF A NONHOMOGENEOUS MEMBRANE

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1. Introduction. We consider a simply connected two dimensional domain D with a nonhomogeneous membrane M stretched across D and fixed at the boundary Γ. Let $p(x, y) \geq 0$ be the density function of the membrane. We shall be concerned with the first eigenvalue λ_0 of the equation

$$u_{xx} + u_{yy} + \lambda p(x, y)u = 0 \tag{1}$$

subject to the condition $u = 0$ on Γ. Let K be the circle with boundary C on which a homogeneous membrane M_1 of the same mass as M is stretched. Let λ_1 be the first eigenvalue of

$$v_{xx} + v_{yy} + \lambda v = 0 \tag{2}$$

with $v = 0$ on C. In a recent paper Nehari [1] established the following interesting result.

THEOREM. (Nehari) *If* $\log p(x,y)$ *is subharmonic then*

$$\lambda_0 \geq \lambda_1 . \tag{3}$$

Nehari further showed that relaxation to the condition that $p(x, y)$ be subharmonic is not possible. In fact for the case that D is a circle and $p(x, y)$ is superharmonic the inequality in (3) is shown to be reversed.

It is the purpose of this paper to establish comparison theorems for the first eigenvalue of homogeneous and nonhomogeneous membranes *of the same shape*. That is, we shall consider the first eigenvalue of equations (1) and (2) in the same domain D subject to the boundary condition $u = 0$ and $v = 0$ on Γ respectively. We denote the first eigenvalue of the latter problem by μ and consider comparisons between λ_0 and μ. We of course have the completely trivial comparison

$$\lambda_0 \geq \mu$$

if $0 \leq p(x, y) \leq 1$ throughout D. Nehari's result pertained to the case where $p(x, y)$ had average value 1 and thus we wish to obtain relations between λ_0 and μ for density functions which may become large.

A general technique for obtaining lower bounds for the first eigenvalue for a homogeneous membrane in a domain D follows from the

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inclusion principle. If D is contained in D_0 then the first eigenvalue for D is larger than that for D_0. If D is bounded then we can enclose D in a rectangle or circle for which the first eigenvalue is known. This technique is also possible for nonhomogeneous membranes as will be readily seen from the basic inequalities established in § 2. In § 3 comparison theorems are established when the density function is assumed to satisfy various conditions involving the behavior of the second derivative of $p(x, y)$. Section 4 discusses comparison theorems between two nonhomogeneous membranes.

2. **Basic inequalities.** Let u be any function which vanishes on Γ, and let $a(x, y)$ be an arbitrary C^2 function in D. We apply Green's theorem to the expression

$$\iint_D au(u_{xx} + u_{yy})dxdy$$

and obtain

$$(4) \quad \iint_D au(u_{xx} + u_{yy})dxdy = -\iint_D a(u_x^2 + u_y^2)dxdy + \frac{1}{2}\iint_D u^2(a_{xx} + a_{yy})dxdy$$

The boundary integrals vanishing in virtue of $u = 0$ on Γ. Further we let $P(x, y)$, $Q(x, y)$ be arbitrary C' functions in D and note that

$$(5) \quad \iint [Pu^2)_x + (Qu^2)_y]dxdy = 0 .$$

Performing the differentiations in (5) and adding the result to (4) we get

$$\begin{aligned} -\iint_D & au(u_{xx} + u_{yy})dxdy \\ = +\iint_D & \Big\{ a(u_x^2 + u_y^2) + 2Puu_x + 2Quu_y \\ & + \Big[P_x + Q_y - \frac{1}{2}(a_{xx} + a_{yy})\Big]u^2\Big\}dxdy . \end{aligned}$$

If u were the first eigenfunction and λ the first eigenvalue of the nonhomogeneous membrane, then (1) would hold and the above expression would be

$$(6) \quad \begin{aligned} \iint_D & \Big\{a(u_x^2 + u_y^2) + 2Puu_x + 2Quu_y \\ & + \Big[P_x + Q_y - \frac{1}{2}(a_{xx} + a_{yy}) - a\lambda p\Big]u^2\Big\}dxdy = 0 \end{aligned}$$

On the other hand this integrand is a quadratic form in u_x, u_y, u. It will be a positive definite form if $a > 0$ and

$$(7) \qquad P_x + Q_y \geq \frac{1}{a}(P^2 + Q^2) + \frac{1}{2}(a_{xx} + a_{yy}) + ap\lambda \ .$$

If a, P, Q, λ happen to satisfy (7) then clearly it is impossible that (6) holds. However if (7) holds for any value $\bar{\lambda}$, it obviously holds for $0 \leq \lambda \leq \bar{\lambda}$ and thus (6) cannnot hold for any function $u(x, y)$ with $0 \leq \lambda \leq \bar{\lambda}$. This implies that $\bar{\lambda}$ is a lower bound for the first eigenvalue of (1).

We shall therefore be concerned with the possibility of selection of functions P, Q, a such that inequality (7) holds for some value $\bar{\lambda}$. For convenience we assume the bounded domain D is in the first quadrant. We select the function $a(x, y)$ to be

$$a(x, y) = \sin \alpha x \sin \beta x$$

where α and β are constants selected so that $a(x, y)$ is positive throughout $\bar{D}$. We define the quantities

$$m_0 = \min_{\bar{D}} a$$

and $M_0 = m_0^{-1}$. Inequality (7) is implied by the inequality

$$(8) \qquad P_x + Q_y \geq M_0(P^2 + Q^2) + \frac{1}{2}(a_{xx} + a_{yy}) + ap\lambda$$

and if we define

$$P_1 = M_0 P, \, Q_1 = M_0 Q$$

(8) is equivalent to

$$(9) \qquad P_{1x} + Q_{1y} \geq P_1^2 + Q_1^2 + \frac{1}{2}M_0(a_{xx} + a_{yy}) + M_0 ap\lambda \ .$$

Let $\phi(x, y)$ be the first eigenfunction for equation (2) in the domain D subject to the condition $v = 0$ on Γ. That is,

$$\phi_{xx} + \phi_{yy} + \mu\phi = 0 \ .$$

We make the following selection:

$$P_1 = -\frac{\phi_x}{\phi}, \quad Q_1 = -\frac{\phi_y}{\phi}$$

and obtain from (9)

$$\mu \geq M_0 \sin \alpha x \sin \beta y \left[-\frac{1}{2}(\alpha^2 + \beta^2) + \lambda p(x, y) \right] \tag{10}$$

Define the quantity

$$N_0 = \max_{\bar{D}} p(x, y) \sin \alpha x \sin \beta y$$

and we obtain the following result.

THEOREM 1. *Let λ_0 be the first eigenvalue for the nonhomogeneous membrane with density function $p(x, y)$ spanning a domain D and μ the first eigenvalue for the homogeneous membrane spanning the same domain. Then*

$$\lambda_0 \geq \frac{\mu + \frac{1}{2}(\alpha^2 + \beta^2)}{M_0 N_0} . \tag{11}$$

The theorem is an immediate consequence of inequality (10) which exhibits the positive definiteness of the integrand (6). Inequality (11) is a statement that (10) must be violated.

We note that (11) is a useful relation if N_0 is particularly small; hence this states that $p(x, y)$ should be small near the center of the membrane, but may be large near the outer edge and still (11) will be a significant lower bound for λ_0. The basic distinction between (11) and other results lies in the fact that $p(x, y)$ has no restriction except positivity.

A word should be said about the selection of the function $a(x, y)$. We chose for this function the first eigenfunction for the equation (2) applied to a rectangle which contains D in its interior. We could have selected for $a(x, y)$ the first eigenfunction for any including domain, e.g., a circle, equilateral triangle, etc. with a resulting inequality similar to (11). Finally the selection $a \equiv 1$ yields the standard result

$$\lambda_0 \geq \frac{\mu}{\max_{\bar{D}} p(x, y)} .$$

3. Bounds with condition on the density function. We return to inequality (7) and the selection of a, P, and Q. We recall that these functions may be arbitrary except that $a(x, y)$ must be positive. We make the choice

$$a(x, y) = \frac{1}{p(x, y)} \tag{12}$$

Then (7) becomes

$$P_x + Q_y \geq p(x, y)(P^2 + Q^2) + \frac{1}{2}\Delta\left(\frac{1}{p}\right) + \lambda .$$

We define

$$p_0 = \max_{\bar{D}} p(x, y) \tag{13}$$

and select

$$P = -\frac{\phi_x}{p_0\phi}, \quad Q = -\frac{\phi_y}{p_0\phi}$$

where, as before, ϕ is the first eigenfunction of (2) for the domain D. We obtain

$$\frac{\mu}{p_0} \geq \frac{1}{2}\Delta\left(\frac{1}{p}\right) + \lambda .$$

If we assume the function $1/p$ is superharmonic and set

$$N_1 = -\max_{\bar{D}} \frac{1}{2}\Delta\left(\frac{1}{p}\right) \tag{14}$$

we obtain the following result.

THEOREM 2. *Let λ_0 be the first eigenvalue for the nonhomogeneous membrane with density function $p(x, y)$ and μ the corresponding first eigenvalue for the homogeneous membrane spanning the same domain D. If $1/p$ is superharmonic in D we have the inequality*

$$\lambda_0 \geq \frac{\mu}{p_0} + N_1 \tag{15}$$

where p_0 and N_1 are given by (13) *and* (14) *respectively.*

It is possible to obtain a comparison theorem for the case where $\log p$ is subharmonic. To see this we make the choice

$$a(x, y) = \log \frac{1}{p}$$

and we assume $0 < p(x, y) < 1$ in $\bar{D}$. With this selection we take

$$P = -\frac{\phi_x}{p_0\phi}, \quad Q = -\frac{\phi_y}{p_0\phi}$$

as before and obtain

$$\frac{\mu}{p_0} \geq \frac{1}{2}\Delta\left(\log \frac{1}{p}\right) + \lambda p \log \frac{1}{p} .$$

We assume $\log p$ is subharmonic and define

$$N_2 = \frac{1}{2} \min_{\bar{D}} \Delta(\log p) \tag{16}$$

$$N_3 = \max_{\bar{D}} p \log \frac{1}{p} . \tag{17}$$

THEOREM 3. *Let* λ_0 *and* μ *be as in Theorem* 2. *If* $\log p$ *is subharmonic in* D *then*

$$\lambda_0 \geq \frac{\mu}{p_0 N_3} + \frac{N_2}{N_3}$$

where N_2 *and* N_3 *are given by* (16) *and* (17).

A final application of this type which we exhibit results from the selection

$$a = e^{\alpha p(x,y)}$$

where α is a constant which remains to be chosen. If we suppose that p is strictly superharmonic and select α so that

$$\frac{1}{2}\Delta p + \alpha(p_x^2 + p_y^2) \leq 0$$

we obtain the relation

$$\lambda_0 \geq \mu \max_{\bar{D}} \left(\frac{e^{-\alpha p}}{p}\right) .$$

4. Comparison of two nonhomogeneous membranes. Let $q(x, y)$ be a second density function corresponding to a membrane spanning D and let ν be the first eigenvalue for

$$w_{xx} + w_{yy} + \nu q(x, y) w = 0 \tag{18}$$

with boundary condition $w = 0$ on Γ. We denote the corresponding first eigenfunction by $\psi(x, y)$. It is possible to compare λ_0 and ν when the functions p and q satisfy various relations. Let

$$q_0 = \max_{\bar{D}} q(x, y) \tag{19}$$

$$r_0 = \max_{\bar{D}} \frac{p(x, y)}{q(x, y)} \tag{20}$$

and

$$N_4 = -\max_{\bar{D}} \Delta\left(\frac{q}{p}\right). \tag{21}$$

We make the selections

$$a = \frac{q}{p}, \quad P = -\frac{\psi_x}{r_0\psi}, \quad Q = -\frac{\psi_y}{r_0\psi}$$

and find

$$\frac{\nu q}{r_0} \geq \frac{1}{2}\Delta\left(\frac{q}{p}\right) + q\lambda .$$

THEOREM 4. *Let* λ_0 *and* ν *be the first eigenvalue corresponding to density functions* p *and* q *respectively. If* q/p *is superharmonic then we have the inequality*

$$\lambda_0 \geq \frac{\nu}{r_0} + \frac{1}{2}\frac{N_4}{q_0}$$

where q_0, r_0 *and* N_4 *are given by* (19), (20) *and* (21).

Additional inequalities, analogous to those obtained in §§ 2 and 3 may be obtained by other selections for a, P and Q.

BIBLIOGRAPHY

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